CHAPTER V

Covariant statistics: the laws for material media

1 *Introduction*

In order to find the covariant macroscopic laws from the corresponding microscopic equations one has to introduce a covariant averaging procedure. To that end covariant distribution functions will be employed that describe the statistical properties of collections of world lines in Minkowski space. In connexion with this, several covariant distribution functions of a particular type will ensue: 'synchronous', 'retarded' and 'advanced' distribution functions, which are useful to describe averages of certain microscopic quantities. The first of these is a direct generalization of the non-relativistic distribution function, while the latter have no non-relativistic counterparts.

With the help of the covariant averaging procedure we then derive the Maxwell equations, the energy–momentum and angular momentum balances and the thermodynamical laws. All macroscopic quantities occurring in these laws will be found as statistical expressions in terms of microscopic quantities. In particular we shall obtain in this way an expression for the macroscopic energy–momentum tensor of a polarized medium in the presence of electromagnetic fields. It will be shown that two ways of splitting the tensor in so-called field and material parts present themselves in a natural way. This result throws light on the much discussed controversy on the 'correct form' of the field part of the energy–momentum tensor. In fact this problem is not well posed if one does not bring into the discussion also the expression for the corresponding material energy–momentum tensor. Only the sum of the two parts of the tensor is physically significant. Nevertheless it is sometimes convenient to introduce a definite splitting to discuss certain physical properties of the system. The situation is analogous to the one encountered in the non-relativistic theory, where we found various expressions for the ponderomotive force density in a polarized medium, each with its corresponding material pressure tensor.

2 *Covariant statistical mechanics*

a. *Covariant distribution functions*

A system of N point particles $i = 1, 2, \ldots$ is completely specified by giving their world lines $R_i^\alpha(s_i)$ in Minkowski space. (The world lines are parametrized by means of their proper times s_i.) The number of world lines that intersect a three-surface element $\mathrm{d}^3\Sigma$ (with time-like normal n^α, $n^0 > 0$) at the position R_1^α, is given by

$$\sum_i \delta^{(3)}\{n; R_1 - R_i(s_i)\}|_{n\cdot\{R_1 - R_i(s_i)\}=0}\, \mathrm{d}^3\Sigma. \tag{1}$$

The bar, with the equation $n\cdot\{R_1 - R_i(s_i)\} = 0$, indicates that one has to take in the delta function the proper times s_{i0} which are the solutions of this equation. The delta function employed here is the generalization of the ordinary delta function $\delta\{\boldsymbol{R}_1 - \boldsymbol{R}_i(s_i)\}$: in fact it is equal to it if n^α is purely time-like: $(1, 0, 0, 0)$. In the general case it is defined by writing

$$\delta^{(4)}(x) = \delta^{(3)}(n; x)\delta(n\cdot x), \tag{2}$$

where x^α is an arbitrary four-vector. If one wants to specify also the four-velocity and higher proper time derivatives as lying in the intervals $(R_1^{[1]\alpha}, R_1^{[1]\alpha} + \mathrm{d}^4 R_1^{[1]\alpha})$ etc.[1], one gets for the number of world lines crossing $\mathrm{d}^3\Sigma$:

$$\mathrm{d}N = \sum_i \delta^{(3)}\{n; R_1 - R_i(s_{i0})\}\delta^{(4)}\{R_1^{[1]} - R_i^{[1]}(s_{i0})\} \ldots \mathrm{d}^3\Sigma\, \mathrm{d}^4 R_1^{[1]} \ldots, \tag{3}$$

where $R_i^{[n]\alpha}(s_i) \equiv \mathrm{d}^n R_i^\alpha(s_i)/\mathrm{d}s_i^n$. By adding a factor $\delta[n\cdot\{R_1 - R_i(s_i)\}]$ and an integration over $n\cdot R_i(s_i)$ one obtains after introduction of the new integration variable s_i:

$$\mathrm{d}N = -\sum_i \int \delta^{(4)}\{R_1 - R_i(s_i)\} n\cdot R_i^{[1]}(s_i) \delta^{(4)}\{R_1^{[1]} - R_i^{[1]}(s_i)\} \ldots \mathrm{d}s_i\, \mathrm{d}^3\Sigma\, \mathrm{d}^4 R_1^{[1]} \ldots. \tag{4}$$

We may write this as

$$\mathrm{d}N = -c^{-1} n\cdot R_1^{[1]} f_1(1) \mathrm{d}^3\Sigma\, \mathrm{d}^4 R_1^{[1]} \ldots \tag{5}$$

with the abbreviation

$$f_1(1) \equiv c \sum_i \int \delta^{(4)}\{R_1 - R_i(s_i)\}\delta^{(4)}\{R_1^{[1]} - R_i^{[1]}(s_i)\} \ldots \mathrm{d}s_i, \tag{6}$$

[1] The upper indices between square brackets indicate the number of differentiations with respect to proper time. The four-velocity $\mathrm{d}R_i^\alpha/\mathrm{d}s_i$ for instance is indicated as $R_i^{[1]\alpha}(s_i)$.

where the argument 1 at the left-hand side stands for the set of variables R_1^α, $R_1^{[1]\alpha}$, ... This distribution function, which is a measure for the density of world lines, is an invariant with respect to Lorentz transformations and moreover independent of the normal vector n^α. If the system contains many particles one may replace the discontinuous function (6) by a distribution function which is a smooth function of R_1^α and the independent components of $R_1^{[1]\alpha}$, etc. Formally such a coarse graining is achieved by employing for the one-point distribution function a weighted average of the right-hand side of (6)

$$f_1(1) = c \sum_\gamma w_\gamma \sum_i \int \delta^{(4)}\{R_1 - R_{i\gamma}(s_i)\}\delta^{(4)}\{R_1^{[1]} - R_{i\gamma}^{[1]}(s_i)\} \ldots \mathrm{d}s_i, \tag{7}$$

with $\sum_\gamma w_\gamma = 1$.

From (6) and (7) it is apparent that the distribution function vanishes if the $R_1^{[1]\alpha}$, $R_1^{[2]\alpha}$, ... do not satisfy simultaneously the set of relations

$$\begin{gathered} R_1^{[1]}\cdot R_1^{[1]} + c^2 = 0, \\ \sum_{i=0}^{j} \binom{j}{i} R_1^{[i+1]}\cdot R_1^{[j-i+1]} = 0, \qquad (j = 1, 2, \ldots). \end{gathered} \tag{8}$$

The components of $R_1^{[1]\alpha}$, $R_1^{[2]\alpha}$, ... are thus not all independent. In other words the distribution function is the product of a number of delta functions with as arguments the left-hand sides of the above relations and a function which is smooth after the coarse graining has been performed.

The distribution function (6) fulfils a continuity equation, which may be derived by writing the identity

$$c \sum_i \int \frac{\mathrm{d}}{\mathrm{d}s_i} [\delta^{(4)}\{R_1 - R_i(s_i)\}\delta^{(4)}\{R_1^{[1]} - R_i^{[1]}(s_i)\} \ldots] \mathrm{d}s_i = 0. \tag{9}$$

The differentiations with respect to s_i follow by applying the chain rule. Then one gets for (9)

$$\left(R_1^{[1]}\cdot \frac{\partial}{\partial R_1} + R_1^{[2]}\cdot \frac{\partial}{\partial R_1^{[1]}} + \ldots\right) f_1(1) = 0, \tag{10}$$

which is the continuity equation. If the distribution function depends on a finite number of variables, i.e. $f_1(1) = f_1(R_1, \ldots, R_1^{[n]})$, the continuity equation gets the form

$$\left(R_1^{[1]}\cdot \frac{\partial}{\partial R_1} + \ldots + R_1^{[n]}\cdot \frac{\partial}{\partial R_1^{[n-1]}}\right) f_1(R_1, \ldots, R_1^{[n]}) + \int \frac{\partial}{\partial R_1^{[n]}} \cdot R_1^{[n+1]} f_1(R_1, \ldots, R_1^{[n+1]}) \mathrm{d}^4 R_1^{[n+1]} = 0, \tag{11}$$

where we supposed that $R_1^{[n+1]\alpha}$ was independent of the other variables[1].

The continuity equation was derived here starting from the representation (6) of the distribution function. The smoothed distribution function (7) will satisfy the same conservation law, since coarse graining, i.e. adding a summation with weights w_γ, will not change the proof.

In a similar way one derives that the joint probability (normalized to $N(N-1)$) to find a world line crossing a surface element $d^3\Sigma_1$ with normal n_1^α at the position R_1^α and four-velocity $R_1^{[1]\alpha}$ etc. and a different world line crossing $d^3\Sigma_2$ with normal n_2^α at R_2^α and $R_2^{[1]\alpha}$ etc. is:

$$c^{-2} n_1 \cdot R_1^{[1]} n_2 \cdot R_2^{[1]} f_2(1,2) d^3\Sigma_1 \, d^3\Sigma_2 \, d^4R_1^{[1]} d^4R_2^{[1]} \ldots, \tag{12}$$

where $f_2(1, 2)$ is the two-point distribution function. Here $f_2(1, 2)$ is given by

$$f_2(1,2) = c^2 \sum_\gamma w_\gamma \sum_{i,j(i\neq j)} \int \delta^{(4)}\{R_1 - R_{i\gamma}(s_i)\}\delta^{(4)}\{R_1^{[1]} - R_{i\gamma}^{[1]}(s_i)\} \ldots$$
$$\delta^{(4)}\{R_2 - R_{j\gamma}(s_j)\}\delta^{(4)}\{R_2^{[1]} - R_{j\gamma}^{[1]}(s_j)\} \ldots ds_i \, ds_j. \tag{13}$$

In the following we shall frequently use the two-point correlation function $c_2(1, 2)$, which is defined as

$$c_2(1,2) = f_2(1,2) - f_1(1) f_1(2). \tag{14}$$

The generalization to particles with structure and to mixtures of different particles is trivial. The distribution function will then depend also on the internal variables and will be labelled by an index numbering the species.

b. *Definition of macroscopic quantities*

The microscopic quantities for which we want to define average values with the help of the covariant distribution function of the preceding subsection are sums of one-particle or two-particle quantities. The one-particle quantities have the form of integrals along world lines, so that their sums read

$$a(R) = \sum_i \int \alpha\{R_i(s_i), R_i^{[1]}(s_i), R_i^{[2]}(s_i), \ldots; R\} ds_i, \tag{15}$$

[1] If however the $R_1^{[n+1]\alpha}$ is dependent on $R_1, \ldots, R_1^{[n]\alpha}$, the integration in (11) may be performed with as a result the continuity equation

$$\left\{R_1^{[1]} \cdot \frac{\partial}{\partial R_1} + \ldots + R_1^{[n]} \cdot \frac{\partial}{\partial R_1^{[n-1]}} + \frac{\partial}{\partial R_1^{[n]}} \cdot R_1^{[n+1]}(R_1, \ldots, R_1^{[n]})\right\}$$
$$f_1(R_1, \ldots, R_1^{[n]}) = 0.$$

This is the case which occurs in the kinetic theory of gases, where $n = 1$.

where $R_i^\mu(s_i)$ is the position four-vector of point particle i with proper time s_i, $R_i^{[1]\mu}(s_i) \equiv dR_i^\mu/ds_i$ its four-velocity, $R_i^{[2]\mu}(s_i) \equiv d^2R_i^\mu/ds_i^2$ its four-acceleration, etc. The average value of such a quantity follows by taking the weighted average

$$A(R) \equiv \langle a(R) \rangle = \sum_\gamma w_\gamma \sum_i \int \alpha\{R_{i\gamma}(s_i), R_{i\gamma}^{[1]}(s_i), \ldots; R\} ds_i \tag{16}$$

with the weights w_γ which have been introduced in the preceding subsection. This average may be written in terms of the distribution function (7):

$$A(R) = c^{-1} \int \alpha(1; R) f_1(1) d1, \tag{17}$$

where the argument 1 stands for the set of variables R_1^μ $R_1^{[1]\mu}$, ... and where d1 stands for $d^4R_1 \, d^4R_1^{[1]} \ldots$.

Likewise for a sum of two-particle quantities

$$a(R) = \sum_{i,j(i\neq j)} \int \alpha\{R_i(s_i), R_i^{[1]}(s_i), \ldots, R_j(s_j), R_j^{[1]}(s_j), \ldots; R\} ds_i \, ds_j \tag{18}$$

one obtains in the same fashion the average

$$A(R) \equiv \langle a(R) \rangle = c^{-2} \int \alpha(1, 2; R) f_2(1, 2) d1 \, d2, \tag{19}$$

where the two-point distribution function $f_2(1, 2)$ has been defined in (13).

Since the distribution functions $f_1(1)$ and $f_2(1, 2)$ are Lorentz invariant, as was shown in subsection *a*, the averages $A(R)$ (17) and (19) have the same tensorial character as the microscopic quantities $a(R)$ (15) and (18).

From the expressions (17) and (19) it is obvious that the average of a derivative of a quantity a with respect to R^μ is equal to the derivative of the average quantity $A = \langle a \rangle$:

$$\langle \partial_\mu a \rangle = \partial_\mu \langle a \rangle \equiv \partial_\mu A. \tag{20}$$

This commutation property will be used frequently in the derivation of the macroscopic laws.

The covariant averages (17) and (19) may be cast in a particular form if the microscopic quantities have special properties, as will be shown in the following subsection.

c. *Synchronous, retarded and advanced distribution functions*

Let us consider first a physical quantity of the form

$$a(n, \tau) = \sum_i \int \alpha\{R_i(s_i), \ldots\} \delta\{n \cdot R_i(s_i) + c\tau\} ds_i, \tag{21}$$

where n^μ is a time-like unit vector ($n^0 > 0$) and where τ is an arbitrary real number. (The charge–current density is an example of such a quantity.) In such a quantity only those points of the world lines of the particles i contribute that lie in a plane three-surface $n \cdot R + c\tau = 0$. In particular if the normal n^μ has the form $\mathring{n}^\mu \equiv (1, 0, 0, 0)$ this equation for the plane reduces to $t = \tau$, i.e. only 'synchronous' points of the world lines contribute.

The integral in (21) may be performed by introducing the integration variable $n \cdot R_i(s_i)$ instead of s_i:

$$a(n, \tau) = -\sum_i \frac{\alpha\{R_i(s_i), \ldots\}}{n \cdot R_i^{[1]}(s_i)} \bigg|_{n \cdot R_i(s_i) + c\tau = 0}, \tag{22}$$

where the suffix means that s_i is the solution of the equation in question.

One often encounters physical quantities of the form (21) or (22) with α depending also explicitly on R and the parameter τ equal to $-c^{-1} n \cdot R$. Moreover space-time derivatives of quantities of this type occur, for instance

$$\partial_\mu \sum_i \int \alpha\{R_i(s_i), \ldots; R\} \delta[n \cdot \{R_i(s_i) - R\}] \mathrm{d}s_i . \tag{23}$$

(The polarization tensor is a quantity of this type.) Such a quantity may be written in a form that shows that it is of the same 'synchronous' type as (21). To that purpose we use the identity for an arbitrary function $f(x)$ and an arbitrary four-vector v^μ

$$\partial_\mu f[n \cdot \{R_i(s_i) - R\}] = -\frac{n_\mu}{n \cdot v} v \cdot \frac{\partial}{\partial R_i} f[n \cdot \{R_i(s_i) - R\}] \tag{24}$$

for the special choice $f(x) = \delta(x)$ and $v^\mu = R_i^{[1]\mu}(s_i)$. Then (23) becomes

$$\sum_i \int [\partial_\mu \alpha\{R_i(s_i), \ldots; R\}] \delta[n \cdot \{R_i(s_i) - R\}] \mathrm{d}s_i$$
$$- \sum_i \int \alpha\{R_i(s_i), \ldots; R\} \frac{n_\mu}{n \cdot R_i^{[1]}(s_i)} \frac{\mathrm{d}}{\mathrm{d}s_i} \delta[n \cdot \{R_i(s_i) - R\}] \mathrm{d}s_i . \tag{25}$$

After a partial integration one obtains

$$\sum_i \int \left\{ \partial_\mu \alpha + n_\mu \frac{\mathrm{d}}{\mathrm{d}s_i} \left(\frac{\alpha}{n \cdot R_i^{[1]}} \right) \right\} \delta\{n \cdot (R_i - R)\} \mathrm{d}s_i , \tag{26}$$

so that it becomes apparent that the space-time derivatives (23) of 'synchronous quantities' are themselves synchronous quantities of the type (21). The latter fact could also have been seen from (22) with $\tau = -c^{-1} n \cdot R$ since differentiation of that function with implicit dependence yields

$$-\partial_\mu \left[\sum_i \frac{\alpha\{R_i(s_i), \ldots; R\}}{n \cdot R_i^{[1]}(s_i)} \bigg|_{n \cdot \{R_i(s_i) - R\} = 0} \right]$$
$$= -\left[\sum_i \mathrm{d}_{i\mu}^{\mathrm{syn}} \frac{\alpha\{R_i(s_i), \ldots; R\}}{n \cdot R_i^{[1]}(s_i)} \right] \bigg|_{n \cdot \{R_i(s_i) - R\} = 0}, \tag{27}$$

where the differentiation $\mathrm{d}_{i\mu}^{\mathrm{syn}}$ stands for

$$\mathrm{d}_{i\mu}^{\mathrm{syn}} \equiv \partial_\mu + \frac{n_\mu}{n \cdot R_i^{[1]}(s_i)} \frac{\mathrm{d}}{\mathrm{d}s_i} . \tag{28}$$

The right-hand side of (27) shows again that the quantity under consideration is of the synchronous type. (As it should be, the right-hand side of (27) may be shown to be equal to (26).)

The average of a quantity of the type (21), which according to (17) is

$$A(n, \tau) = c^{-1} \int \alpha(1) \delta(n \cdot R_1 + c\tau) f_1(1) \mathrm{d}1, \tag{29}$$

may be written as

$$A(n, \tau) = -\int \frac{\alpha(1)}{n \cdot R_1^{[1]}} f_1^{\mathrm{syn}}(1; n, \tau) \mathrm{d}1, \tag{30}$$

where we introduced the *synchronous distribution function*

$$f_1^{\mathrm{syn}}(1; n, \tau) \equiv -c^{-1} n \cdot R_1^{[1]} \delta(n \cdot R_1 + c\tau) f_1(1). \tag{31}$$

From (30) it is apparent that the average of a 'synchronous' quantity in the form (22) may be obtained by replacing $R_i^\mu(s_i)$, $R_i^{[1]\mu}(s_i)$, ... by the variables R_1^μ, $R_1^{[1]\mu}$, ..., multiplying by the synchronous distribution function and integrating over all variables. From (31) and (5) the interpretation of $f_1^{\mathrm{syn}}(1; n, \tau)$ follows immediately: $f_1^{\mathrm{syn}}(1; n, \tau)\mathrm{d}1$ is the number of atoms with position four-vector satisfying the relation $n \cdot R_1 + c\tau = 0$, that lie in the volume element $\mathrm{d}1 \equiv \mathrm{d}^4 R_1 \mathrm{d}^4 R_1^{[1]} \ldots$.

Let us next consider a 'retarded quantity' of the form:

$$a(R) = \sum_i \int \alpha\{R_i(s_i), \ldots\} \theta\{R - R_i(s_i)\} \delta[\{R - R_i(s_i)\}^2] \mathrm{d}s_i . \tag{32}$$

(The four-potential due to a point charge is a quantity of this type.) The combination of θ- and δ-function selects indeed world line points which lie on the negative light-cone with R^μ as top. If $R_i^\mu(s_i) \neq R^\mu$ for all i and s_i, one may perform the integration in (32) by introducing the integration variable $\{R - R_i(s_i)\}^2$ instead of s_i. One obtains then:

$$a(R) = \sum_i \frac{\alpha\{R_i(s_i), \ldots\}}{2 R_i^{[1]}(s_i) \cdot \{R_i(s_i) - R\}} \bigg|_{\mathrm{ret}}, \tag{33}$$

where the suffix ret denotes the fact that one has to take s_i as solution of the equations $\{R - R_i(s_i)\}^2 = 0$ and $R^0 - R_i^0(s_i) > 0$.

Apart from quantities of the type (32), there occur also quantities which are space-time derivatives:

$$\partial_\mu \sum_i \int \alpha\{R_i(s_i), \ldots; R\}\theta\{R-R_i(s_i)\}\delta[\{R-R_i(s_i)\}^2]\mathrm{d}s_i. \tag{34}$$

(The four-potentials due to electromagnetic multipoles and the electromagnetic fields are of this type). For $R^\mu \neq R_i^\mu(s_i)$ such a quantity may be written in a form which shows that it is a retarded quantity of the type (32). To this end one must use an identity valid for an arbitrary function f and an arbitrary four-vector v^μ:

$$\partial_\mu f[\{R-R(s_i)\}^2] = -\frac{\{R-R_i(s_i)\}_\mu}{v\cdot\{R-R_i(s_i)\}}\, v\cdot\frac{\partial}{\partial R_i} f[\{R-R_i(s_i)\}^2]. \tag{35}$$

If this relation for the choice $f(x) = \delta(x)$ and $v^\mu = R_i^{[1]\mu}(s_i)$ is used in (34) for $R^\mu \neq R_i^\mu(s_i)$ and a partial integration is performed one gets

$$\sum_i \int \left[\partial_\mu \alpha + \frac{\mathrm{d}}{\mathrm{d}s_i}\left\{\frac{(R-R_i)_\mu \alpha}{R_i^{[1]}\cdot(R-R_i)}\right\}\right]\theta(R-R_i)\delta\{(R-R_i)^2\}\mathrm{d}s_i, \tag{36}$$

so that for $R^\mu \neq R_i^\mu(s_i)$ the quantity (34) is indeed of the retarded type (32). This could also have been seen by differentiating, for $R^\mu \neq R_i^\mu(s_i)$, the implicit function (33) with respect to R_μ:

$$\partial_\mu\left[\sum_i \frac{\alpha\{R_i(s_i), \ldots; R\}}{2R_i^{[1]}\cdot\{R_i(s_i)-R\}}\bigg|_{\mathrm{ret}}\right] = \left[\sum_i \mathrm{d}_{i\mu}^{\mathrm{ret}} \frac{\alpha\{R_i(s_i), \ldots; R\}}{2R_i^{[1]}\cdot\{R_i(s_i)-R\}}\right]\bigg|_{\mathrm{ret}}, \tag{37}$$

where the differentiation $\mathrm{d}_{i\mu}^{\mathrm{ret}}$ is (for $R^\mu \neq R_i^\mu(s_i)$)

$$\mathrm{d}_{i\mu}^{\mathrm{ret}} = \partial_\mu + \frac{\{R-R_i(s_i)\}_\mu}{R_i^{[1]}(s_i)\cdot\{R-R_i(s_i)\}}\frac{\mathrm{d}}{\mathrm{d}s_i}. \tag{38}$$

The right-hand side of (37) shows again that the quantity involved has retarded character for $R^\mu \neq R_i^\mu$. (One may prove that the right-hand side of (37) is indeed equal to (36).)

The average of the quantity (32), which reads according to (17)

$$A(R) = c^{-1}\int \alpha(1)\theta(R-R_1)\delta\{(R-R_1)^2\}f_1(1)\mathrm{d}1, \tag{39}$$

may be written as

$$A(R) = \int \frac{\alpha(1)}{2R_1^{[1]}\cdot(R_1-R)} f_1^{\mathrm{ret}}(1; R)\mathrm{d}1. \tag{40}$$

Here we introduced the *retarded distribution function*[1] defined as

$$f_1^{\mathrm{ret}}(1; R) \equiv 2c^{-1}R_1^{[1]}\cdot(R_1-R)\theta(R-R_1)\delta\{(R-R_1)^2\}f_1(1). \tag{41}$$

[1] Such a distribution function, but in its three-dimensional form (56), has been introduced by S. R. de Groot and J. Vlieger, Physica **31**(1965)254 and in four-dimensional form by L. G. Suttorp, thesis, Amsterdam (1968).

(It would seem from (40) that the integrand has a singularity for $R_1^\mu = R^\mu$, but the denominator appearing in front of f_1^{ret} is compensated by a factor in f_1^{ret} itself.) From (40) it appears that the average of a 'retarded' quantity which is given by (33) for $R_i^\mu(s_i) \neq R^\mu$ may be obtained by replacing $R_i^\mu(s_i)$, $R_i^{[1]\mu}(s_i)$, ... by the variables R_1^μ, $R_1^{[1]\mu}$, ..., multiplying by the retarded distribution function and integrating over all variables. From (41) and (5) follows the interpretation of $f_1^{\mathrm{ret}}(1; R)$, namely: $f_1^{\mathrm{ret}}(1; R)\mathrm{d}1$ is the number of atoms with position four-vector R_1^μ satisfying $(R-R_1)^2 = 0$ and $R^0 > R_1^0$, that lies in the volume element $\mathrm{d}1 \equiv \mathrm{d}^4R_1\mathrm{d}^4R_1^{[1]}\ldots$.

We may treat in an analogous way the average of an 'advanced quantity'

$$a(R) = \sum_i \int \alpha\{R_i(s_i), \ldots\}\theta\{R_i(s_i)-R\}\delta[\{R-R_i(s_i)\}^2]\mathrm{d}s_i, \tag{42}$$

which may be written for $R_i^\mu(s_i) \neq R^\mu$ as

$$a(R) = \sum_i \frac{\alpha\{R_i(s_i), \ldots\}}{2R_i^{[1]}(s_i)\cdot\{R-R_i(s_i)\}}\bigg|_{\mathrm{adv}}, \tag{43}$$

where the suffix adv indicates that s_i is the solution of the positive light-cone equations $\{R-R_i(s_i)\}^2 = 0$ and $R_i^0(s_i)-R^0 > 0$. The average of such a quantity (42) reads according to (17)

$$A(R) = c^{-1}\int \alpha(1)\theta(R_1-R)\delta\{(R-R_1)^2\}f_1(1)\mathrm{d}1. \tag{44}$$

It may be written as

$$A(R) = \int \frac{\alpha(1)}{2R_1^{[1]}\cdot(R-R_1)} f_1^{\mathrm{adv}}(1; R)\mathrm{d}1, \tag{45}$$

with the help of the *advanced distribution function*

$$f_1^{\mathrm{adv}}(1; R) \equiv 2c^{-1}R_1^{[1]}\cdot(R-R_1)\theta(R_1-R)\delta\{(R-R_1)^2\}f_1(1\,. \tag{46}$$

The retarded, and likewise the advanced, distribution function may be written in terms of the synchronous distribution function. To that purpose we write first the identity valid for a four-vector x^μ ($\neq 0$):

$$\theta(x)\delta(x^2) = \frac{\delta(x^0-|\boldsymbol{x}|)}{2|\boldsymbol{x}|}, \tag{47}$$

where we used the property of the delta function

$$\delta\{f(x)\} = \sum_i \frac{1}{|\partial f/\partial x|}\delta(x-x_i) \tag{48}$$

with x_i the (non-degenerate) roots of $f(x) = 0$. One may obtain the covariant form of (47) by noting that

$$x^2 = -(n\cdot x)^2 + |\Delta_n\cdot x|^2, \tag{49}$$

where n^μ is a time-like unit vector ($n^0 > 0$), $\Delta_n^{\mu\nu} \equiv g^{\mu\nu} + n^\mu n^\nu$ and $|y| \equiv \sqrt{(y\cdot y)}$ for a space-like vector y^μ. If one substitutes (49) into the left-hand side of (47) and then uses the property (48) one obtains

$$\theta(x)\delta(x^2) = -\frac{\delta(n\cdot x + |\Delta_n\cdot x|)}{2n\cdot x}, \tag{50}$$

where the right-hand side is the covariant generalization of the right-hand side of (47). We now use relation (50) with $x^\mu = R^\mu - R_1^\mu$ in the expression (41) for the retarded distribution function. Then by comparison with the expression (31) for the synchronous distribution function one finds for $R^\mu \neq R_1^\mu$:

$$f_1^{\rm ret}(1; R) = -\frac{R_1^{[1]}\cdot(R-R_1)}{n\cdot(R-R_1)n\cdot R_1^{[1]}} f_1^{\rm syn}(1; n, -c^{-1}n\cdot R - c^{-1}|\Delta_n\cdot(R-R_1)|). \tag{51}$$

In particular if one chooses the unit vector n^μ as $R_1^{[1]\mu}/c$ one obtains unity for the factor in front of $f_1^{\rm syn}$. A different choice of n^μ which is often convenient, is $n^\mu = \mathring{n}^\mu \equiv (1, 0, 0, 0)$. Then (51) gets the form (for $R^\mu \neq R_1^\mu$)

$$f_1^{\rm ret}(1; R) = \kappa(1) f_1^{\rm syn}(1; \mathring{n}, t - c^{-1}|\boldsymbol{R}-\boldsymbol{R}_1|), \tag{52}$$

where we used the abbreviation

$$\kappa(1) \equiv 1 - \frac{\boldsymbol{\beta}_1\cdot(\boldsymbol{R}-\boldsymbol{R}_1)}{|\boldsymbol{R}-\boldsymbol{R}_1|} \tag{53}$$

and the fact that $ct - ct_1 = |\boldsymbol{R}-\boldsymbol{R}_1|$, since $R_1^\mu - R^\mu$ lies on the negative light-cone. (The quantity $\boldsymbol{\beta}_1 \equiv \boldsymbol{R}_1^{[1]}/R_1^{[1]0}$ is the particle velocity divided by the speed of light.)

The synchronous, the retarded as well as the advanced distribution functions, given in (31), (41) and (46) respectively, depend on the four-vectors R_1^μ, $R_1^{[1]\mu}, \ldots, R_1^{[n]\mu}$ of which not all components are independent. In fact the velocity and the higher derivatives satisfy the relations (8). Moreover the synchronous distribution function vanishes, unless $n\cdot R_1 + c\tau = 0$, whereas the retarded and advanced distribution functions vanish unless $(R-R_1)^2 = 0$ and $R^0 - R_1^0 > 0$ or < 0 respectively. Therefore it is often convenient to introduce distribution functions which result from the functions discussed above by integrating over the time-components of the four-vectors. Using then also the three-velocities, three-accelerations etc. instead of the space-components of the corresponding four-vectors, one writes the number of particles with position $\boldsymbol{R}_1$, velocity $c\boldsymbol{\beta}_1$, acceleration $c^2\partial_0\boldsymbol{\beta}_1$ etc. at time τ as

$$f_1^{\rm syn}(\boldsymbol{R}_1, \boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots; \tau)\mathrm{d}\boldsymbol{R}_1\,\mathrm{d}\boldsymbol{\beta}_1\,\mathrm{d}\partial_0\boldsymbol{\beta}_1 \ldots, \tag{54}$$

where

$$f_1^{\rm syn}(\boldsymbol{R}_1, \boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots; \tau) \equiv \frac{\partial(\boldsymbol{R}_1^{[1]}, \boldsymbol{R}_1^{[2]}, \ldots)}{\partial(\boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots)} \int f_1^{\rm syn}(1; \mathring{n}, \tau)\mathrm{d}R_1^0\,\mathrm{d}R_1^{[1]0} \ldots, \tag{55}$$

where a Jacobian appears and where $\mathring{n}^\mu$ is the vector $(1, 0, 0, 0)$. This shows that the synchronous distribution function is the generalization of the ordinary distribution function of non-relativistic theory.

Likewise one may write the number of particles with position $\boldsymbol{R}_1$, velocity $c\boldsymbol{\beta}_1$, acceleration $c^2\partial_0\boldsymbol{\beta}_1$, etc. at a time, related to the observer's time t by the light-cone equation $t_1 = t - c^{-1}|\boldsymbol{R}-\boldsymbol{R}_1|$, as:

$$f_1^{\rm ret}(\boldsymbol{R}_1, \boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots; \boldsymbol{R}, t)\mathrm{d}\boldsymbol{R}_1\mathrm{d}\boldsymbol{\beta}_1\mathrm{d}\partial_0\boldsymbol{\beta}_1 \ldots, \tag{56}$$

where

$$f_1^{\rm ret}(\boldsymbol{R}_1, \boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots; \boldsymbol{R}, t) \equiv \frac{\partial(\boldsymbol{R}_1^{[1]}, \boldsymbol{R}_1^{[2]}, \ldots)}{\partial(\boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots)} \int f_1^{\rm ret}(1; R)\mathrm{d}R_1^0\,\mathrm{d}R_1^{[1]0} \ldots. \tag{57}$$

The advanced distribution functions may be treated in a completely similar way.

The connexion between retarded and synchronous distribution functions in the form (52) may now be translated (for $R^\mu \neq R_1^\mu$) into:

$$f_1^{\rm ret}(\boldsymbol{R}_1, \boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots; \boldsymbol{R}, t) = \kappa(1) f_1^{\rm syn}(\boldsymbol{R}_1, \boldsymbol{\beta}_1, \partial_0\boldsymbol{\beta}_1, \ldots; t - c^{-1}|\boldsymbol{R}-\boldsymbol{R}_1|), \tag{58}$$

as follows from (55) and (57).

3 *The Maxwell equations*

a. *Derivation of the macroscopic field equations*

The atomic field tensor $f^{\alpha\beta}$ which has been given explicitly in (IV.14) is a quantity of the form (15), i.e. a sum of integrals along world lines of the atoms of the system. Therefore one may define the average in the way as defined in formula (17) of the preceding section:

$$F^{\alpha\beta} = \langle f^{\alpha\beta} \rangle \tag{59}$$

with components $(F^{01}, F^{02}, F^{03}) = \boldsymbol{E}$ and $(F^{23}, F^{31}, F^{12}) = \boldsymbol{B}$. Since the atomic fields $f^{\alpha\beta}$ are retarded quantities, one may alternatively write the macroscopic fields in terms of retarded distribution functions, as discussed in the preceding section[1].

The atomic charge–current density vector and the atomic polarization tensor which have been given in (IV.5) and (IV.8) are again quantities of the type (15). So again the average of these quantities may be defined as in formula (13):

$$J^\alpha = \langle j^\alpha \rangle, \tag{60}$$

$$M^{\alpha\beta} = \langle m^{\alpha\beta} \rangle, \tag{61}$$

with components $J^0 = c\varrho^e$, $(J^1, J^2, J^3) = \boldsymbol{J}$, $(M^{01}, M^{02}, M^{03}) = -\boldsymbol{P}$ and $(M^{23}, M^{31}, M^{12}) = \boldsymbol{M}$.

The equations which govern these macroscopic quantities are obtained by averaging the atomic field equations (IV.7,13). This gives

$$\begin{aligned} \langle \partial_\beta f^{\alpha\beta} \rangle &= c^{-1} \langle j^\alpha \rangle + \langle \partial_\beta m^{\alpha\beta} \rangle, \\ \langle \partial^\alpha f^{\beta\gamma} \rangle + \langle \partial^\beta f^{\gamma\alpha} \rangle + \langle \partial^\gamma f^{\alpha\beta} \rangle &= 0. \end{aligned} \tag{62}$$

Since differentiation and averaging commute according to (20), one may write these equations as

$$\begin{aligned} \partial_\beta \langle f^{\alpha\beta} \rangle &= c^{-1} \langle j^\alpha \rangle + \partial_\beta \langle m^{\alpha\beta} \rangle, \\ \partial^\alpha \langle f^{\beta\gamma} \rangle + \partial^\beta \langle f^{\gamma\alpha} \rangle + \partial^\gamma \langle f^{\alpha\beta} \rangle &= 0. \end{aligned} \tag{63}$$

Now from these equations, with the notations (59), (60) and (61) for the macroscopic quantities, one obtains

$$\begin{aligned} \partial_\beta F^{\alpha\beta} &= c^{-1} J^\alpha + \partial_\beta M^{\alpha\beta}, \\ \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} &= 0. \end{aligned} \tag{64}$$

These are precisely Maxwell's equations

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \varrho^e - \nabla \cdot \boldsymbol{P}, \\ -\partial_0 \boldsymbol{E} + \nabla \wedge \boldsymbol{B} &= c^{-1} \boldsymbol{J} + \partial_0 \boldsymbol{P} + \nabla \wedge \boldsymbol{M}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \partial_0 \boldsymbol{B} + \nabla \wedge \boldsymbol{E} &= 0. \end{aligned} \tag{65}$$

One can also introduce the macroscopic 'displacement tensor' (cf. (IV.24)):

$$H^{\alpha\beta} = F^{\alpha\beta} - M^{\alpha\beta} \tag{66}$$

[1] S. R. de Groot and J. Vlieger, op. cit.

with components $(H^{01}, H^{02}, H^{03}) = \boldsymbol{D}$ and $(H^{23}, H^{31}, H^{12}) = \boldsymbol{H}$. In other words (66) reads

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{E} + \boldsymbol{P}, \\ \boldsymbol{H} &= \boldsymbol{B} - \boldsymbol{M}. \end{aligned} \tag{67}$$

Then the equations (64) become

$$\begin{aligned} \partial_\beta H^{\alpha\beta} &= c^{-1} J^\alpha, \\ \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} &= 0, \end{aligned} \tag{68}$$

or in three-dimensional notation:

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \varrho^e, \\ -\partial_0 \boldsymbol{D} + \nabla \wedge \boldsymbol{H} &= c^{-1} \boldsymbol{J}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \partial_0 \boldsymbol{B} + \nabla \wedge \boldsymbol{E} &= 0. \end{aligned} \tag{69}$$

The covariant nature of the Maxwell equations has now been obtained as a consequence of the covariant nature of the microscopic field equations. It needs no longer be postulated as in the traditional expositions of the Maxwell theory [1].

Finally one may derive the macroscopic law of conservation of charge by averaging the atomic conservation law (IV.26). One finds then, using the fact that differentiation and averaging commute and the notation (60):

$$\partial_\alpha J^\alpha = 0, \tag{70}$$

or

$$\frac{\partial \varrho^e}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{71}$$

in three-dimensional notation.

[1] Earlier, incomplete attempts to derive Maxwell's equations in a covariant way are due to Ph. Frank, Ann. Physik **27**(1908)1059; H. Minkowski and M. Born, Math. Ann. **68**(1910) 526; A. D. Fokker, Phil. Mag. **39**(1920)404; W. Dällenbach, Ann. Physik **58**(1919)523; J. Frenkel, Lehrbuch der Elektrodynamik II, (Springer, Berlin 1928); S. R. de Groot and J. Vlieger, Physica **31**(1965)254. For a discussion see: W. Pauli, Theory of relativity (Pergamon Press, London 1958); L. G. Suttorp, thesis, Amsterdam (1968); S. R. de Groot, The Maxwell equations (North-Holland Publ. Co., Amsterdam 1969).

b. *Explicit forms of the macroscopic current vector and polarization tensor*

The atomic four-current density given in (IV.5) is an example of a synchronous quantity of the type (21). Its form (22) with $n^\alpha = \mathring{n}^\alpha \equiv (1, 0, 0, 0)$ and $\tau = t$ was explicited in (IV.42) and (IV.43). Therefore according to (30) the average may be expressed in terms of the synchronous distribution function $f_1^{\text{syn}}(1; \mathring{n}, t)$ given in (31). Instead of this synchronous distribution function depending on the position and velocity four-vectors it is convenient to introduce the three-dimensional functions of the type (55). This gives for the macroscopic charge and current densities:

$$\begin{aligned} \varrho^e &= \sum_a e_a \int \delta^{(3)}(\boldsymbol{R}_1 - \boldsymbol{R}) f_1^{\text{syn},a}(\boldsymbol{R}_1; t) \mathrm{d}\boldsymbol{R}_1, \\ \boldsymbol{J} &= c \sum_a e_a \int \boldsymbol{\beta}_1 \delta^{(3)}(\boldsymbol{R}_1 - \boldsymbol{R}) f_1^{\text{syn},a}(\boldsymbol{R}_1, \boldsymbol{\beta}_1; t) \mathrm{d}\boldsymbol{R}_1 \mathrm{d}\boldsymbol{\beta}_1, \end{aligned} \tag{72}$$

(where we added a summation over different species a) or

$$\begin{aligned} \varrho^e &= \sum_a e_a f_1^{\text{syn},a}(\boldsymbol{R}; t), \\ \boldsymbol{J} &= c \sum_a e_a \int \boldsymbol{\beta}_1 f_1^{\text{syn},a}(\boldsymbol{R}, \boldsymbol{\beta}_1; t) \mathrm{d}\boldsymbol{\beta}_1. \end{aligned} \tag{73}$$

The polarization tensor, given in (IV.8) is likewise a synchronous quantity, this time of the type (23). Its form (27) with $n^\alpha = \mathring{n}^\alpha \equiv (1, 0, 0, 0)$ and $\tau = t$ has been evaluated in (IV.36–39). The average may thus be expressed. according to (30), with the help of the synchronous distribution function (31) or its three-dimensional form (55). In particular one finds with (IV.48) and (IV.49) for the dipole contribution to the polarization tensor

$$\begin{aligned} \boldsymbol{P}^{(1)} &= \int (\underline{\boldsymbol{\mu}}_1^{(1)} - \underline{\boldsymbol{\nu}}_1^{(1)} \wedge \boldsymbol{\beta}_1) f_1^{\text{syn}}(\boldsymbol{R}, \boldsymbol{\beta}_1, \boldsymbol{\mu}_1^{(1)}, \boldsymbol{\nu}_1^{(1)}; t) \mathrm{d}\boldsymbol{\beta}_1 \mathrm{d}\boldsymbol{\mu}_1^{(1)} \mathrm{d}\boldsymbol{\nu}_1^{(1)}, \\ \boldsymbol{M}^{(1)} &= \int (\underline{\boldsymbol{\nu}}_1^{(1)} + \underline{\boldsymbol{\mu}}_1^{(1)} \wedge \boldsymbol{\beta}_1) f_1^{\text{syn}}(\boldsymbol{R}, \boldsymbol{\beta}_1, \boldsymbol{\mu}_1^{(1)}, \boldsymbol{\nu}_1^{(1)}; t) \mathrm{d}\boldsymbol{\beta}_1 \mathrm{d}\boldsymbol{\mu}_1^{(1)} \mathrm{d}\boldsymbol{\nu}_1^{(1)}. \end{aligned} \tag{74}$$

The underlined quantities have been given in (IV.50, 51) as:

$$\begin{aligned} \underline{\boldsymbol{\mu}}_1^{(1)} &\equiv \boldsymbol{\Omega}_1 \cdot \boldsymbol{\mu}_1^{(1)} = \boldsymbol{\mu}_{1,\perp}^{(1)} + \sqrt{1 - \beta_1^2}\, \boldsymbol{\mu}_{1,//}^{(1)}, \\ \underline{\boldsymbol{\nu}}_1^{(1)} &\equiv \boldsymbol{\Omega}_1 \cdot \boldsymbol{\nu}_1^{(1)} = \boldsymbol{\nu}_{1,\perp}^{(1)} + \sqrt{1 - \beta_1^2}\, \boldsymbol{\nu}_{1,//}^{(1)}, \end{aligned} \tag{75}$$

where the electric and magnetic dipole moments $\boldsymbol{\mu}_1^{(1)}$ and $\boldsymbol{\nu}_1^{(1)}$ have been split into parts perpendicular and parallel to the atomic velocity $\boldsymbol{\beta}_1$.

The formulae (74–75) show the effects of the atomic velocity $\boldsymbol{\beta}_1 c$. Relativistic effects are *a*: the Lorentz contractions of the parallel components of both the electric and magnetic dipole moments, and furthermore *b*: the last term of $\boldsymbol{P}$, which describes the effect of moving magnetic dipoles on the electric polarization vector. Such an effect has been observed experimentally[1]. It forms the basis of the so-called unipolar induction machine in which a cylindrical permanent magnet with magnetization parallel to the axis is rotated around this axis. Then a potential difference arises between the mantle and the axis which can cause a current if the material is a conductor, as is the case for the iron used in such machines.

In fluids under ordinary circumstances β_1 is of the order of the sound velocity divided by the velocity of light, that is of the order of 10^{-6}. The vibrations in solids may have circular frequencies ω of about $3 \times 10^{13} \mathrm{s}^{-1}$ (optical branch). The atomic velocity $\dot{R}_1$ is equal to about $R_1 \omega$ with R_1 of the order of 10^{-8} cm. Thus $\beta_1 = \dot{R}_1 / c$ is then about 10^{-5}. The Lorentz contractions are then negligibly small, because $\sqrt{(1 - \beta_1^2)}$ differs from unity only by an amount of the order of 10^{-12} for fluids, and 10^{-10} for solids. The effect of moving magnetic dipoles is not so small since it is proportional to β_1. Its magnitude compared to the main term is $\beta_1 \nu_1^{(1)} / \mu_1^{(1)}$. (Since we are concerned with upper limits on the orders of magnitude of the various effects, only absolute values are considered and no attention is paid to the vectorial character of the quantities.) The proportion $\nu_1^{(1)} / \mu_1^{(1)}$ is, according to (IV.27) and (IV.28), of the order of 10^{-2} (the fine structure constant). So the effect of moving magnetic dipoles has a relative magnitude as compared to the leading term of 10^{-8} for fluids and 10^{-7} for solids. (The corresponding effect of moving electric dipoles in the magnetization, which is a non-relativistic effect, is of order 10^{-4} for fluids and 10^{-3} for solids.)

If all atoms have the same velocity, say $\boldsymbol{\beta} c$, then the polarization may be expressed in this velocity and the macroscopic electric and magnetic dipole densities, defined as

$$\begin{aligned} \mathscr{P}^{(1)} &\equiv \int \boldsymbol{\mu}_1^{(1)} f_1^{\text{syn}}(\boldsymbol{R}, \boldsymbol{\mu}_1^{(1)}; t) \mathrm{d}\boldsymbol{\mu}_1^{(1)}, \\ \mathscr{M}^{(1)} &\equiv \int \boldsymbol{\nu}_1^{(1)} f_1^{\text{syn}}(\boldsymbol{R}, \boldsymbol{\nu}_1^{(1)}; t) \mathrm{d}\boldsymbol{\nu}_1^{(1)}. \end{aligned} \tag{76}$$

Indeed with these definitions and $\boldsymbol{\beta}_1 = \boldsymbol{\beta}$, one gets for (74), using also the first equalities of (75):

[1] H. A. Wilson, Phil. Trans. A **204**(1904)121; H. A. Wilson and M. Wilson, Proc. Roy. Soc. A **89**(1913)99.

$$\begin{aligned} \boldsymbol{P}^{(1)} &= \boldsymbol{\Omega}\cdot\mathscr{P}^{(1)}-\mathscr{M}^{(1)}\wedge\boldsymbol{\beta}, \\ \boldsymbol{M}^{(1)} &= \boldsymbol{\Omega}\cdot\mathscr{M}^{(1)}+\mathscr{P}^{(1)}\wedge\boldsymbol{\beta}, \end{aligned} \tag{77}$$

where the tensor $\boldsymbol{\Omega} \equiv \mathbf{U}+(\gamma^{-1}-1)\boldsymbol{\beta\beta}/\beta^2$ with $\gamma^{-1} \equiv \sqrt{(1-\beta^2)}$ contains the common velocity $\boldsymbol{\beta}c$. These formulae may be applied to the case of rigid crystal lattices, where all carriers of electromagnetic moments have the same velocity if the vibrations are neglected.

The contributions from electric and magnetic quadrupole moments to the polarization tensor (61) become with (IV.52–53)

$$\begin{aligned} \boldsymbol{P}^{(2)} &= -\nabla\cdot\int(\boldsymbol{\mu}_1^{(2)}-\underline{\boldsymbol{\nu}}_1^{(2)}\wedge\boldsymbol{\beta}_1)f_1^{\mathrm{syn}}(\boldsymbol{R},1;t)\mathrm{d}1 \\ &\quad -\int[\gamma_1\boldsymbol{\beta}_1\cdot\partial_0\{\gamma_1(\underline{\boldsymbol{\mu}}_1^{(2)}-\underline{\boldsymbol{\nu}}_1^{(2)}\wedge\boldsymbol{\beta}_1)\}+\tfrac{1}{2}\gamma_1\,\partial_0(\gamma_1\boldsymbol{\beta}_1)\cdot\underline{\boldsymbol{\nu}}_1^{(2)}\wedge\boldsymbol{\beta}_1]f_1^{\mathrm{syn}}(\boldsymbol{R},1;t)\mathrm{d}1, \\ \boldsymbol{M}^{(2)} &= -\nabla\cdot\int(\underline{\boldsymbol{\nu}}_1^{(2)}+\underline{\boldsymbol{\mu}}_1^{(2)}\wedge\boldsymbol{\beta}_1)f_1^{\mathrm{syn}}(\boldsymbol{R},1;t)\mathrm{d}1 \\ &\quad -\int[\gamma_1\boldsymbol{\beta}_1\cdot\partial_0\{\gamma_1(\underline{\boldsymbol{\nu}}_1^{(2)}+\underline{\boldsymbol{\mu}}_1^{(2)}\wedge\boldsymbol{\beta}_1)\}-\tfrac{1}{2}\gamma_1\,\partial_0(\gamma_1\boldsymbol{\beta}_1)\cdot\underline{\boldsymbol{\nu}}_1^{(2)}]f_1^{\mathrm{syn}}(\boldsymbol{R},1;t)\mathrm{d}1, \end{aligned} \tag{78}$$

where 1 stands for the atomic quantities occurring in the integrands. The underlined quadrupole moments follow from (IV.54) as:

$$\begin{aligned} \underline{\boldsymbol{\mu}}_1^{(2)} &\equiv \boldsymbol{\Omega}_1\cdot\boldsymbol{\mu}_1^{(2)}\cdot\boldsymbol{\Omega}_1 = \boldsymbol{\mu}_{1,\perp\perp}^{(2)}+\sqrt{1-\beta_1^2}(\boldsymbol{\mu}_{1,\perp//}^{(2)}+\boldsymbol{\mu}_{1,//\perp}^{(2)})+(1-\beta_1^2)\boldsymbol{\mu}_{1,////}^{(2)}, \\ \underline{\boldsymbol{\nu}}_1^{(2)} &\equiv \boldsymbol{\Omega}_1\cdot\boldsymbol{\nu}_1^{(2)}\cdot\boldsymbol{\Omega}_1 = \boldsymbol{\nu}_{1,\perp\perp}^{(2)}+\sqrt{1-\beta_1^2}(\boldsymbol{\nu}_{1,\perp//}^{(2)}+\boldsymbol{\nu}_{1,//\perp}^{(2)})+(1-\beta_1^2)\boldsymbol{\nu}_{1,////}^{(2)}. \end{aligned} \tag{79}$$

This formula shows that the electric and magnetic quadrupole moments suffer a Lorentz contraction.

The first term of $\boldsymbol{P}$ and the first two terms of $\boldsymbol{M}$ occurred also in the non-relativistic theory (chapter II), but with the non-relativistic quadrupole moments $\overline{\boldsymbol{\mu}}_1^{(2)}$ and $\overline{\boldsymbol{\nu}}_1^{(2)}$. Thus relativistic effects of four different types appear in the expressions given here. In the first place two effects similar to those found for dipole substances occur:

1°: the Lorentz contractions of the longitudinal components of the quadrupole moments $\boldsymbol{\mu}_1^{(2)}$ and $\boldsymbol{\nu}_1^{(2)}$. Under common circumstances, the effects are quite small, for the same reasons as explained in the dipole case.

2°: the effect of moving magnetic quadrupoles, described by the second term of $\boldsymbol{P}$. Its relative magnitude as compared to the leading term of $\boldsymbol{P}$ is (again under normal circumstances) of the order of 10^{-8} for fluids and 10^{-7} for solids.

Two more relativistic effects, which were absent in the dipole case, are encountered in the expressions of $\boldsymbol{P}$ and $\boldsymbol{M}$:

3°: the multipole fluxion effect, which is connected with the occurrence of time derivatives of the quadrupole moments.

4°: the acceleration effect, which is due to the presence of terms with the atomic acceleration $\partial_0\boldsymbol{\beta}_1$.

The effects depend on the magnitude of the atomic velocities and accelerations. Let us give some numerical estimations of the various effects in systems under normal circumstances. Let us first consider a solid with atomic vibrations in the optical branch. As mentioned above $\omega \simeq 3\times10^{13}\mathrm{s}^{-1}$ and $\beta_1 \simeq 10^{-5}$. The vibration acceleration is $\ddot{R}_1 \simeq R_1\omega^2$ and therefore $\partial_0\beta_1 = R_1\omega^2c^{-2} \simeq 10^{-2}\,\mathrm{cm}^{-1}$. The time derivative of the electric quadrupole moment $\partial_0\mu_1^{(2)}$ is of the order $e_{ki}r_{ki}\dot{r}_{ki}/c$ or $10^6e_{ki}r_{ki}^2$ (since $r_{ki} \simeq 10^{-8}$ cm and $\dot{r}_{ki}/c$ is of the order of the fine structure constant) or $10^6\mu_1^{(2)}$. Similarly, since roughly $\ddot{r}_{ki}/\dot{r}_{ki} \simeq \dot{r}_{ki}/r_{ki}$, the time derivative of the magnetic quadrupole moment $\partial_0\nu_1^{(2)}$ is of the order $10^6\nu_1^{(2)}$. These numbers will allow us to estimate the relativistic effects in the polarization tensor. Just as in the dipole case the effects 1° and 2° are small compared to the leading terms which they accompany. Let us therefore consider here the other effects. The main terms in $\boldsymbol{P}$ and $\boldsymbol{M}$, including the non-relativistic effect, contain gradients. Their magnitude, if compared to the dipole terms, depends thus on the inhomogeneities of the material. A very rough guess is obtained in the following way. Let the electric quadrupole moment $\mu_1^{(2)}$ be of the order of 10^{-8} cm times $\mu_1^{(1)}$, because it contains one more factor r_{ki} than the dipole moment $\mu_1^{(1)}$. Let the 'inhomogeneity length' be 10^{-2} to 1 cm. Then the magnitude of the (non-relativistic) $\nabla\cdot\boldsymbol{\mu}_1^{(2)}$ effect expressed in the $\mu_1^{(1)}$ effect is 10^{-6} to $10^{-8}\mu_1^{(1)}$. The relativistic effects are more interesting because they do not contain gradients and are thus independent of the inhomogeneities. The multipole fluxion effect in $\boldsymbol{P}$ is $\beta_1\partial_0\mu_1^{(2)} \simeq 10^{-7}\mu_1^{(1)}$. The acceleration effect in $\boldsymbol{P}$ is $(\partial_0\beta_1)\nu_1^{(2)}\beta_1$. It contains the magnetic quadrupole moment which is about 10^{-2} (the fine structure constant $\simeq \dot{r}_{ki}/c$) of the electric quadrupole moment. The acceleration effect becomes of the order $10^{-17}\mu_1^{(1)}$.

Similarly in $\boldsymbol{M}$ the main (non-relativistic) term with $\nabla\cdot\boldsymbol{\nu}_1^{(2)}$ becomes (with the use of the same figures as above) 10^{-8} to $10^{-10}\mu_1^{(1)}$, the multipole fluxion effect $\beta_1\partial_0\nu_1^{(2)}$ becomes $10^{-9}\mu_1^{(1)}$ and the acceleration effect $(\partial_0\beta_1)\nu_1^{(2)}$ (note that this effect contains the magnetic quadrupole moment both in $\boldsymbol{P}$ and $\boldsymbol{M}$) becomes $10^{-12}\mu_1^{(1)}$. The conclusion can be that all quadrupolar effects are very small, if compared to dipolar effects. But if a substance is studied which contains quadrupoles, but no dipoles, then the quadrupolar effects can only be compared amongst each other. The conclusion which one may draw from the figures given above is then that the relativistic multipole fluxion effect may exceed the non-relativistic effects under favourable physical circumstances.

The preceding concerned the case of a solid. In fluids β_1 is about ten times smaller and the collision frequency is perhaps a hundred times smaller. The relativistic effects are then exceedingly small.

If all atoms have the same velocity $\boldsymbol{\beta}$ and acceleration $\partial_0 \boldsymbol{\beta}$, the polarization can be expressed in terms of these quantities and of the macroscopic quadrupolar densities:

$$\begin{aligned} \mathscr{P}^{(2)} &\equiv \int \boldsymbol{\mu}_1^{(2)} f_1^{\mathrm{syn}}(\boldsymbol{R}, \boldsymbol{\mu}_1^{(2)}; t) \mathrm{d}\boldsymbol{\mu}_1^{(2)}, \\ \mathscr{M}^{(2)} &\equiv \int \boldsymbol{\nu}_1^{(2)} f_1^{\mathrm{syn}}(\boldsymbol{R}, \boldsymbol{\nu}_1^{(2)}; t) \mathrm{d}\boldsymbol{\nu}_1^{(2)}. \end{aligned} \tag{80}$$

Indeed with (79) and the abbreviations

$$\begin{aligned} \underline{\mathscr{P}}^{(2)} &\equiv \boldsymbol{\Omega}\cdot\mathscr{P}^{(2)}\cdot\boldsymbol{\Omega} = \mathscr{P}^{(2)}_{\perp\perp} + \sqrt{1-\beta^2}(\mathscr{P}^{(2)}_{\perp//} + \mathscr{P}^{(2)}_{//\perp}) + (1-\beta^2)\mathscr{P}^{(2)}_{////}, \\ \underline{\mathscr{M}}^{(2)} &\equiv \boldsymbol{\Omega}\cdot\mathscr{M}^{(2)}\cdot\boldsymbol{\Omega} = \mathscr{M}^{(2)}_{\perp\perp} + \sqrt{1-\beta^2}(\mathscr{M}^{(2)}_{\perp//} + \mathscr{M}^{(2)}_{//\perp}) + (1-\beta^2)\mathscr{M}^{(2)}_{//}, \end{aligned} \tag{81}$$

it follows that the quadrupole contributions to the polarization tensor (78) get the form

$$\begin{aligned} \boldsymbol{P}^{(2)} &= -\nabla\cdot(\underline{\mathscr{P}}^{(2)} - \underline{\mathscr{M}}^{(2)} \wedge \boldsymbol{\beta}) - \gamma\boldsymbol{\beta}\cdot\mathrm{d}_0\{\gamma(\underline{\mathscr{P}}^{(2)} - \underline{\mathscr{M}}^{(2)} \wedge \boldsymbol{\beta})\} \\ &\qquad - \tfrac{1}{2}\gamma\partial_0(\gamma\boldsymbol{\beta})\cdot(\underline{\mathscr{M}}^{(2)} \wedge \boldsymbol{\beta}), \qquad (82) \\ \boldsymbol{M}^{(2)} &= -\nabla\cdot(\underline{\mathscr{M}}^{(2)} + \underline{\mathscr{P}}^{(2)} \wedge \boldsymbol{\beta}) - \gamma\boldsymbol{\beta}\cdot\mathrm{d}_0\{\gamma(\underline{\mathscr{M}}^{(2)} + \underline{\mathscr{P}}^{(2)} \wedge \boldsymbol{\beta})\} + \tfrac{1}{2}\gamma\partial_0(\gamma\boldsymbol{\beta})\cdot\underline{\mathscr{M}}^{(2)}, \end{aligned}$$

where $\mathrm{d}_0 \equiv \partial_0 + \boldsymbol{\beta}\cdot\nabla$ is the substantial time derivative. It should be remarked that in $\boldsymbol{M}$ the acceleration effect subsists even if the velocity $\boldsymbol{\beta} = 0$.

4 *The conservation of energy–momentum*

a. *The conservation of rest mass*

The atomic conservation law (IV.176) of rest mass contains the mass flow four-vector, the average of which is, according to (17) with (15):

$$m_1^{\mathrm{rest}} \int u_1^\alpha \delta^{(4)}(X_1 - R) f_1(1) \mathrm{d}1. \tag{83}$$

Since this is a time-like four-vector it may be used to define the macroscopic four-velocity U^α (with $U_\alpha U^\alpha = -c^2$) and the macroscopic rest mass density

ϱ' by writing

$$\varrho' U^\alpha = m_1^{\mathrm{rest}} \int u_1^\alpha \delta^{(4)}(X_1 - R) f_1(1) \mathrm{d}1. \tag{84}$$

From the atomic mass conservation law (IV.176) it follows immediately, with (20), that

$$\partial_\alpha(\varrho' U^\alpha) = 0, \tag{85}$$

which is the macroscopic law of mass conservation.

b. *Energy–momentum conservation for a fluid system of neutral atoms*

The conservation law of energy–momentum will follow by taking the average of the atomic energy–momentum law (IV.177):

$$\begin{aligned} c\partial_\beta \left\langle \sum_k \int m_k u_k^\alpha u_k^\beta \delta^{(4)}(X_k - R) \mathrm{d}s_k \right\rangle &= c \left\langle \sum_k \int \mathfrak{f}_k^\alpha \delta^{(4)}(X_k - R) \mathrm{d}s_k \right\rangle \\ &\quad - c^{-1}\partial_\beta \left\langle \sum_k \int (s_k^{\alpha\gamma} \mathfrak{f}_{k\gamma}/m_k + \mathfrak{d}_k^{\alpha\gamma} u_{k\gamma}) u_k^\beta \delta^{(4)}(X_k - R) \mathrm{d}s_k \right\rangle. \end{aligned} \tag{86}$$

The total force and torque $\mathfrak{f}_k^\alpha$ and $\mathfrak{d}_k^{\alpha\beta}$ have been given in (IV.191–192) with (IV.193–194) and (IV.206–207). We shall confine ourselves in this subsection to systems of neutral atoms, since we are interested in the effects due to electromagnetic multipole moments. In the following we shall show that (86) may be cast into the form of a conservation law, i.e. as $\partial_\beta T^{\alpha\beta} = 0$, where $T^{\alpha\beta}$ is the energy–momentum tensor.

At the left-hand side of (86) appears the divergence $\partial_\beta T^{\alpha\beta}_{(\mathrm{m})\mathrm{I}}$ of the tensor

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{I}} \equiv \int m_1 u_1^\alpha u_1^\beta \delta^{(4)}(X_1 - R) f_1(1) \mathrm{d}1 \tag{87}$$

as follows from (17) with (15). This relativistic generalization of the kinetic energy and momentum densities and flows forms a first contribution (index I) to the macroscopic energy–momentum tensor. In particular, in view of its form, it is said to contribute to the *material* part of the energy–momentum tensor (denoted by the index (m))[1].

[1] Of course the characterization of (87) as being purely material does not exclude the fact that its value will depend in general on the macroscopic fields, since the distribution function will depend on these quantities. The classification by means of the index (m) for material, and later on (f) for field parts of the energy–momentum tensor is made only for convenience. The physical laws contain the *total* energy–momentum tensor, which will be specified in the following.

The right-hand side of (86) contains the total force and torque $\mathfrak{f}_k^\alpha$ and $\mathfrak{d}_k^{\alpha\beta}$ which have been split into long range and short range parts in (IV.191–192). We shall consider first the long range part which is obtained by introducing the total long range force and torque, specified in (IV.193–194). One gets thus the sum of external field and interatomic two-particle contributions. The latter, which contain the two-point distribution function $f_2(1, 2)$ (13), may be written as the sum of an uncorrelated term with $f_1(1)f_1(2)$ and a correlated term with $c_2(1, 2)$ (14). In this way one obtains in the first place the uncorrelated long range part of the right-hand side of (86). It contains the Maxwell fields (59) and reads

$$\int \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\mathfrak{m}_{1\beta\gamma}\,\delta^{(4)}(X_1-R)f_1(1)\mathrm{d}1$$

$$-c^{-2}\partial_\beta\int\Big[(F^{\alpha\gamma}\mathfrak{m}_{1\gamma\varepsilon}u_1^\varepsilon-\Delta_1^{\alpha\gamma}\mathfrak{m}_{1\gamma\varepsilon}F^{\varepsilon\zeta}u_{1\zeta})u_1^\beta$$

$$+\frac{s_1^{\alpha\gamma}}{m_1}\left\{\tfrac{1}{2}(\partial_\gamma F_{\varepsilon\zeta})\mathfrak{m}_1^{\varepsilon\zeta}-c^{-2}\frac{\mathrm{d}}{\mathrm{d}s_1}(F_{\gamma\varepsilon}\mathfrak{m}_1^{\varepsilon\zeta}u_{1\zeta})\right\}u_1^\beta\Big]\,\delta^{(4)}(X_1-R)f_1(1)\mathrm{d}1, \quad (88)$$

where (IV.193–194) with (IV.182–183) have been used and where only dipole contributions have been taken into account. The latter limitation gives us the leading terms in the uncorrelated part of the macroscopic energy–momentum tensor. (In the correlated part such a limitation is not possible, since virtual multipoles of all orders cannot be excluded.) With the definitions (61) of the macroscopic polarization and (84) of the macroscopic velocity one may write (88) as

$$\tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})M_{\beta\gamma}-c^{-2}\partial_\beta\{F^{\alpha\gamma}M_{\gamma\varepsilon}U^\varepsilon-\Delta^{\alpha\gamma}M_{\gamma\varepsilon}F^{\varepsilon\zeta}U_\zeta)U^\beta\}-\partial_\beta T^{\alpha\beta}_{(\mathrm{m})\mathrm{II}}, \quad (89)$$

with $\Delta^{\alpha\beta}\equiv g^{\alpha\beta}+c^{-2}U^\alpha U^\beta$. Here a second contribution to the material energy–momentum tensor appears:

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{II}}\equiv c^{-2}\int\Big[F^{\alpha\gamma}\mathfrak{m}_{1\gamma\varepsilon}(u_1^\varepsilon u_1^\beta-U^\varepsilon U^\beta)-\mathfrak{m}_{1\gamma\varepsilon}F^{\varepsilon\zeta}(\Delta_1^{\alpha\gamma}u_{1\zeta}u_1^\beta-\Delta^{\alpha\gamma}U_\zeta U^\beta)$$

$$+\frac{s_1^{\alpha\gamma}}{m_1}\left\{\tfrac{1}{2}(\partial_\gamma F_{\varepsilon\zeta})\mathfrak{m}_1^{\varepsilon\zeta}-c^{-2}\frac{\mathrm{d}}{\mathrm{d}s_1}(F_{\gamma\varepsilon}\mathfrak{m}_1^{\varepsilon\zeta}u_{1\zeta})\right\}u_1^\beta\Big]\,\delta^{(4)}(X_1-R)f_1(1)\mathrm{d}1. \quad (90)$$

This contribution contains in its first two terms velocity fluctuations $u_1^\alpha-U^\alpha$ while in the other two terms the inner angular momentum $s_1^{\alpha\beta}$ appears. On the other hand the first three terms of (89) contain exclusively macroscopic fields, polarizations and velocities. With the Maxwell equations (64) the first term of (89) may be written in the form of a divergence:

$$\tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})M_{\beta\gamma}=F^{\alpha\gamma}\partial_\beta M_\gamma^{\cdot\beta}-\partial_\beta(F^{\alpha\gamma}M_\gamma^{\cdot\beta})=\partial_\beta(F^{\alpha\gamma}F_\gamma^{\cdot\beta}-F^{\alpha\gamma}M_\gamma^{\cdot\beta}+\tfrac{1}{4}g^{\alpha\beta}F_{\gamma\varepsilon}F^{\gamma\varepsilon}), \quad (91)$$

where both the homogeneous and the inhomogeneous field equations have been employed. With (91) and (66) the first three terms of (89) become a divergence $-\partial_\beta T^{\alpha\beta}_{(\mathrm{f})}$, with the field energy–momentum tensor

$$T^{\alpha\beta}_{(\mathrm{f})}\equiv -F^{\alpha\gamma}H_\gamma^{\cdot\beta}-\tfrac{1}{4}g^{\alpha\beta}F_{\gamma\varepsilon}F^{\gamma\varepsilon}+c^{-2}(F^{\alpha\gamma}M_{\gamma\varepsilon}U^\varepsilon-\Delta^{\alpha\gamma}M_{\gamma\varepsilon}F^{\varepsilon\zeta}U_\zeta)U^\beta. \quad (92)$$

This is the only part of the total energy–momentum tensor which depends, in its explicit form, exclusively on the macroscopic fields, polarizations and velocities. Therefore we labelled it with the index (f), although, as remarked before, the division of the total energy–momentum tensor into a field part and a material part is not essential. We shall discuss the contents of (92) later on.

The uncorrelated long range part of the right-hand side of (86) has now been found, so that the correlated long range and the short range parts remain to be discussed. Let us start with the latter. The last terms of (86) are already in the form of a divergence; introducing a distribution function according to (19) and substituting (IV.206–207), one finds for them $-\partial_\beta T^{\alpha\beta}_{(\mathrm{m})\mathrm{III}}$, with a third contribution to the material energy–momentum tensor

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{III}}\equiv c^{-3}\int(s_1^{\alpha\gamma}\hat{\mathfrak{f}}^{\mathrm{S}}_{1;2\gamma}/m_1+\hat{\mathfrak{d}}^{\mathrm{S}\alpha\gamma}_{1;2}u_{1\gamma})u_1^\beta\,\delta^{(4)}(X_1-R)f_2(1,2)\mathrm{d}1\,\mathrm{d}2. \quad (93)$$

In the short range part of the first term at the right-hand side of (86) we shall consider separately the contributions from the 'plus' and 'minus' fields. The plus part of the total force on atom k is given explicitly in (IV.206) with (IV.204) and (IV.210). Introducing an extra variable s^α (with differential $\mathrm{d}^4s\equiv\mathrm{d}s$) and a four-dimensional delta function we may write the short range plus field contribution as

$$c^{-1}\int\hat{\mathfrak{f}}^{\mathrm{S}\alpha}_{+1;2}(s)\delta^{(4)}(X_1-R)\delta^{(4)}(X_2-R+s)f_2(1,2)\mathrm{d}1\,\mathrm{d}2\,\mathrm{d}s, \quad (94)$$

where we employed the abbreviation

$$\hat{\mathfrak{f}}^{\mathrm{S}\alpha}_{+1;2}(s)\equiv\frac{c^{-1}}{4\pi}\sum_{i,j}e_{1i}e_{2j}(u_{1i}\cdot u_{2j}\,\partial_s^\alpha-u_{2j}^\alpha u_{1i}\cdot\partial_s)\delta\{(s+r_{1i}-r_{2j})^2\}$$

$$-\frac{c}{4\pi}\sum_{n,m=1}^{\infty}(-1)^m\left\{\mathfrak{m}_1^{\alpha_1\ldots\alpha_{n+1}}\partial_{s\alpha_n}+c^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}s_1}+u_1\cdot\partial_s\right)\mathfrak{m}_1^{\alpha_1\ldots\alpha_{n+1}}u_{1\alpha_n}\right\}$$

$$\left\{\mathfrak{m}_2^{\beta_1\ldots\beta_{m+1}}\partial_{s\beta_m}-c^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}s_2}-u_2\cdot\partial_s\right)\mathfrak{m}_2^{\beta_1\ldots\beta_{m+1}}u_{2\beta_m}\right\}\partial_{s\alpha_1\ldots\alpha_{n-1}\beta_1\ldots\beta_{m-1}}$$

$$(\partial_s^\alpha g_{\alpha_{n+1}\beta_{m+1}}-\partial_{s\alpha_{n+1}}g^\alpha_{\beta_{m+1}})\delta(s^2). \quad (95)$$

In the non-relativistic theory the expression corresponding to (94) could be transformed into a divergence (a three-divergence and a time derivative) by making an appropriate Taylor expansion, which could be broken off after the first term since the integrand has short range as a function of s^α. In fact it is the difference between the unexpanded and the multipole expanded atomic force so that it vanishes if the atoms are sufficiently far apart. We now have to generalize this procedure, due to Irving and Kirkwood, to the relativistic case. In (94) the two-point distribution function $f_2(1,2)$ appears. As a result of the presence of the two delta functions, it contains the position four-vectors R^α and $R^\alpha - s^\alpha$ of the two atoms, so that its form is $f_2(R, 1, R-s, 2)$ (where now 1 and 2 denote the other dynamical properties of the atoms). As the relativistic generalization of the Irving–Kirkwood procedure we expand the two-point distribution function as a function of R^α in a Taylor series around $R^\alpha + \frac{1}{2}s^\alpha$:

$$f_2(R, 1, R-s, 2) = f_2(R+\tfrac{1}{2}s, 1, R-\tfrac{1}{2}s, 2) - \tfrac{1}{2}s^\alpha \partial_\alpha f_2(R+\tfrac{1}{2}s, 1, R-\tfrac{1}{2}s, 2) + \ldots . \tag{96}$$

Only the first few terms contribute significantly since the integrand has short range as a function of s^α.

If one introduces the expansion (96) into (94) one gets a sum of two terms: one term has the same form as (94), but with different delta functions:

$$c^{-1} \int \mathfrak{f}^{s\alpha}_{+1;2}(s) \delta^{(4)}(X_1 - R - \tfrac{1}{2}s) \delta^{(4)}(X_2 - R + \tfrac{1}{2}s) f_2(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s; \tag{97}$$

the other term may be written as $-\partial_\beta T^{\alpha\beta}_{(\mathrm{m})\mathrm{IV}}$, with a fourth contribution to the energy–momentum tensor

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{IV}} \equiv \tfrac{1}{2} c^{-1} \int s^\beta \mathfrak{f}^{s\alpha}_{+1;2}(s) \delta^{(4)}(X_1 - R - \tfrac{1}{2}s) \delta^{(4)}(X_2 - R + \tfrac{1}{2}s) f_2(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s. \tag{98}$$

If (95) is inserted into (97) one finds that the terms which contain ∂_s^α vanish for reasons of symmetry: by making the substitutions $s^\alpha \leftrightarrow -s^\alpha$ and $1 \leftrightarrow 2$ the integral changes into its opposite. As to the remaining terms of (97) with (95), one finds for the unexpanded part by integrating partially with respect to s^α:

$$- \frac{c^{-2}}{4\pi} \int \sum_{i,j} e_{1i} e_{2j} u^\alpha_{2j} \left[\frac{\mathrm{d}r_{1i}}{\mathrm{d}s_1} \cdot \partial_s \delta\{(s + r_{1i} - r_{2j})^2\} - \delta\{(s + r_{1i} - r_{2j})^2\} u_1 \cdot \partial_s \right]$$
$$\delta^{(4)}(X_1 - R - \tfrac{1}{2}s) \delta^{(4)}(X_2 - R + \tfrac{1}{2}s) f_2(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s$$
$$= - \frac{c^{-2}}{4\pi} \int \sum_{i,j} e_{1i} e_{2j} u^\alpha_{2j} \frac{\mathrm{d}}{\mathrm{d}s_1} [\delta^{(4)}\{(s + r_{1i} - r_{2j})^2\} \delta^{(4)}(X_1 - R - \tfrac{1}{2}s)]$$
$$\delta^{(4)}(X_2 - R + \tfrac{1}{2}s) f_2(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s$$
$$- \frac{c^{-2}}{8\pi} \partial_\beta \int u_1^\beta \sum_{i,j} e_{1i} e_{2j} u^\alpha_{2j} \delta\{(s + r_{1i} - r_{2j})^2\} \delta^{(4)}(X_1 - R - \tfrac{1}{2}s)$$
$$\delta^{(4)}(X_2 - R + \tfrac{1}{2}s) f_2(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s. \tag{99}$$

The first term at the right-hand side vanishes as follows by integrating partially and employing the conservation of probability in the form (10), while the second term gives a contribution to $-\partial_\beta T^{\alpha\beta}_{(\mathrm{m})\mathrm{V}}$. The fifth part of the material energy–momentum tensor occurring here reads, if the multipole expanded part of the terms without ∂_s^α of (97) with (95) are treated in a similar fashion:

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{V}} \equiv \frac{c^{-2}}{8\pi} \int u_1^\beta \left[\sum_{i,j} e_{1i} e_{2j} u^\alpha_{2j} \delta\{(s + r_{1i} - r_{2j})^2\} + \sum_{n,m=1}^{\infty} (-1)^m \mathfrak{m}_1^{\alpha_1 \ldots \alpha_{n+1}} \right.$$
$$\left. u_{1\alpha_{n+1}} \left\{ \mathfrak{m}_2^{\beta_1 \ldots \beta_m \alpha} \partial_{s\beta_m} - c^{-2} \left(\frac{\mathrm{d}}{\mathrm{d}s_2} - u_2 \cdot \partial_s \right) \mathfrak{m}_2^{\beta_1 \ldots \beta_m \alpha} u_{2\beta_m} \right\} \partial_{s\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_{m-1}} \delta(s^2) \right]$$
$$\delta^{(4)}(X_1 - R - \tfrac{1}{2}s) \delta^{(4)}(X_2 - R + \tfrac{1}{2}s) f_2(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s. \tag{100}$$

The plus field short range contributions have thus been written in the form of a divergence. The essential step consisted in showing that owing to symmetry part of the terms of (97) with (95) vanished. This symmetry was intimately connected with the appearance of the delta function $\delta(s^2)$ which is invariant under the transformation $s^\alpha \leftrightarrow -s^\alpha$. If we had used from the beginning the complete retarded field, instead of only its plus part, an extra factor $\theta(s)$ would have appeared, which would have destroyed this invariance. Thus if we consider the minus field contribution a different procedure will have to be followed to transform it into a divergence[1].

The minus field short range part of the first term at the right-hand side of (86) is obtained by inserting the atomic expressions (IV.206) with (IV.208) and (IV.202). If one employs the inhomogeneous atomic field equations (IV.20–23) for the partial fields $f_\pm^{\alpha\beta}$ together with the homogeneous ones,

[1] In the non-relativistic theory this problem did not arise, because the non-relativistic terms were due exclusively to the plus field.

one finds for the unexpanded minus field part a divergence $-\partial_\beta T^{\alpha\beta}_{(m)VI}$, with the abbreviation

$$T^{\alpha\beta}_{(m)VI} \equiv -c^{-2}\sum_{i,j}\int\{\hat{f}^{\alpha\gamma}_{-2j}(R+r_{1i})\hat{f}^{\cdot\beta}_{+1i\gamma}(R+r_{1i})+\hat{f}^{\alpha\gamma}_{+1i}(R+r_{1i})\hat{f}^{\cdot\beta}_{-2j\gamma}(R+r_{1i}) + \tfrac{1}{2}g^{\alpha\beta}\hat{f}_{+1i\gamma\varepsilon}(R+r_{1i})\hat{f}^{\gamma\varepsilon}_{-2j}(R+r_{1i})\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2 + \frac{c^{-2}}{4\pi}\sum_{i,j}e_{1i}\int \hat{f}^{\alpha\beta}_{-2j}(R+r_{1i})u_{1i}\cdot\partial\delta\{(R-X_1)^2\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2, \quad (101)$$

Likewise one arrives by similar steps at the expression $-\partial_\beta T^{\alpha\beta}_{(m)VII}$ for the multipole expanded minus field short range part of the first term at the right-hand side of (86). The seventh contribution to the material energy–momentum tensor, which occurs here, has the form

$$T^{\alpha\beta}_{(m)VII} \equiv -\frac{c^{-1}}{4\pi}\sum_{n=1}^{\infty}\int\left[\left\{\mathfrak{m}_1^{\alpha_1\ldots\alpha_n\zeta}\partial_{\alpha_n}+c^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}s_1}+u_1\cdot\partial\right)\mathfrak{m}_1^{\alpha_1\ldots\alpha_n\zeta}u_{1\alpha_n}\right\}\partial_{\alpha_1\ldots\alpha_{n-1}}\hat{f}^{\gamma\varepsilon}_{-2(m)}\right] \{g^\alpha_\gamma(\partial^\beta g_{\varepsilon\zeta}+g^\beta_\varepsilon\partial_\zeta-g^\beta_\zeta\partial_\varepsilon)+g^\beta_\gamma(\partial^\alpha g_{\varepsilon\zeta}-g^\alpha_\zeta\partial_\varepsilon)+g^{\alpha\beta}g_{\gamma\zeta}\partial^\varepsilon\} \delta\{(R-X_1)^2\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2. \quad (102)$$

In this way all short range parts of the right-hand side of (86) have been evaluated as divergences of various contributions to the material energy–momentum tensor.

The correlated long range parts of the right-hand side of (86) may be discussed along similar lines, the only difference being that one has to confine oneself to systems with short range correlations[1]. For such systems one may assume the validity of the generalized Irving–Kirkwood approximation, which states that the correlation function is slowly varying over distances that may be compared with the correlation length[2]. Then one may write a Taylor expansion for the correlation function (cf. (96)) and break it off after the second term. As a result one obtains for the correlated long range part of (86) a term $-\partial_\beta T^{\alpha\beta}_{(m)VIII}$. The material energy–momentum tensor appearing here consists of various contributions: first a term like (93) but with $\mathfrak{f}^{S\alpha}_{1,2}$ and $\mathfrak{d}^{S\alpha\beta}_{1;2}$ replaced by $\mathfrak{f}^{L\alpha}_{1;2(mm)}$ and $\mathfrak{d}^{L\alpha\beta}_{1;2(mm)}$ (v. (IV.202–203)), and $f_2(1,2)$ replaced by $c_2(1,2)$; secondly a term like (98) with (95) with similar alterations; thirdly a term like the multipole expanded part of (100) but with $f_2(1,2)$ replaced by $-c_2(1,2)$; and fourthly a term like (102) again with the same replacement.

[1] The name long range referred to the atomic quantity. If such a quantity is multiplied by the correlation function, the long range character need no longer prevail: in fact if no long range correlations are present in the system, the correlation function and therefore also the product with the long range atomic quantity will have short range character.
[2] If long range correlations are present one may employ an artifice like that of chapter II, section 5*h* (cf. problem 9).

In this way all contributions to the energy–momentum law (86) have been written in the form of divergences, so that we have obtained a conservation law of energy–momentum for a system of neutral atoms without long range correlations in an external electromagnetic field:

$$\partial_\beta T^{\alpha\beta} = 0 \quad (103)$$

with

$$T^{\alpha\beta} = T^{\alpha\beta}_{(f)} + T^{\alpha\beta}_{(m)}. \quad (104)$$

The energy–momentum tensor $T^{\alpha\beta}$ consists of nine contributions, which have been classified as one contribution $T^{\alpha\beta}_{(f)}$ (92), which depended solely on the macroscopic (Maxwell) fields, the polarizations and the velocities, and eight other terms. The latter form together what has been called here the material energy–momentum tensor $T^{\alpha\beta}_{(m)}$.

The case $\alpha = 0$ of (103) represents the energy conservation law which may be written (with $\partial_0 = \partial/\partial ct$ and $\partial_i = \nabla_i = \partial/\partial R^i$, $i = 1, 2, 3$) as

$$\frac{\partial}{\partial t}T^{00}+\nabla_i\cdot cT^{0i} = 0, \quad (105)$$

where T^{00} is the energy density and where cT^{0i} ($i = 1, 2, 3$) are the components of the energy flow. The cases $\alpha = i = 1, 2, 3$ of (103) form the law of momentum conservation:

$$\frac{\partial}{\partial t}c^{-1}T^{i0}+\nabla_j T^{ij} = 0, \quad (106)$$

where $c^{-1}T^{i0}$ is the momentum density and T^{ij} the momentum flow.

In subsection *d* we shall discuss the components of the energy–momentum tensor in more detail.

c. *Energy–momentum conservation for a neutral plasma*

In the preceding we considered only one-component systems. The extension to a mixture is in particular necessary if one wants to study neutral plasmas, in which particles with different charges occur. The various species will be labelled by a special index. The starting point for the derivation of the conservation law of energy–momentum for plasmas is the atomic equation

(IV.177), where one has to take the inner angular momentum $s_k^{\alpha\beta}$ and the total torque $\mathfrak{d}_k^{\alpha\beta}$ as zero since the particles are considered to be point charges without structure (for the same reason we may denote the position of the particles simply by R_k instead of X_k). The average of this atomic equation becomes then:

$$c\partial_\beta \left\langle \sum_k \int m_k u_k^\alpha u_k^\beta \delta^{(4)}(R_k - R) \mathrm{d}s_k \right\rangle = c \left\langle \sum_k \int \mathfrak{f}_k^\alpha \delta^{(4)}(R_k - R) \mathrm{d}s_k \right\rangle. \quad (107)$$

From the definition (15) with (17) of an average quantity it follows that the left-hand side of this equation can be written as $\partial_\beta T_{(\mathrm{m})\mathrm{I}}^{\alpha\beta}$ with

$$T_{(\mathrm{m})\mathrm{I}}^{\alpha\beta} \equiv \sum_a \int m_a u_1^\alpha u_1^\beta \delta^{(4)}(R_1 - R) f_1^a(1) \mathrm{d}1, \quad (108)$$

where a labels the species. This quantity forms part of the material energy–momentum tensor. At the right-hand side of (107) the total force $\mathfrak{f}_k^\alpha$ on atom k appears. It is given by (IV.191) with (IV.193), (IV.182) and (IV.195), but without the short range and the multipole terms:

$$\mathfrak{f}_k^\alpha = c^{-1} e_k F_{\mathrm{e}}^{\alpha\beta}(R_k) u_{k\beta} + c^{-1} \sum_{l(\neq k)} e_k f_l^{\alpha\beta}(R_k) u_{k\beta}, \quad (109)$$

where $F_{\mathrm{e}}^{\alpha\beta}$ is the external field and $f_l^{\alpha\beta}$ the retarded field generated by particle l. The latter was given in (IV.14, 15) as:

$$f_l^{\alpha\beta} = -\frac{e_l}{2\pi} \int (u_l^\alpha \partial^\beta - u_l^\beta \partial^\alpha) \delta\{(R - R_l)^2\} \theta(R - R_l) \mathrm{d}s_l. \quad (110)$$

We substitute (109) with (110) into the right-hand side of (107) and make use of the splitting of the two-point distribution function into the product of two one-point distribution functions and a correlation function. The uncorrelated part becomes

$$c^{-1} \sum_a e_a \int F^{\alpha\beta}(R) u_{1\beta} \delta^{(4)}(R_1 - R) f_1^a(1) \mathrm{d}1, \quad (111)$$

where $F^{\alpha\beta}(R)$ is the macroscopic (Maxwell) field. With the definition (60) with (IV.5) of the macroscopic charge–current density J^γ this expression becomes

$$c^{-1} F^{\alpha\beta} J_\beta, \quad (112)$$

which is the macroscopic Lorentz force density. By using the inhomogeneous Maxwell equation one then finds $F^{\gamma\beta}\partial_\gamma F_\beta^{\cdot\gamma}$ or, with the homogeneous Maxwell equation, $-\partial_\beta T_{(\mathrm{f})}^{\alpha\beta}$ with the field energy–momentum tensor:

$$T_{(\mathrm{f})}^{\alpha\beta} \equiv -F^{\alpha\gamma} F_\gamma^{\cdot\beta} - \tfrac{1}{4} g^{\alpha\beta} F_{\gamma\varepsilon} F^{\gamma\varepsilon}. \quad (113)$$

These are the contributions which depend solely on the Maxwell fields. We now turn to the correlated part of the right-hand side of (107). Just as in the preceding subsection it will be convenient to split the interatomic field (110) into a plus and a minus part (cf. (IV.17) and (IV.18)). The plus field contribution to the correlated part at the right-hand side of (107) is:

$$-c^{-2} \sum_{a,b} \frac{e_a e_b}{4\pi} \int \{(u_2^\alpha \partial_s^\beta - u_2^\beta \partial_s^\alpha) u_{1\beta} \delta(s^2)\} \delta^{(4)}(R_1 - R) \delta^{(4)}(R_2 - R + s) c_2^{ab}(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s, \quad (114)$$

where we introduced an extra integration over a variable s^α and a four-dimensional delta function $\delta^{(4)}(R_2 - R + s)$. We confine ourselves now to the case without long range correlations[1]. For a plasma which is neutral in its proper frame (i.e. in the frame in which $U^\alpha = (c, 0, 0, 0)$) and which is not too far from equilibrium this seems a reasonable assumption. Then we may make a Taylor expansion of the correlation function, just as in the preceding section, and retain only the first few terms. This procedure, which is the relativistic generalization of Irving and Kirkwood's method, brings (114) into the form

$$c^{-2} \sum_{a,b} \frac{e_a e_b}{4\pi} \int \{(u_1 \cdot u_2 \partial_s^\alpha - u_2^\alpha u_1 \cdot \partial_s) \delta(s^2)\} \delta^{(4)}(R_1 - R - \tfrac{1}{2}s) \delta^{(4)}(R_2 - R + \tfrac{1}{2}s) c_2^{ab}(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s - \partial_\beta T_{(\mathrm{m})\mathrm{II}}^{\alpha\beta}, \quad (115)$$

where a second contribution to the material energy–momentum tensor arises:

$$T_{(\mathrm{m})\mathrm{II}}^{\alpha\beta} \equiv c^{-2} \sum_{a,b} \frac{e_a e_b}{8\pi} \int s^\beta \{(u_1 \cdot u_2 \partial_s^\alpha - u_2^\alpha u_1 \cdot \partial_s) \delta(s^2)\} \delta^{(4)}(R_1 - R - \tfrac{1}{2}s) \delta^{(4)}(R_2 - R + \tfrac{1}{2}s) c_2^{ab}(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s. \quad (116)$$

The first part of the first term of (115) may be shown to vanish by using the transformation $s^\alpha \leftrightarrow -s^\alpha$, $1 \leftrightarrow 2$. The second part of the first term of (115) may be written after partial integrations, first with respect to s^α, and subsequently with respect to R_1^α, in the form

$$c^{-2} \sum_{a,b} \frac{e_a e_b}{4\pi} \int u_2^\alpha \delta(s^2) \delta^{(4)}(R_1 - R - \tfrac{1}{2}s) \delta^{(4)}(R_2 - R + \tfrac{1}{2}s) u_1 \cdot \frac{\partial}{\partial R_1} c_2^{ab}(1,2) \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}s - \partial_\beta T_{(\mathrm{m})\mathrm{III}}^{\alpha\beta}. \quad (117)$$

[1] The case with long range correlations may be treated by making use of an artifice as employed in chapter II, section 5*h*.

With the help of the continuity equation for $c_2^{ab}(1,2)$, which has the form (10), it appears that the first term of (117) vanishes. The second term contains a further contribution to the material energy–momentum tensor

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{III}} \equiv c^{-2}\sum_{a,b}\frac{e_a e_b}{8\pi}\int u_2^{\alpha} u_1^{\beta}\,\delta(s^2)\delta^{(4)}(R_1-R-\tfrac{1}{2}s)\,\delta^{(4)}(R_2-R+\tfrac{1}{2}s)c_2^{ab}(1,2)\mathrm{d}1\,\mathrm{d}2\,\mathrm{d}s. \quad (118)$$

We finally have to treat the minus field contribution to the correlated part at the right-hand side of (107). It reads, according to (109) and the definition (19) with (18) of an average quantity,

$$c^{-2}\sum_a e_a\int \hat{f}_{-2}^{\alpha\beta}(R_1)u_{1\beta}\,\delta^{(4)}(R_1-R)c_2^{ab}(1,2)\mathrm{d}1\,\mathrm{d}2. \quad (119)$$

With the help of the atomic field equations (IV.13), (IV.20) and (IV.22) for the plus and minus fields $\hat{f}_{+}^{\alpha\beta}$ and $\hat{f}_{-}^{\alpha\beta}$, this expression may be transformed into a divergence $-\partial_\beta T^{\alpha\beta}_{(\mathrm{m})\mathrm{IV}}$, where a last contribution to the material energy–momentum tensor appears:

$$T^{\alpha\beta}_{(\mathrm{m})\mathrm{IV}} \equiv -c^{-2}\int\{\hat{f}_{-2}^{\alpha\gamma}\hat{f}_{+1\gamma}^{\cdot\beta}+\hat{f}_{+1}^{\alpha\gamma}\hat{f}_{-2\gamma}^{\cdot\beta}+\tfrac{1}{2}g^{\alpha\beta}\hat{f}_{+1\gamma\varepsilon}\hat{f}_{-2}^{\gamma\varepsilon}\}c_2(1,2)\mathrm{d}1\,\mathrm{d}2. \quad (120)$$

To summarize the results: the conservation law of energy–momentum for a neutral plasma

$$\partial_\beta T^{\alpha\beta} = 0, \quad (121)$$

with

$$T^{\alpha\beta} = T^{\alpha\beta}_{(\mathrm{f})}+T^{\alpha\beta}_{(\mathrm{m})} \quad (122)$$

has been found. The energy–momentum tensor $T^{\alpha\beta}$ consists of a 'field part' $T^{\alpha\beta}_{(\mathrm{f})}$ (113) and four contributions (108), (116), (118) and (120) to the 'material part' $T^{\alpha\beta}_{(\mathrm{m})}$. The law (121) contains the energy–momentum conservation law and the momentum conservation law as the $\alpha = 0$, and $\alpha = i = 1, 2, 3$ components respectively, as explained at the end of the preceding subsection.

d. *The macroscopic energy–momentum tensor*

The macroscopic energy–momentum tensors (104) and (122) consist of field and material parts, which have been specified in the preceding.

The macroscopic field energy–momentum tensor for a fluid of dipole atoms in an external field is given in (92) as an expression involving the field and polarization tensors $F^{\alpha\beta}$, $H^{\alpha\beta}$ and $M^{\alpha\beta}$. Its components are the field energy density $T^{00}_{(\mathrm{f})}$, the field energy flow $cT^{0i}_{(\mathrm{f})}$, the field momentum density $c^{-1}T^{i0}_{(\mathrm{f})}$ and the field momentum flow $T^{ij}_{(\mathrm{f})}$. An alternative expression for the field energy–momentum tensor may be obtained if we define the four-vectors E^α and B^α in terms of the field tensor $F^{\alpha\beta}$ as:

$$E^\alpha = c^{-1}F^{\alpha\beta}U_\beta, \quad (123)$$

$$B^\alpha = -\tfrac{1}{2}c^{-1}\varepsilon^{\alpha\beta\gamma\zeta}F_{\beta\gamma}U_\zeta, \quad (124)$$

where U^α is the bulk velocity and $\varepsilon^{\alpha\beta\gamma\zeta}$ the Levi-Civita tensor with $\varepsilon^{0123} = -1$. From these definitions and the antisymmetry of $F^{\alpha\beta}$ and $\varepsilon^{\alpha\beta\gamma\zeta}$ follow the orthogonality relations

$$E_\alpha U^\alpha = 0, \qquad B_\alpha U^\alpha = 0. \quad (125)$$

Equations (123) and (124) may be inverted, with the result

$$F^{\alpha\beta} = c^{-1}(U^\alpha E^\beta - U^\beta E^\alpha + \varepsilon^{\alpha\beta\gamma\zeta}U_\gamma B_\zeta). \quad (126)$$

In the local momentary rest frame (denoted by (0)), where $U^\alpha = (c, 0, 0, 0)$ the definitions (123) and (124) reduce to

$$E^{\alpha(0)} = (0, \boldsymbol{E}^{(0)}); \qquad B^{\alpha(0)} = (0, \boldsymbol{B}^{(0)}), \quad (127)$$

so that E^α and B^α are four-vectors of which the space components in the local momentary rest frame are the electric and magnetic field respectively. In the observer's frame, where $U^\alpha = (\gamma c, \gamma c\boldsymbol{\beta})$, the four-vectors E^α and B^α read in three-dimensional notation:

$$E^\alpha = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{E}, \gamma\boldsymbol{E}+\gamma\boldsymbol{\beta}\wedge\boldsymbol{B}), \quad (128)$$

$$B^\alpha = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{B}, \gamma\boldsymbol{B}-\gamma\boldsymbol{\beta}\wedge\boldsymbol{E}). \quad (129)$$

In an analogous way we define D^α and H^α in terms of the excitation tensor $H^{\alpha\beta}$ as:

$$D^\alpha = c^{-1}H^{\alpha\beta}U_\beta, \quad (130)$$

$$H^\alpha = -\tfrac{1}{2}c^{-1}\varepsilon^{\alpha\beta\gamma\zeta}H_{\beta\gamma}U_\zeta, \quad (131)$$

with the properties

$$D_\alpha U^\alpha = 0, \qquad H_\alpha U^\alpha = 0 \quad (132)$$

and the inverse relation

$$H^{\alpha\beta} = c^{-1}(U^\alpha D^\beta - U^\beta D^\alpha + \varepsilon^{\alpha\beta\gamma\zeta}U_\gamma H_\zeta). \quad (133)$$

In the local momentary rest frame we have

$$D^{\alpha(0)} = (0, \boldsymbol{D}^{(0)}); \qquad H^{\alpha(0)} = (0, \boldsymbol{H}^{(0)}), \quad (134)$$

and in the observer's frame:

$$D^\alpha = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{D}, \gamma\boldsymbol{D}+\gamma\boldsymbol{\beta}\wedge\boldsymbol{H}), \tag{135}$$

$$H^\alpha = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{H}, \gamma\boldsymbol{H}-\gamma\boldsymbol{\beta}\wedge\boldsymbol{D}). \tag{136}$$

Finally we introduce P^α and M^α by the definitions involving the macroscopic polarization tensor $M^{\alpha\beta}$:

$$P^\alpha = -c^{-1}M^{\alpha\beta}U_\beta, \tag{137}$$

$$M^\alpha = -\tfrac{1}{2}c^{-1}\varepsilon^{\alpha\beta\gamma\zeta}M_{\beta\gamma}U_\zeta, \tag{138}$$

with the properties

$$P_\alpha U^\alpha = 0, \qquad M_\alpha U^\alpha = 0 \tag{139}$$

and the inverse relation

$$M^{\alpha\beta} = c^{-1}(-U^\alpha P^\beta + U^\beta P^\alpha + \varepsilon^{\alpha\beta\gamma\zeta}U_\gamma M_\zeta). \tag{140}$$

In the local momentary rest frame we have

$$P^{\alpha(0)} = (0, \boldsymbol{P}^{(0)}), \qquad M^{\alpha(0)} = (0, \boldsymbol{M}^{(0)}) \tag{141}$$

and in the observer's frame:

$$P^\alpha = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{P}, \gamma\boldsymbol{P}-\gamma\boldsymbol{\beta}\wedge\boldsymbol{M}), \tag{142}$$

$$M^\alpha = (\gamma\boldsymbol{\beta}\cdot\boldsymbol{M}, \gamma\boldsymbol{M}+\gamma\boldsymbol{\beta}\wedge\boldsymbol{P}). \tag{143}$$

Since $H^{\alpha\beta} = F^{\alpha\beta} - M^{\alpha\beta}$ the four-vectors E^α, B^α, D^α, H^α, P^α and M^α are connected by the identities

$$E^\alpha + P^\alpha = D^\alpha, \qquad B^\alpha - M^\alpha = H^\alpha. \tag{144}$$

If we introduce the expressions (126), (133) and (140) into the macroscopic field energy–momentum tensor (92) we obtain an alternative expression for $T^{\alpha\beta}_{(\mathrm{f})}$:

$$T^{\alpha\beta}_{(\mathrm{f})} = -E^\alpha D^\beta - H^\alpha B^\beta + \Delta^{\alpha\beta}(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - M_\gamma B^\gamma)$$
$$+\tfrac{1}{2}c^{-2}U^\alpha U^\beta(E_\gamma E^\gamma + B_\gamma B^\gamma) - c^{-2}U^\alpha \varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta\varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta. \tag{145}$$

This expression shows that $T^{\alpha\beta}_{(\mathrm{f})}$ is in general asymmetric since $E^\alpha D^\beta + H^\alpha B^\beta$ is asymmetric. If however the medium is isotropic as far as polarization and magnetization are concerned, which means that $\boldsymbol{P}^{(0)} = \kappa\boldsymbol{E}^{(0)}$ and $\boldsymbol{M}^{(0)} = \chi\boldsymbol{B}^{(0)}$ (with susceptibilities κ and χ, which may depend on $\boldsymbol{E}^{(0)2}$ and $\boldsymbol{B}^{(0)2}$), it follows from (127) and (141) that

$$P^\alpha = \kappa E^\alpha, \qquad M^\alpha = \chi B^\alpha. \tag{146}$$

As a consequence of (144) these equalities imply that

$$D^\alpha = \varepsilon E^\alpha, \qquad H^\alpha = \mu^{-1}B^\alpha, \tag{147}$$

with the dielectric constant $\varepsilon = 1+\kappa$ and the (reciprocal) permeability $\mu^{-1} = 1-\chi$. With these relations it follows that $T^{\alpha\beta}_{(\mathrm{f})}$ is symmetric for substances that are isotropic as far as polarization and magnetization are concerned.

In the local momentary rest frame the components of the energy–momentum tensor (145) read in three-dimensional notation (with $i, j = 1, 2, 3$):

$$\begin{pmatrix} T^{00}_{(\mathrm{f})} & T^{0i}_{(\mathrm{f})} \\ T^{i0}_{(\mathrm{f})} & T^{ij}_{(\mathrm{f})} \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2) & (\boldsymbol{E}\wedge\boldsymbol{H})^i \\ (\boldsymbol{E}\wedge\boldsymbol{H})^i & -E^iD^j - H^iB^j + (\tfrac{1}{2}\boldsymbol{E}^2+\tfrac{1}{2}\boldsymbol{B}^2 - \boldsymbol{M}\cdot\boldsymbol{B})g^{ij} \end{pmatrix}, \tag{148}$$

where we omitted the superscript (0) for brevity's sake.

For electric dipole substances ($\boldsymbol{M} = 0$ in the local momentary rest frame) the results (148) for $T^{i0}_{(\mathrm{f})}$ and $T^{ij}_{(\mathrm{f})}$ were already given by Lorentz[1] and by Einstein and Laub[1] on the basis of electron-theoretical arguments. Minkowski's[1] and Abraham's[1] tensors differ essentially from (148); both have for $T^{00}_{(\mathrm{f})}$ and in the bracket of $T^{ij}_{(\mathrm{f})}$ the expression $\tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}+\tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$. Minkowski writes for $T^{i0}_{(\mathrm{f})}$ the vector $(\boldsymbol{D}\wedge\boldsymbol{B})^i$ and Abraham symmetrizes the pressure tensor $T^{ij}_{(\mathrm{f})}$ even for anisotropic substances. (For a discussion and for later literature v. section 7.)

The simple expression (148) is valid only in the local momentary rest frame. The general expression (145) contains the velocity $c\boldsymbol{\beta}$, taken at the observer's point $(ct, \boldsymbol{R})$. Its components read in three-dimensional notation (with $i, j = 1, 2, 3$):

$$T^{00}_{(\mathrm{f})} = \tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{2}\boldsymbol{B}^2 + \boldsymbol{P}\cdot\boldsymbol{E} - \gamma^2\boldsymbol{\beta}\cdot(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E}) - \gamma^4(\boldsymbol{P}-\boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}), \tag{149}$$

$$T^{0i}_{(\mathrm{f})} = (\boldsymbol{E}\wedge\boldsymbol{H})^i - \gamma^2\boldsymbol{\beta}\cdot(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E})\beta^i - \gamma^4(\boldsymbol{P}-\boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B})\beta^i, \tag{150}$$

$$T^{i0}_{(\mathrm{f})} = (\boldsymbol{E}\wedge\boldsymbol{H})^i - \gamma^2\beta^2(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E})^i + \gamma^2\{\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{E}+\boldsymbol{M}\wedge\boldsymbol{B})\}^i$$
$$-\gamma^4(\boldsymbol{P}-\boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B})\beta^i, \tag{151}$$

[1] H. A. Lorentz, Enc. Math. Wiss. V 2, fasc. 1 (Teubner, Leipzig 1904) 245; A. Einstein and J. Laub, Ann. Physik **26**(1908)541; H. Minkowski, Nachr. Ges. Wiss. Göttingen (1908) 53; Math. Ann. **68**(1910)472; M. Abraham, R. C. Circ. Mat. Palermo **28**(1909)1, **30**(1910) 33; Theorie der Elektrizität II (Teubner, Leipzig 1923) 300.

$$\begin{aligned} T^{ij}_{(f)} = & -E^i D^j - H^i B^j + (\tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{2}\boldsymbol{B}^2 - \boldsymbol{M}\cdot\boldsymbol{B})g^{ij} \\ & + \gamma^2 \{\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{E} + \boldsymbol{M}\wedge\boldsymbol{B}) - \boldsymbol{P}\wedge\boldsymbol{B} + \boldsymbol{M}\wedge\boldsymbol{E}\}^i \beta^j \\ & - \gamma^4 (\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B})\beta^i\beta^j, \end{aligned} \quad (152)$$

where $\gamma = (1-\boldsymbol{\beta}^2)^{-\frac{1}{2}}$ and $\boldsymbol{\Omega}^2 = \mathsf{U} - \boldsymbol{\beta\beta}$ (v. (IV.A12)). In the non-relativistic limit one is interested in the quantities $T^{00}_{(f)}$, $cT^{0i}_{(f)}$, $c^{-1}T^{i0}_{(f)}$ and $T^{ij}_{(f)}$ up to order c^{-1}. Using the fact that the magnetization $\boldsymbol{M}$ is of order c^{-1}, one finds then from the above formulae the expressions that occur in the non-relativistic energy and momentum laws (II.109) and (II.118).

For a neutral plasma the macroscopic field energy–momentum tensor (113) has a simple form, as compared to (92), because now no polarization terms enter into the expression. The introduction of electric and magnetic field four-vectors (123) and (124) by means of (126) would be unpractical here, because then the four-velocity, which is absent from the original expression (113), would be artificially introduced. The fact that (113) depends only on the fields, not on the four-velocity, implies that its components

$$\begin{pmatrix} T^{00}_{(f)} & T^{0i}_{(f)} \\ T^{i0}_{(f)} & T^{ij}_{(f)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2) & (\boldsymbol{E}\wedge\boldsymbol{B})^i \\ (\boldsymbol{E}\wedge\boldsymbol{B})^i & -E^iE^j - B^iB^j + \frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)g^{ij} \end{pmatrix} \quad (153)$$

are form-invariant if a different Lorentz frame is chosen as coordinate system.

The remaining part of the total energy–momentum tensor has been called its material part. For a fluid system of dipole atoms it has been specified as a sum of eight contributions, while in the case of a plasma it consists of four terms. In order to get some insight into their structure it is instructive to study the way in which their non-relativistic limit is reached. It then turns out that for the dipole case the contributions (87), (90), (98–100) and the corresponding correlation terms lead (apart from rest energy terms) to the kinetic terms (labelled by K in chapter II), the field-dependent terms (F), the short range terms (S) and the correlation terms (C) of the non-relativistic approximation (cf. problem 8). The other terms of the material energy–momentum tensor give no contributions in the non-relativistic limit. Likewise one finds for neutral plasmas that the contributions (108) and (116–118) lead to the non-relativistic kinetic (K) and correlation terms (C) respectively (v. problem 7).

e. *The ponderomotive force density*

The macroscopic conservation laws of energy and momentum (103) or (121) may be formulated in the form of a balance equation

$$\partial_\beta T^{\alpha\beta}_{(m)} = F^\alpha, \quad (154)$$

where F^α is defined as

$$F^\alpha \equiv -\partial_\beta T^{\alpha\beta}_{(f)}, \quad (155)$$

and is called the ponderomotive force density.

For a fluid system of neutral atoms the ponderomotive force that corresponds – according to (155) – to the field energy–momentum tensor (92) follows with (91):

$$F^\alpha = \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})M_{\beta\gamma} - c^{-2}\partial_\beta\{U^\beta(F^{\alpha\gamma}M_{\gamma\varepsilon}U^\varepsilon - \Delta^{\alpha\gamma}M_{\gamma\varepsilon}F^{\varepsilon\zeta}U_\zeta)\}, \quad (156)$$

where the projector $\Delta^{\alpha\beta}$ was defined as $g^{\alpha\beta} + c^{-2}U^\alpha U^\beta$. If we introduce the operator

$$D \equiv U^\alpha \partial_\alpha \quad (157)$$

and the specific volume

$$v' \equiv (\varrho')^{-1}, \quad (158)$$

we may write (156) in the form:

$$F^\alpha = \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})M_{\beta\gamma} - c^{-2}\varrho' D\{v'(F^{\alpha\beta}M_{\beta\gamma}U^\gamma - \Delta^{\alpha\beta}M_{\beta\gamma}F^{\gamma\varepsilon}U_\varepsilon)\}, \quad (159)$$

where (85) has been used.

If the four-vectors E^α, B^α, $P^\alpha_v \equiv v'P^\alpha$ and $M^\alpha_v \equiv v'M^\alpha$ are introduced with the help of (126) and (140) we obtain an alternative form for the ponderomotive force density:

$$\begin{aligned} F^\alpha = & \varrho'[(\partial^\alpha E_\beta)P^\beta_v + (\partial^\alpha B_\beta)M^\beta_v - c^{-2}\varepsilon^{\alpha\beta\gamma\zeta}D\{(P_{v\beta}B_\gamma - M_{v\beta}E_\gamma)U_\zeta\} \\ & - c^{-2}(\partial^\alpha U^\beta)\varepsilon_{\beta\gamma\zeta\eta}(P^\gamma_v B^\zeta - M^\gamma_v E^\zeta)U^\eta + c^{-2}D(U^\alpha E_\beta P^\beta_v), \end{aligned} \quad (160)$$

where we used (125), (139) and the identity $(\partial^\alpha U^\beta)U_\beta = 0$, which follows from $U_\alpha U^\alpha = -c^2$. Contraction with U^α yields the relation:

$$U_\alpha F^\alpha = -\varrho' E_\alpha DP^\alpha_v + (DB_\alpha)M^\alpha. \quad (161)$$

The components of the ponderomotive force density may be written in three-dimensional notation. From (160) with (128–129) and (142–143), or directly from (159), one finds with $U^\alpha = c\gamma(1, \boldsymbol{\beta})$:

$$\begin{aligned} F^0 = & -(\partial_0 \boldsymbol{E})\cdot\boldsymbol{P} - (\partial_0 \boldsymbol{B})\cdot\boldsymbol{M} + \varrho'\gamma \mathrm{d}_0\{\gamma\boldsymbol{\beta}\cdot(\boldsymbol{P}_v\wedge\boldsymbol{B} - \boldsymbol{M}_v\wedge\boldsymbol{E})\} \\ & + \varrho'\gamma \mathrm{d}_0\{\gamma^3(\boldsymbol{P}_v - \boldsymbol{\beta}\wedge\boldsymbol{M}_v)\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B})\}, \end{aligned} \quad (162)$$

$$\boldsymbol{F} = (\nabla \boldsymbol{E})\cdot\boldsymbol{P} + (\nabla \boldsymbol{B})\cdot\boldsymbol{M} + \varrho'\gamma \mathrm{d}_0\{\gamma(\boldsymbol{P}_v \wedge \boldsymbol{B} - \boldsymbol{M}_v \wedge \boldsymbol{E})\}$$
$$- \varrho'\gamma \mathrm{d}_0\{\gamma\boldsymbol{\beta} \wedge (\boldsymbol{P}_v \wedge \boldsymbol{E} + \boldsymbol{M}_v \wedge \boldsymbol{B})\}$$
$$+ \varrho'\gamma \mathrm{d}_0\{\gamma^3\boldsymbol{\beta}(\boldsymbol{P}_v - \boldsymbol{\beta} \wedge \boldsymbol{M}_v)\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E} + \boldsymbol{\beta} \wedge \boldsymbol{B})\}, \quad (163)$$

where $c\mathrm{d}_0$ is the substantial time derivative $c(\partial_0 + \boldsymbol{\beta}\cdot\nabla)$, $\boldsymbol{\Omega}^2 = \mathbf{U} - \boldsymbol{\beta}\boldsymbol{\beta}$ and $\boldsymbol{P}_v$, $\boldsymbol{M}_v$ are the specific polarizations $v'\boldsymbol{P}$ and $v'\boldsymbol{M}$. These expressions get simple forms in the local momentary rest frame in which the local macroscopic velocity vanishes:

$$F^0 = \varrho'\boldsymbol{E}\cdot\partial_0 \boldsymbol{P}_v - (\partial_0 \boldsymbol{B})\cdot\boldsymbol{M} + 2(\partial_0 \boldsymbol{\beta})\cdot(\boldsymbol{E} \wedge \boldsymbol{M}), \quad (164)$$

$$\boldsymbol{F} = (\nabla \boldsymbol{E})\cdot\boldsymbol{P} + (\nabla \boldsymbol{B})\cdot\boldsymbol{M} + \varrho'\partial_0(\boldsymbol{P}_v \wedge \boldsymbol{B} - \boldsymbol{M}_v \wedge \boldsymbol{E})$$
$$- (\partial_0 \boldsymbol{\beta}) \wedge (\boldsymbol{P} \wedge \boldsymbol{E} + \boldsymbol{M} \wedge \boldsymbol{B}) + (\partial_0 \boldsymbol{\beta})\boldsymbol{E}\cdot\boldsymbol{P}. \quad (165)$$

In these expressions relativistic effects containing the acceleration occur. In the special case that $\boldsymbol{\beta}$ is constant in time and space one finds for the components of the ponderomotive force density in the rest frame:

$$F^0 = \boldsymbol{E}\cdot\partial_0 \boldsymbol{P} - (\partial_0 \boldsymbol{B})\cdot\boldsymbol{M}, \quad (166)$$

$$\boldsymbol{F} = (\nabla \boldsymbol{E})\cdot\boldsymbol{P} + (\nabla \boldsymbol{B})\cdot\boldsymbol{M} + \partial_0(\boldsymbol{P} \wedge \boldsymbol{B} - \boldsymbol{M} \wedge \boldsymbol{E}), \quad (167)$$

where we used the fact that $\partial_0 v'$ vanishes (since it is equal to $v'\nabla\cdot\boldsymbol{\beta}$ as follows from mass conservation in the rest frame).

From the general expressions (162) and (163) one may derive the non-relativistic and semi-relativistic expressions for the ponderomotive force. The latter follow by retaining terms of order c^{-1} in cF^0 and $\boldsymbol{F}$ and considering the polarization and magnetization as being of order c^0. By using mass conservation one finds for cF^0 and $\boldsymbol{F}$ the expressions:

$$cF^0 = \frac{\partial \boldsymbol{P}}{\partial t}\cdot\boldsymbol{E} - \frac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{M} + \nabla\cdot(\boldsymbol{v}\boldsymbol{P}\cdot\boldsymbol{E}) + \frac{2\partial}{c\partial t}\{\boldsymbol{v}\cdot(\boldsymbol{E} \wedge \boldsymbol{M})\} + \frac{2}{c}\nabla\cdot\{\boldsymbol{v}\boldsymbol{v}\cdot(\boldsymbol{E} \wedge \boldsymbol{M})\}, \quad (168)$$

$$\boldsymbol{F} = (\nabla \boldsymbol{E})\cdot\boldsymbol{P} + (\nabla \boldsymbol{B})\cdot\boldsymbol{M} + \frac{\partial}{c\partial t}(\boldsymbol{P} \wedge \boldsymbol{B} - \boldsymbol{M} \wedge \boldsymbol{E}) + \frac{1}{c}\nabla\cdot\{\boldsymbol{v}(\boldsymbol{P} \wedge \boldsymbol{B} - \boldsymbol{M} \wedge \boldsymbol{E})\}. \quad (169)$$

The non-relativistic expressions (II.114) and (II.106) follow from these by considering the magnetization as being of order c^{-1} and again retaining terms of order c^{-1}. The difference between the non-relativistic and the semi-relativistic ponderomotive force densities is that the latter contains the magnetodynamic effect with the vector product of the magnetization $\boldsymbol{M}$ and the electric field $\boldsymbol{E}$.

For plasmas the ponderomotive force density that corresponds – according to (155) – to the field energy–momentum tensor (113) follows with (112) as:

$$F^\alpha = c^{-1}F^{\alpha\beta}J_\beta. \quad (170)$$

This is the Lorentz force density with components:

$$F^0 = c^{-1}\boldsymbol{E}\cdot\boldsymbol{J}, \quad (171)$$

$$\boldsymbol{F} = \varrho^e\boldsymbol{E} + c^{-1}\boldsymbol{J} \wedge \boldsymbol{B}. \quad (172)$$

(The charge density in the proper frame of U^α was supposed to vanish; this need not be the case in other Lorentz frames.)

5 *The conservation of angular momentum*

a. *The balance equation of inner angular momentum*

The inner angular momentum law for a system of neutral atoms will follow by taking the average of the atomic law (IV.178):

$$c\partial_\gamma \left\langle \sum_k \int s_k^{\alpha\beta} u_k^\gamma \delta^{(4)}(X_k - R)\mathrm{d}s_k \right\rangle$$
$$= c\left\langle \sum_k \int \{\Delta_{k\gamma}^{\alpha}\Delta_{k\varepsilon}^{\beta}\mathfrak{d}_k^{\gamma\varepsilon} + c^{-2}(s_k^{\alpha\gamma}u_k^\beta - s_k^{\beta\gamma}u_k^\alpha)\mathfrak{f}_{k\gamma}/m_k\}\delta^{(4)}(X_k - R)\mathrm{d}s_k \right\rangle. \quad (173)$$

The averages in this equation may be written with the help of the covariant distribution functions that have been defined in section 2. The left-hand side becomes according to (15) with (17)

$$\partial_\gamma \int s_1^{\alpha\beta}u_1^\gamma\, \delta^{(4)}(X_1 - R)f_1(1)\mathrm{d}1, \quad (174)$$

where $f_1(1)$ is the one-point distribution function (7). The macroscopic inner angular momentum density is defined as

$$S^{\alpha\beta} \equiv \int s_1^{\alpha\beta}\delta^{(4)}(X_1 - R)f_1(1)\mathrm{d}1. \quad (175)$$

By splitting the atomic velocity u_1^α into the macroscopic velocity U^α defined in (84) and a velocity fluctuation $u_1^\alpha - U^\alpha$ we get for (174)

$$\partial_\gamma(S^{\alpha\beta}U^\gamma + J_1^{\alpha\beta\gamma}), \quad (176)$$

with the abbreviation

$$J_1^{\alpha\beta\gamma} \equiv \int s_1^{\alpha\beta}(u_1^\gamma - U^\gamma)\delta^{(4)}(X_1 - R)f_1(1)\mathrm{d}1. \tag{177}$$

The forces and torques $\mathfrak{f}_k^\alpha$ and $\mathfrak{d}_k^{\alpha\beta}$, which appear at the right-hand side of (173) have been specified in (IV.191–192) as sums of long range and short range contributions. The long range parts are given in (IV.193–194) as the sum of an external field and an interatomic field term. The latter part, being a two-point quantity, is multiplied by a two-point distribution function in (173) if the average is expressed with the help of distribution functions. The two-point distribution function is written in (14) as the sum of an uncorrelated and a correlated part. The sum of the external field part and the uncorrelated part of the long range contribution to the right-hand side follows from (IV.182–183) with (IV.200–203). Taking only dipole contributions and introducing the Maxwell fields we obtain then for this sum of terms

$$\int \Big[\Delta_{1\gamma}^\alpha \Delta_{1\varepsilon}^\beta (F^{\gamma\zeta}\mathfrak{m}_{1\zeta}^{\cdot\varepsilon} - \mathfrak{m}_1^{\gamma\zeta}F_\zeta^{\cdot\varepsilon}) + \frac{1}{m_1 c^2}(s_1^{\alpha\gamma}u_1^\beta - s_1^{\beta\gamma}u_1^\alpha)\Big\{\tfrac{1}{2}(\partial_\gamma F_{\varepsilon\zeta})\mathfrak{m}_1^{\varepsilon\zeta} - c^{-2}\frac{\mathrm{d}}{\mathrm{d}s_1}(F_{\gamma\varepsilon}\mathfrak{m}_1^{\varepsilon\zeta}u_{1\zeta})\Big\}\Big] \delta^{(4)}(X_1 - R)f_1(1)\mathrm{d}1. \tag{178}$$

The terms with the macroscopic velocity in the first part of (178) may be written as (twice) the antisymmetric part

$$T_{(\mathrm{f})}^{\alpha\beta} - T_{(\mathrm{f})}^{\beta\alpha} \tag{179}$$

of the field energy–momentum tensor $T_{(\mathrm{f})}^{\alpha\beta}$ (92). The remaining terms of (178) may likewise be identified with (twice) the antisymmetric part of a term[1] of the total energy–momentum tensor, namely

$$T_{(\mathrm{m})\mathrm{II}}^{\alpha\beta} - T_{(\mathrm{m})\mathrm{II}}^{\beta\alpha}, \tag{180}$$

as follows from inspection of (90).

Now that the uncorrelated long range part of the right-hand side of (173) has been found, we consider its short range contribution. By employing two-point distribution functions according to (19) and inserting the atomic expressions (IV.206–207) one obtains for this contribution

$$c^{-1}\int \hat{\mathfrak{d}}_{1;2}^{S\alpha\beta}\,\delta^{(4)}(X_1 - R)f_2(1.\ 2)\mathrm{d}1\,\mathrm{d}2 + T_{(\mathrm{m})\mathrm{III}}^{\alpha\beta} - T_{(\mathrm{m})\mathrm{III}}^{\beta\alpha}, \tag{181}$$

[1] Note that the kinetic term $T_{(\mathrm{m})\mathrm{I}}^{\alpha\beta}$ of the material energy–momentum tensor does not contribute here, since it is symmetric.

where the tensor $T_{(\mathrm{m})\mathrm{III}}^{\alpha\beta}$ appears, which has been given in (93). To discuss the first term of this expression, we split it into a plus field and a minus field part. The former follows by insertion of the atomic formulae (IV.211) with (IV. 205). If the generalized Irving–Kirkwood expansion (96) is employed one may write the plus field part of the first term of (181) as

$$c^{-1}\int \hat{\mathfrak{d}}_{+1;2}^{S\alpha\beta}(s)\delta^{(4)}(X_1 - R - \tfrac{1}{2}s)\delta^{(4)}(X_2 - R + \tfrac{1}{2}s)f_2(1,2)\mathrm{d}1\,\mathrm{d}2\,\mathrm{d}s - \partial_\gamma J_{\mathrm{II}}^{\alpha\beta\gamma} \tag{182}$$

with the abbreviations

$$\begin{aligned}\hat{\mathfrak{d}}_{+1;2}^{S\alpha\beta}(s) \equiv{}& \frac{c^{-1}}{4\pi}\sum_{i,j} e_{1i}e_{2j}r_{1i}^\alpha(u_{1i}\cdot u_{2j}\,\partial_s^\beta - u_{2j}^\beta\, u_{1i}\cdot\partial_s)\delta\{(s + r_{1i} - r_{2j})^2\} \\ &+ \frac{c}{4\pi}\sum_{n,m=1}^{\infty}(-1)^m\Big\{\Delta_{1\alpha_{n+1}}^\alpha\,\mathfrak{m}_1^{\alpha_1\ldots\alpha_{n+1}}\partial_{s\alpha_{n-1}} - (n-1)\mathfrak{m}_1^{\alpha\alpha_1\ldots\alpha_n}\partial_{s\alpha_{n-1}} \\ &\qquad - c^{-2}(n-1)\Big(\frac{\mathrm{d}}{\mathrm{d}s_1} + u_1\cdot\partial_s\Big)\mathfrak{m}_1^{\alpha\alpha_1\ldots\alpha_n}u_{1\alpha_{n-1}}\Big\} \\ &\Big\{\mathfrak{m}_2^{\beta_1\ldots\beta_{m+1}}\partial_{s\beta_m} - c^{-2}\Big(\frac{\mathrm{d}}{\mathrm{d}s_2} - u_2\cdot\partial_s\Big)\mathfrak{m}_2^{\beta_1\ldots\beta_{m+1}}u_{2\beta_m}\Big\}\,\partial_{s\alpha_1\ldots\alpha_{n-2}\beta_1\ldots\beta_{m-1}} \\ &\qquad (\partial_s^\beta g_{\alpha_n\beta_{m+1}} - g_{\beta_{m+1}}^\beta\,\partial_{s\alpha_n})\delta(s^2) - (\alpha,\beta),\end{aligned} \tag{183}$$

and

$$J_{\mathrm{II}}^{\alpha\beta\gamma} \equiv \tfrac{1}{2}c^{-1}\int s^\gamma\hat{\mathfrak{d}}_{+1;2}^{S\alpha\beta}(s)\delta^{(4)}(X_1 - R - \tfrac{1}{2}s)\delta^{(4)}(X_2 - R + \tfrac{1}{2}s)f_2(1,2)\mathrm{d}1\,\mathrm{d}2\,\mathrm{d}s. \tag{184}$$

The last expression is a contribution to the inner angular momentum flow. If (183) is substituted into the first term of (182) one may distinguish various contributions. In the first place we consider the unexpanded term with ∂_s^β. Making use of the symmetry with respect to an interchange of 1 and 2, one may write it as

$$-\frac{c^{-2}}{8\pi}\int\sum_{i,j}e_{1i}e_{2j}s^\alpha u_{1i}\cdot u_{2j}[\partial_s^\beta\delta\{(s + r_{1i} - r_{2j})^2\}]\delta^{(4)}(X_1 - R - \tfrac{1}{2}s)\,\delta^{(4)}(X_2 - R + \tfrac{1}{2}s)f_2(1,2)\mathrm{d}1\,\mathrm{d}2\,\mathrm{d}s - (\alpha,\beta), \tag{185}$$

where we used moreover the fact that $(s^\alpha + r_{1i}^\alpha - r_{2j}^\alpha)\partial_s^\beta - (\alpha,\beta)$ acting on the delta function gives a vanishing result. Comparing (185) to (98) with (95) one obtains the result that it is equal to the unexpanded part with ∂_s^γ or ∂_s^β of $T_{(\mathrm{m})\mathrm{IV}}^{\alpha\beta} - T_{(\mathrm{m})\mathrm{IV}}^{\beta\alpha}$. Likewise one may derive that the corresponding multipole expanded part of (182) with (183) (i.e. again the part with ∂_s^β) is equal to the antisymmetric part of the multipole expanded terms with ∂_s^α or ∂_s^β in $T^{\alpha\beta}_{(m)IV} - T^{\beta\alpha}_{(m)IV}$ given by (98) with (95). The latter result follows in the simplest way by making use of the identity

$$s^\beta \partial^\alpha_{s\alpha_1\ldots\alpha_n\beta_1\ldots\beta_m}\delta(s^2) - (\alpha,\beta) = -\sum_{i=1}^{n} \partial^\alpha_{s\alpha_1\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_n\beta_1\ldots\beta_m}\delta(s^2)\delta^\beta_{\alpha_i}$$

$$-\sum_{j=1}^{m} \partial^\alpha_{s\alpha_1\ldots\alpha_n\beta_1\ldots\beta_{j-1}\beta_{j+1}\ldots\beta_m}\delta(s^2)\delta^\beta_{\beta_j} - (\alpha,\beta), \quad (186)$$

(which follows from $\partial_\alpha\delta(s^2) = 2s_\alpha\delta'(s^2)$). The two parts of $T^{\alpha\beta}_{(m)IV}$ which we have encountered up to now will be denoted by $T^{\alpha\beta}_{(m)IV'}$ so that we have found for part of the first term of (182):

$$T^{\alpha\beta}_{(m)IV'} - T^{\beta\alpha}_{(m)IV'}. \quad (187)$$

We now consider the remaining parts of (183) (inserted into (182)). They may be transformed (by making use of (186) and of the conservation of probability) in such a way that they become

$$T^{\alpha\beta}_{(m)IV''} + T^{\alpha\beta}_{(m)V} - (\alpha,\beta) - \partial_\gamma J^{\alpha\beta\gamma}_{III}. \quad (188)$$

Here the second part of the fourth together with the fifth part of the energy–momentum tensor appears (v. (98) and (100)). Furthermore we introduced a contribution to the inner angular momentum flow:

$$J^{\alpha\beta\gamma}_{III} \equiv \frac{c^{-2}}{8\pi}\int u_1^\gamma \Big[\sum_{i,j} e_{1i}e_{2j}(r^\alpha_{1i} + \tfrac{1}{2}s^\alpha)u^\beta_{2j}\,\delta\{(s + r_{1i} - r_{2j})^2\}$$

$$+ \sum_{n,m=1}^{\infty} (-1)^n \{ n\,\mathfrak{m}_1^{\alpha\alpha_1\ldots\alpha_n}u_{1\alpha_n} + \tfrac{1}{2}s^\alpha \mathfrak{m}_1^{\alpha_1\ldots\alpha_{n+1}}u_{1\alpha_{n+1}}\partial_{s\alpha_n}\}$$

$$\Big\{ \mathfrak{m}_2^{\beta_1\ldots\beta_m\beta}\partial_{s\beta_m} - c^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}s_2} - u_2\cdot\partial_s\right)\mathfrak{m}_2^{\beta_1\ldots\beta_m\beta}u_{2\beta_m}\Big\}\,\partial_{s\alpha_1\ldots\alpha_{n-1}\beta_1\ldots\beta_{m-1}}\delta(s^2)\Big]$$

$$\delta^{(4)}(X_1 - R - \tfrac{1}{2}s)\delta^{(4)}(X_2 - R + \tfrac{1}{2}s)f_2(1,2)\mathrm{d}1\,\mathrm{d}2\,\mathrm{d}s - (\alpha,\beta). \quad (189)$$

As the plus field part of the first term of (181) we have found now from (182), (187) and (188):

$$T^{\alpha\beta}_{(m)IV} + T^{\alpha\beta}_{(m)V} - (\alpha,\beta) - \partial_\gamma(J^{\alpha\beta\gamma}_{II} + J^{\alpha\beta\gamma}_{III}). \quad (190)$$

Next we consider the minus field contribution to the first term of (181) in which one has to insert (IV.209) with (IV.203). For the unexpanded part (i.e. the part resulting from the minus field part of the first term of (IV.209)) one finds by making use of the atomic field equations (IV.20–23) and of the

conservation of probability:

$$T^{\alpha\beta}_{(m)VI} - (\alpha,\beta) - \partial_\gamma J^{\alpha\beta\gamma}_{IV}, \quad (191)$$

where $T^{\alpha\beta}_{(m)VI}$ (101) appears together with the divergence of a further contribution to the inner angular momentum flow:

$$J^{\alpha\beta\gamma}_{IV} \equiv -c^{-2}\sum_{i,j}\int r^\alpha_{1i}\{\hat{f}^{\beta\varepsilon}_{-2j}(R + r_{1i})\hat{f}^{\cdot\gamma}_{+1i\varepsilon}(R + r_{1i}) + \hat{f}^{\beta\varepsilon}_{+1i}(R + r_{1i})\hat{f}^{\cdot\gamma}_{-2j\varepsilon}(R + r_{1i})$$

$$+ \tfrac{1}{2}g^{\beta\gamma}\hat{f}_{-2j\varepsilon\zeta}(R + r_{1i})\hat{f}^{\varepsilon\zeta}_{+1i}(R + r_{1i})\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2$$

$$+ c^{-2}\sum_{i,j}\frac{e_{1i}}{4\pi}\int r^\alpha_{1i}\hat{f}^{\beta\gamma}_{-2j}(R + r_{1i})u_{1i}\cdot\partial\delta\{(R - X_1)^2\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2 - (\alpha,\beta)$$

$$+ c^{-2}\sum_{i,j}\frac{e_{1i}}{2\pi}\int \hat{f}^{\alpha\beta}_{-2j}(R + r_{1i})\frac{\mathrm{d}r^\gamma_{1i}}{\mathrm{d}s_1}\delta\{(R - X_1)^2\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2. \quad (192)$$

For the multipole expanded part of the minus field contribution to the first term of (181) one obtains along similar lines:

$$T^{\alpha\beta}_{(m)VII} - (\alpha,\beta) - \partial_\gamma J^{\alpha\beta\gamma}_{V} \quad (193)$$

with the material energy–momentum tensor (102) and the inner angular momentum flow

$$J^{\alpha\beta\gamma}_{V} \equiv -\frac{c^{-1}}{4\pi}\sum_{n=1}^{\infty}\int\Big[\Big[\Big\{\Delta^\alpha_{1\alpha_n}\mathfrak{m}_1^{\alpha_1\ldots\alpha_{n-1}\varepsilon\alpha_n}\partial_{\alpha_{n-1}} - (n-1)\mathfrak{m}_1^{\alpha\alpha_1\ldots\alpha_{n-1}\varepsilon}\partial_{\alpha_{n-1}}$$

$$+ c^{-2}(n-1)\left(\frac{\mathrm{d}}{\mathrm{d}s_1} + u_1\cdot\partial\right)\mathfrak{m}_1^{\alpha\alpha_1\ldots\alpha_{n-2}\varepsilon\alpha_n}u_{1\alpha_n}\Big\}\,\partial_{\alpha_1\ldots\alpha_{n-2}}\hat{f}^{\zeta\eta}_{-2(m)}\Big]$$

$$\{\partial^\beta g_{\varepsilon\zeta}g^\gamma_\eta - \partial^\gamma g^\beta_\zeta g_{\varepsilon\eta} - \partial_\varepsilon g^\beta_\zeta g^\gamma_\eta + \partial_\zeta(g^{\beta\gamma}g_{\varepsilon\eta} - g^\beta_\eta g^\gamma_\varepsilon - g^\beta_\varepsilon g^\gamma_\eta)\}$$

$$\delta\{(R - X_1)^2\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2 - (\alpha,\beta)$$

$$- \frac{c^{-1}}{2\pi}\sum_{n=1}^{\infty}\int\Big[\Big[\Big\{\mathfrak{m}_1^{\alpha_1\ldots\alpha_n\gamma}\partial_{\alpha_n} + c^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}s_1} + u_1\cdot\partial\right)\mathfrak{m}_1^{\alpha_1\ldots\alpha_n\gamma}u_{1\alpha_n}$$

$$+ c^{-2}u_1^\gamma\,\mathfrak{m}_1^{\alpha_1\ldots\alpha_{n+1}}u_{1\alpha_{n+1}}\partial_{\alpha_n}\Big\}\,\partial_{\alpha_1\ldots\alpha_{n-1}}\hat{f}^{\alpha\beta}_{-2(m)}\Big]\,\delta\{(R - X_1)^2\}f_2(1,2)\mathrm{d}1\,\mathrm{d}2.$$

(194)

Now we have found the complete short range part of the right-hand side of (173):

$$T^{\alpha\beta}_{(m)III-VII} - (\alpha,\beta) - \partial_\gamma J^{\alpha\beta\gamma}_{II-V}, \quad (195)$$

where roman indices indicate sums of terms. The correlated part of the long range contribution to the right-hand side of (173) may now readily be found, since these terms have the same structure as the multipole expanded part of the short range contribution. By considering separately plus and minus field contributions one finds for systems of which the correlation function has short range character – so that the generalized Kirkwood approximation is valid[1] – as the correlated long range contribution to (173):

$$T^{\alpha\beta}_{(m)\mathrm{VIII}}-(\alpha,\beta)-\partial_\gamma J^{\alpha\beta\gamma}_{\mathrm{VI}}, \tag{196}$$

where the material energy–momentum contribution $T^{\alpha\beta}_{(m)\mathrm{VIII}}$ has been described in section 4*b*, while the inner angular momentum flow $J^{\alpha\beta\gamma}_{\mathrm{VI}}$ consists of three contributions: first a term like (184), but with $\delta^{\mathrm{L}\alpha\beta}_{+1;2}(s)$ instead of $\delta^{\mathrm{S}\alpha\beta}_{+1;2}(s)$ and the correlation function $c_2(1,2)$ instead of the two-point distribution function $f_2(1,2)$; secondly a term like (189), but only the multipole expanded part of it and $f_2(1,2)$ replaced by $-c_2(1,2)$; thirdly a term like (194), again with $-c_2(1,2)$ instead of $f_2(1,2)$.

Collecting the results we have reached the balance equation of inner angular momentum:

$$\partial_\gamma(S^{\alpha\beta}U^\gamma) = -\partial_\gamma J^{\alpha\beta\gamma}+T^{\alpha\beta}-T^{\beta\alpha}, \tag{197}$$

where the inner angular momentum flow consists of six contributions, given above, and where the source term is equal to twice the antisymmetric part of the total energy–momentum tensor.

The inner angular momentum law (197) has the form of a local balance equation, not of a conservation law, since in general the total energy–momentum tensor will not be symmetric. This is what one would expect since the total angular momentum contains an orbital part as well. The balance law for the orbital angular momentum density $R^\alpha T^{\beta\gamma}-R^\beta T^{\alpha\gamma}$ follows directly from the conservation of total energy–momentum $\partial_\beta T^{\gamma\beta}=0$ (cf. (103)):

$$\partial_\gamma(R^\alpha T^{\beta\gamma}-R^\beta T^{\alpha\gamma}) = T^{\beta\alpha}-T^{\alpha\beta}. \tag{198}$$

By taking the sum of (197) and (198) one obtains:

$$\partial_\gamma(R^\alpha T^{\beta\gamma}-R^\beta T^{\alpha\gamma}+S^{\alpha\beta}U^\gamma+J^{\alpha\beta\gamma}) = 0, \tag{199}$$

which is the law of conservation of total angular momentum.

From the local laws (197–199) one may obtain global laws by integrating over three-space and using Gauss's theorem.

If one studies the non-relativistic limit of the inner angular momentum equation (197) one recovers indeed the equation (II.196) of the non-relativistic theory. In particular one finds for the non-relativistic limits of the space–space components of $S^{\alpha\beta}$ (175) the expression (II.166) of the non-relativistic treatment (v. problem 10). Furthermore the space–space–space components of $J^{\alpha\beta\gamma}_{\mathrm{I}}$ (177) reduce to $\mathbf{J}^{\mathrm{K}}_s$ given by (II.169) (in fact J^{ijk}_{I} reduces to $\varepsilon^{ijm}J^{\mathrm{K}k}_{s\cdot m}$ with ε^{ijk} the Levi-Civita tensor) while the space–space–time component of $J^{\alpha\beta\gamma}_{\mathrm{I}}$ gives no contribution to the non-relativistic inner angular momentum law. Similarly one finds that $J^{\alpha\beta\gamma}_{\mathrm{II}}$ (184) and $J^{\alpha\beta\gamma}_{\mathrm{VI}}$ reduce to $\mathbf{J}^{\mathrm{S}}_s$ (II.180) and $\mathbf{J}^{\mathrm{C}}_s$ (II.183) respectively. The other parts of $J^{\alpha\beta\gamma}$ give no contribution in the non-relativistic limit (cf. problem 12).

[1] We note again that the extension to systems with long range correlation presents no difficulties.

For plasmas, where the internal structure of the particles is disregarded, the angular momentum laws reduce to simple forms since no inner angular momentum exists. Correspondingly (twice) the antisymmetric part $T^{\alpha\beta}-T^{\beta\alpha}$ of the total energy–momentum tensor may be written as $\partial_\gamma J^{\alpha\beta\gamma}$, as follows from inspection of its various terms (108), (113), (116), (118) and (120). In fact, only the part (116) is asymmetric; its antisymmetric part may be written as a divergence by making use of the conservation of probability. Therefore one finds analogously to (197) for a plasma the equation

$$T^{\alpha\beta}-T^{\beta\alpha} = \partial_\gamma J^{\alpha\beta\gamma}. \tag{200}$$

In spite of its resemblance to the inner angular momentum law (197) this equation has a different character; it is in fact only an identity valid for the antisymmetric part of the energy–momentum tensor[1]. Combining the identity (200) with the energy–momentum law (121) one finds for the angular momentum the conservation law

$$\partial_\gamma(R^\alpha T^{\beta\gamma}-R^\beta T^{\alpha\gamma}+J^{\alpha\beta\gamma}) = 0. \tag{201}$$

In the non-relativistic limit both the left-hand and the right-hand side of (200) tend separately to zero (v. problem 11).

b. *The ponderomotive torque density*

The conservation law of total angular momentum (199) contains the total energy–momentum tensor $T^{\alpha\beta}$, which consists of two parts that we have called the 'field' and the 'material' energy–momentum tensors $T^{\alpha\beta}_{(f)}$ and $T^{\alpha\beta}_{(m)}$.

[1] It is possible to symmetrize the energy–momentum tensor by adding a divergenceless part (v. problem 13). To preserve the analogy one would then also have to change in a corresponding way the expressions for the dipole case. Such a change is feasible, but it leads to lengthy expressions. Moreover for the long range correlation case the non-relativistic limit of the symmetrized tensor would not have the same form as that of chapter II.

An alternative form of the conservation law is thus

$$\partial_\gamma(R^\alpha T^{\beta\gamma}_{(m)} - R^\beta T^{\alpha\gamma}_{(m)} + S^{\alpha\beta}U^\gamma + J^{\alpha\beta\gamma}) = R^\alpha F^\beta - R^\beta F^\alpha + T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)}, \quad (202)$$

where we introduced the ponderomotive force density defined in (155) and given explicitly in (156) or (159). This formula shows at the right-hand side in the first place the torque exerted by the ponderomotive force density and in the second place a 'ponderomotive torque density'

$$D^{\alpha\beta} \equiv T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)}. \quad (203)$$

Its explicit forms follow from the field energy–momentum tensor (92):

$$D^{\alpha\beta} = \Delta^\alpha_\varepsilon \Delta^\beta_\zeta (M^{\varepsilon\gamma}F^\zeta_{\cdot\gamma} - F^{\varepsilon\gamma}M^\zeta_{\cdot\gamma}) \quad (204)$$

or, written in terms of the field and polarization four-vectors (123–124) and (137–138)

$$D^{\alpha\beta} = P^\alpha E^\beta - P^\beta E^\alpha + M^\alpha B^\beta - M^\beta B^\alpha. \quad (205)$$

Its components read in three-dimensional notation (where $U^\alpha = c\gamma(1, \boldsymbol{\beta})$):

$$D^{0i} = [-\gamma^2\boldsymbol{\beta}\wedge\{\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{B}-\boldsymbol{M}\wedge\boldsymbol{E})\} - \gamma^2\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{E}+\boldsymbol{M}\wedge\boldsymbol{B})]^i, \quad (206)$$

$$D^{ij} = \{\gamma^2\boldsymbol{\Omega}^2\cdot(\boldsymbol{P}\wedge\boldsymbol{E}+\boldsymbol{M}\wedge\boldsymbol{B}) + \gamma^2\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{B}-\boldsymbol{M}\wedge\boldsymbol{E})\}^k, \quad (207)$$

where $i, j, k = 1, 2, 3$ (cycl.) and $\boldsymbol{\Omega}^2 = \mathbf{U} - \boldsymbol{\beta\beta}$. In the local momentary rest frame (in which the local macroscopic velocity vanishes) these expressions reduce to

$$D^{0i} = 0, \quad (208)$$

$$D^{ij} = (\boldsymbol{P}\wedge\boldsymbol{E}+\boldsymbol{M}\wedge\boldsymbol{B})^k. \quad (209)$$

For substances which are isotropic as far as the electric and magnetic polarizations are concerned the torque density (209) vanishes.

From the general formula (207) one finds for the ponderomotive torque density in semi-relativistic approximation

$$D^{ij} = \{\boldsymbol{P}\wedge\boldsymbol{E}+\boldsymbol{M}\wedge\boldsymbol{B}+\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{B}-\boldsymbol{M}\wedge\boldsymbol{E})\}^k, \quad (210)$$

with $i, j, k = 1, 2, 3$ (cycl.). The non-relativistic limit follows by taking into account that $\boldsymbol{M}$ is of order c^{-1}. Then one recovers (II.189).

6 *Relativistic thermodynamics of polarized fluids and plasmas*

a. *The first law*

In section 4 the macroscopic conservation laws of energy and momentum in a polarized medium have been derived from the atomic conservation laws. For systems with a correlation length which is small compared to macroscopic dimensions the conservation laws read:

$$\partial_\beta(T^{\alpha\beta}_{(f)} + T^{\alpha\beta}_{(m)}) = 0, \qquad (\alpha = 0, 1, 2, 3), \quad (211)$$

with a field energy–momentum tensor $T^{\alpha\beta}_{(f)}$ and a material energy–momentum tensor $T^{\alpha\beta}_{(m)}$. In terms of this material tensor $T^{\alpha\beta}_{(m)}$ we now define a scalar energy density u'_v, a heat flow four-vector J^α_q, a momentum density four-vector I^α and a pressure four-tensor $P^{\alpha\beta}$ as:

$$u'_v = c^{-2}U_\alpha U_\beta T^{\alpha\beta}_{(m)} - \varrho' c^2, \quad (212)$$

$$J^\alpha_q = -U_\beta T^{\beta\gamma}_{(m)} \Delta^\alpha_\gamma, \quad (213)$$

$$I^\alpha = -c^{-2}\Delta^\alpha_\beta T^{\beta\gamma}_{(m)} U_\gamma, \quad (214)$$

$$P^{\alpha\beta} = \Delta^\alpha_\gamma \Delta^\beta_\varepsilon T^{\varepsilon\gamma}_{(m)}, \quad (215)$$

where U^α and ϱ' are the bulk four-velocity and bulk rest mass density defined in (84), while Δ^α_β stands for $\delta^\alpha_\beta + c^{-2}U^\alpha U_\beta$. From (213–215) the orthogonality relations

$$J^\alpha_q U_\alpha = 0, \qquad I^\alpha U_\alpha = 0, \qquad P^{\alpha\beta}U_\alpha = 0, \qquad P^{\alpha\beta}U_\beta = 0 \quad (216)$$

follow. In the local momentary rest frame, in which U^α has the components $(c, 0, 0, 0)$, the four-vectors I^α, J^α_q and the four-tensor $P^{\alpha\beta}$ are hence purely space-like. In this frame the components of $T^{\alpha\beta}_{(m)}$ read:

$$T^{00(0)}_{(m)} = u'_v + \varrho' c^2, \quad (217)$$

$$T^{0i(0)}_{(m)} = c^{-1}J^{i(0)}_q, \quad (218)$$

$$T^{i0(0)}_{(m)} = cI^{i(0)}, \quad (219)$$

$$T^{ij(0)}_{(m)} = P^{ji(0)}, \quad (220)$$

(with $i, j = 1, 2, 3$), as follows from (212–215). In the $(ct, \boldsymbol{R})$-frame the expression for $T^{\alpha\beta}_{(m)}$ in terms of u'_v, J^α_q, I^α and $P^{\alpha\beta}$ is:

$$T^{\alpha\beta}_{(m)} = c^{-2}(\varrho' c^2 + u'_v)U^\alpha U^\beta + c^{-2}U^\alpha J^\beta_q + I^\alpha U^\beta + P^{\beta\alpha}. \quad (221)$$

If the conservation law (211) is multiplied by U_α and (221) is introduced we get:

$$U_\alpha \partial_\beta \{T^{\alpha\beta}_{(f)} + c^{-2}(\varrho' c^2 + u'_v) U^\alpha U^\beta + c^{-2} U^\alpha J^\beta_q + I^\alpha U^\beta + P^{\beta\alpha}\} = 0. \quad (222)$$

The first term is equal to $-U_\alpha F^\alpha$ as follows from the definition (155). An explicit expression for this term has been given in (161):

$$-U_\alpha F^\alpha = \varrho' E_\alpha D(v' P^\alpha) - (DB_\alpha) M^\alpha, \quad (223)$$

where E^α and B^α are the field four-vectors defined in (123) and (124), while the polarization four-vectors P^α and M^α have been defined in (137) and (138). The symbol D stands for the operator $U^\alpha \partial_\alpha$. The remaining terms of (222) may be put into the form:

$$-\varrho' D(v' u'_v) - \partial_\alpha J^\alpha_q - I_\alpha D U^\alpha - P_{\alpha\beta} \partial^\alpha U^\beta, \quad (224)$$

where we used (85) and the orthogonality properties (216). Introducing the energy per unit rest mass

$$u' = v' u'_v \quad (225)$$

and inserting (223) and (224) we get from (222):

$$\varrho' Du' = -\partial_\alpha J^\alpha_q - I_\alpha D U^\alpha - P_{\alpha\beta} \partial^\alpha U^\beta + \varrho' E_\alpha D(v' P^\alpha) - (DB_\alpha) M^\alpha. \quad (226)$$

This is the first law of relativistic thermodynamics for polarized media; it gives an expression for the change in time of the energy u'. The right-hand side contains in the first place the divergence of the heat flow J^α_q together with Eckart's relativistic correction[1] and a term with the pressure tensor $P_{\alpha\beta}$. Furthermore terms with the electromagnetic fields E^α, B^α and the polarizations P^α, M^α occur.

Likewise one may derive the first law for a relativistic neutral plasma. One obtains

$$\varrho' Du' = -\partial_\alpha J^\alpha_q - I_\alpha D U^\alpha - P_{\alpha\beta} \partial^\alpha U^\beta + J_\alpha E^\alpha, \quad (227)$$

with J^α the electric four-current density.

b. *The second law*

In chapter II the non-relativistic Gibbs relation has been derived from equilibrium statistical thermodynamics with the help of a canonical ensemble. Since no statistical derivation of a second law for relativistic systems (with

[1] C. Eckart, Phys. Rev. **58**(1940)919.

interactions) in equilibrium is available, we postulate in analogy with the non-relativistic law:

$$T' Ds' = Du' + p' Dv' - E_\alpha D(v' P^\alpha) + v' M_\alpha DB^\alpha \quad (228)$$

as the relativistic second law of thermodynamics for a dipole fluid of neutral atoms in local equilibrium. Here T', s', u', p', v' are the temperature, specific entropy, specific energy, scalar equilibrium pressure and specific volume in the permanent local rest frame (denoted by a prime). (This frame in which matter is locally at rest all the time is a succession of Lorentz frames, not a Lorentz frame itself.) Furthermore E^α, B^α and P^α, M^α are the field and polarization four-vectors defined in (123), (124), (137) and (138). The derivative D stands for $U^\alpha \partial_\alpha$, where U^α is the local bulk four-velocity. The quantities u' and p' are connected with the energy–momentum tensor $T^{\alpha\beta}_{(m)}$. The expression for the specific energy u' follows from (225) with (212). The scalar pressure p' will be connected to the pressure four-tensor $P^{\alpha\beta}$ (215). In fact the space–space part of the material energy–momentum tensor $T^{\alpha\beta}_{(m)}$ in the rest frame reduces in the non-relativistic limit to the pressure tensor **P** (v. problem 8), which is a scalar quantity p for a fluid in equilibrium (II, section 7*b*). Taking over this property in the present theory one finds that for a fluid in equilibrium $T^{\alpha\beta}_{(m)}$ is a scalar p' in the local momentary rest frame, as far as its space–space components are concerned. In the observer's frame we may express this, with (215), as:

$$p' \Delta^{\alpha\beta} = (\Delta^\alpha_\gamma \Delta^\beta_\varepsilon T^{\varepsilon\gamma}_{(m)})_{eq} \equiv P^{\alpha\beta}_{eq}. \quad (229)$$

From the combination of the first and second law the relativistic entropy balance may be obtained. In fact substitution of (226) in the right-hand side of (228) leads to the entropy balance equation for a polarized fluid of neutral atoms:

$$\varrho' Ds' = -\partial_\alpha S^\alpha + \sigma, \quad (230)$$

where we introduced the entropy flux:

$$S^\alpha = \frac{1}{T'} J^\alpha_q \quad (231)$$

and the entropy source strength σ given by:

$$T'\sigma = -\frac{1}{T'} J_{q\alpha} \partial^\alpha T' - I_\alpha D U^\alpha - (P_{\alpha\beta} - p' \Delta_{\alpha\beta}) \partial^\alpha U^\beta + \varrho' (E_\alpha - E_{eq.\alpha}) D(v' P^\alpha) - (M_\alpha - M_{eq.\alpha}) DB^\alpha. \quad (232)$$

Here we used $\partial_\alpha U^\alpha = \varrho' Dv'$, which is a consequence of the rest mass conservation (85). Furthermore we took into account that in the second law (228) the equilibrium values $E_{\mathrm{eq},\alpha}$ and $M_{\mathrm{eq},\alpha}$ are to be read for E_α and M_α.

The entropy flux (231) is equal to the heat flow divided by the local temperature; the entropy source strength (232) contains contributions due to heat conduction, viscous phenomena and electric and magnetic relaxation.

For neutral plasmas we write in analogy with the non-relativistic theory the second law in the form

$$T'Ds' = Du' + p'Dv' \tag{233}$$

with the same connexions between u' and p' and the energy–momentum tensor $T^{\alpha\beta}_{(\mathrm{m})}$ as given above. For the entropy balance one finds the same form as (230) with the entropy flux (231) and the entropy source strength

$$T'\sigma = -\frac{1}{T'} J_{q\alpha}\partial^\alpha T' - I_\alpha DU^\alpha - (P_{\alpha\beta} - p'\Delta_{\alpha\beta})\partial^\alpha U^\beta + J_\alpha E^\alpha, \tag{234}$$

where an extra term that represents the effect of Joule heat production appears.

c. *The free energy for systems with linear constitutive relations*

In this and the following sections some consequences of the relativistic first and second law of thermodynamics will be discussed, especially in connexion with the conservation laws of energy and momentum. The treatment will to some extent be similar to that of the non-relativistic theory of chapter II. § 8*a*. We shall confine ourselves to a polarized fluid of neutral atoms with linear constitutive relations of the form:

$$\begin{aligned} P^\alpha &= \kappa(v', T')E^\alpha, \\ M^\alpha &= \chi(v', T')B^\alpha. \end{aligned} \tag{235}$$

In three-dimensional notation and in the permanent local rest frame these relations read:

$$\begin{aligned} \boldsymbol{P}' &= \kappa(v', T')\boldsymbol{E}', \\ \boldsymbol{M}' &= \chi(v', T')\boldsymbol{B}'. \end{aligned} \tag{236}$$

From the Gibbs relation (228) we may derive an expression for the time derivative of the specific free energy

$$f' \equiv u' - T's'; \tag{237}$$

we obtain:

$$Df' = -p'Dv' - s'DT' + E_\alpha D(v'P^\alpha) - v'M_\alpha DB^\alpha. \tag{238}$$

This relation may be integrated at constant v' and T' with the result:

$$f' = f'_0 + v'\int_{\mathrm{const.}\,v',T'} (E_\alpha \mathrm{d}P^\alpha - M_\alpha \mathrm{d}B^\alpha), \tag{239}$$

where f'_0 is the specific free energy for the same specific volume v' and temperature T' but at zero fields and polarizations. If the constitutive relations (235) are inserted, the integral in (239) may be carried out with the result

$$f' = f'_0 + \tfrac{1}{2}v'\kappa^{-1}P_\alpha P^\alpha - \tfrac{1}{2}v'\chi B_\alpha B^\alpha; \tag{240}$$

if κ and χ are eliminated an alternative form is obtained:

$$f' = f'_0 + \tfrac{1}{2}v'E_\alpha P^\alpha - \tfrac{1}{2}v'B_\alpha M^\alpha. \tag{241}$$

The scalar equilibrium pressure follows from the specific free energy by differentiation with respect to the specific volume v' at constant T', specific polarization $v'P^\alpha$ and field B^α, as (228) shows. Hence the pressure $p' = -\partial f'/\partial v'$ is connected with the pressure $p'_0 = -\partial f'_0/\partial v'$ for the same values of v' and T', but with switched-off fields, by a relation following from (240):

$$p' = p'_0 + \tfrac{1}{2}\kappa^{-1}P_\alpha P^\alpha + \tfrac{1}{2}\chi B_\alpha B^\alpha + \frac{v'}{2\kappa^2}\frac{\partial\kappa}{\partial v'}P_\alpha P^\alpha + \tfrac{1}{2}v'\frac{\partial\chi}{\partial v'}B_\alpha B^\alpha, \tag{242}$$

or with (235):

$$p' = p'_0 + \tfrac{1}{2}E_\alpha P^\alpha + \tfrac{1}{2}B_\alpha M^\alpha + \tfrac{1}{2}v'\frac{\partial\kappa}{\partial v'}E_\alpha E^\alpha + \tfrac{1}{2}v'\frac{\partial\chi}{\partial v'}B_\alpha B^\alpha. \tag{243}$$

The specific entropy follows from the specific free energy by differentiation with respect to temperature T' at constant v', $v'P^\alpha$ and B^α. From (240) we obtain:

$$s' = s'_0 + \frac{v'}{2\kappa^2}\frac{\partial\kappa}{\partial T'}P_\alpha P^\alpha + \tfrac{1}{2}v'\frac{\partial\chi}{\partial T'}B_\alpha B^\alpha, \tag{244}$$

where $s' = -\partial f'/\partial T'$ and $s'_0 = -\partial f'_0/\partial T'$. Introducing E^α and B^α with (235) we may write this expression as:

$$s' = s'_0 + \tfrac{1}{2}v'\frac{\partial\kappa}{\partial T'}E_\alpha E^\alpha + \tfrac{1}{2}v'\frac{\partial\chi}{\partial T'}B_\alpha B^\alpha. \tag{245}$$

An expression for the specific energy follows from (237) with (241) and (245):

$$u' = u'_0 + \tfrac{1}{2}v'E_\alpha P^\alpha - \tfrac{1}{2}v'B_\alpha M^\alpha + \tfrac{1}{2}v'T'\frac{\partial\kappa}{\partial T'}E_\alpha E^\alpha + \tfrac{1}{2}v'T'\frac{\partial\chi}{\partial T'}B_\alpha B^\alpha, \qquad (246)$$

where $u'_0 = f'_0 + T's'_0$ (cf. (237)). The energy density $u'_v = (v')^{-1}u'$ as compared with the energy density $u'_{v0} = (v')^{-1}u'_0$ at zero fields reads:

$$u'_v = u'_{v0} + \tfrac{1}{2}E_\alpha P^\alpha - \tfrac{1}{2}B_\alpha M^\alpha + \tfrac{1}{2}T'\frac{\partial\kappa}{\partial T'}E_\alpha E^\alpha + \tfrac{1}{2}T'\frac{\partial\chi}{\partial T'}B_\alpha B^\alpha. \qquad (247)$$

We have obtained now the expressions (243) and (247) for the equilibrium pressure p' and the energy density u'_v. The method which is employed here to derive these results is analogous to that given for the non-relativistic case. The formulae (243) and (247) will be used in the next subsection for a discussion of the material energy–momentum tensor.

d. *The energy–momentum tensor for a polarized fluid at local equilibrium*

The material energy–momentum tensor for a polarized fluid of neutral atoms has the general form (221). In view of the expression (232) for the entropy production, we shall suppose that in local equilibrium all thermodynamic flows J^α_q, I^α, $P^{\gamma\beta} - p'\Delta^{\alpha\beta}$, $E^\alpha - E^\alpha_{\text{eq}}$ and $M^\alpha - M^\alpha_{\text{eq}}$ vanish. Then (221) reduces to

$$T^{\alpha\beta}_{(\text{m})\text{eq}} = c^{-2}(\varrho' c^2 + u'_v)U^\alpha U^\beta + p'\Delta^{\alpha\beta}, \qquad (248)$$

which is the energy–momentum tensor for a perfect fluid. The total energy–momentum tensor for a polarized fluid in local equilibrium is the sum of the field part (145) and the material part just given:

$$\begin{aligned} T^{\alpha\beta}_{\text{eq}} = & -E^\alpha D^\beta - H^\alpha B^\beta + \Delta^{\alpha\beta}(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma + p') \\ & + c^{-2}U^\alpha U^\beta(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma + \varrho' c^2 + u'_v) \\ & - c^{-2}U^\alpha \varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta \varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta . \qquad (249)\end{aligned}$$

If the fields are switched off, while the temperature T' and density ϱ' are kept constant, the energy–momentum tensor becomes:

$$T^{\alpha\beta}_{\text{eq},0} = c^{-2}(\varrho' c^2 + u'_{v0})U^\alpha U^\beta + p'_0\Delta^{\alpha\beta}. \qquad (250)$$

With the expressions (243) and (247) for p' and u'_v, derived for a fluid with linear constitutive relations, the difference $T^{\alpha\beta}_{\text{eq}} - T^{\alpha\beta}_{\text{eq},0}$ between the energy–momentum tensor in the presence and in the absence of electromagnetic fields may be obtained. This tensor contains the complete effect of the switching-on of the fields. In view of this property it may be considered as the field part of the energy–momentum tensor; it will be called $T^{\alpha\beta}_{[\text{f}]}$ to distinguish it from $T^{\alpha\beta}_{(\text{f})}$, that has been introduced earlier. The corresponding material energy–momentum tensor $T^{\alpha\beta}_{[\text{m}]}$ is then $T^{\alpha\beta}_{\text{eq},0}$. Hence for a polarized fluid of neutral atoms in local equilibrium we have:

$$T^{\alpha\beta} = T^{\alpha\beta}_{[\text{f}]} + T^{\alpha\beta}_{[\text{m}]}, \qquad (251)$$

$$T^{\alpha\beta}_{[\text{m}]} \equiv T^{\alpha\beta}_{\text{eq},0}, \qquad T^{\alpha\beta}_{[\text{f}]} \equiv T^{\alpha\beta}_{\text{eq}} - T^{\alpha\beta}_{\text{eq},0}. \qquad (252)$$

We have introduced thus a second way of splitting the total energy–momentum tensor $T^{\alpha\beta}$ into a 'field' and a 'material' part. The difference between this splitting and that of (104) is that here we confine ourselves to (equilibrium) systems with linear constitutive relations. For such systems it is possible to specify the effect of the turning on of the electromagnetic fields. The splitting (104) was valid under more general circumstances, but did not permit the kind of disentangling, achieved with the present splitting of $T^{\alpha\beta}$. We may note here that a similar situation arose already in the non-relativistic theory in connexion with the material pressure tensor which has been defined there in two different ways (v. chapter II, section 8*a*).

From (251) and (252) with (243), (247), (249) and (250) we obtain explicit expressions for $T^{\alpha\beta}_{[\text{f}]}$ and $T^{\alpha\beta}_{[\text{m}]}$:

$$\begin{aligned} T^{\alpha\beta}_{[\text{f}]} = & -E^\alpha D^\beta - H^\alpha B^\beta + \tfrac{1}{2}\Delta^{\alpha\beta}\left(E_\gamma D^\gamma + B_\gamma H^\gamma + v'\frac{\partial\kappa}{\partial v'}E_\gamma E^\gamma + v'\frac{\partial\chi}{\partial v'}B_\gamma B^\gamma\right) \\ & + \tfrac{1}{2}c^{-2}U^\alpha U^\beta\left(E_\gamma D^\gamma + B_\gamma H^\gamma + T'\frac{\partial\kappa}{\partial T'}E_\gamma E^\gamma + T'\frac{\partial\chi}{\partial T'}B_\gamma B^\gamma\right) \\ & - c^{-2}U^\alpha \varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta \varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta, \qquad (253)\end{aligned}$$

$$T^{\alpha\beta}_{[\text{m}]} = c^{-2}(\varrho' c^2 + u'_{v0})U^\alpha U^\beta + p'_0\Delta^{\alpha\beta}, \qquad (254)$$

where (144) has been used.

Often the magnetic susceptibility $\tilde{\chi}$ is defined by

$$M^\alpha = \tilde{\chi}(v', T')H^\alpha. \qquad (255)$$

It is connected to χ, defined in (235), by the relation:

$$\tilde{\chi} = \chi/(1-\chi), \qquad (256)$$

as follows from (144). From (235), (255) and (256) one proves

$$B_\alpha B^\alpha\frac{\partial\chi}{\partial v'} = H_\alpha H^\alpha\frac{\partial\tilde{\chi}}{\partial v'}; \qquad B_\alpha B^\alpha\frac{\partial\chi}{\partial T'} = H_\alpha H^\alpha\frac{\partial\tilde{\chi}}{\partial T'}. \qquad (257)$$

Hence the field energy–momentum tensor (253) may be written alternatively as:

$$T^{\alpha\beta}_{[f]} = -E^\alpha D^\beta - H^\alpha B^\beta + \tfrac{1}{2}\Delta^{\alpha\beta}\left(E_\gamma D^\gamma + B_\gamma H^\gamma + v'\frac{\partial\kappa}{\partial v'}E_\gamma E^\gamma + v'\frac{\partial\tilde{\chi}}{\partial v'}H_\gamma H^\gamma\right)$$
$$+\tfrac{1}{2}c^{-2}U^\alpha U^\beta\left(E_\gamma D^\gamma + B_\gamma H^\gamma + T'\frac{\partial\kappa}{\partial T'}E_\gamma E^\gamma + T'\frac{\partial\tilde{\chi}}{\partial T'}H_\gamma H^\gamma\right)$$
$$-c^{-2}U^\alpha\varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta\varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta. \quad (258)$$

This expression shows the symmetry of $T^{\alpha\beta}_{[f]}$ with respect to electric and magnetic phenomena: the equations (144), the first of (235), (255) and (258) are invariant under the transformations:

$$\begin{aligned} &E^\alpha \to H^\alpha; \quad P^\alpha \to M^\alpha; \quad D^\alpha \to B^\alpha; \quad \kappa \to \tilde{\chi};\\ &H^\alpha \to -E^\alpha; \quad M^\alpha \to -P^\alpha; \quad B^\alpha \to -D^\alpha; \quad \tilde{\chi} \to \kappa. \end{aligned} \quad (259)$$

(The Maxwell equations (68) without sources ($J^\alpha = 0$) are also invariant with respect to these transformations as may be proved with the help of (126) and (133).)

In the local permanent rest frame the tensors (253) and (254) take the form:

$$T'^{\alpha\beta}_{[f]} = \begin{bmatrix} \frac{1}{2}\left(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}' + T'\frac{\partial\kappa}{\partial T'}\boldsymbol{E}'^2 + T'\frac{\partial\chi}{\partial T'}\boldsymbol{B}'^2\right) & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + \frac{1}{2}\left(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}' + v'\frac{\partial\kappa}{\partial v'}\boldsymbol{E}'^2 + v'\frac{\partial\chi}{\partial v'}\boldsymbol{B}'^2\right)\mathbf{U} \end{bmatrix}, \quad (260)$$

$$T'^{\alpha\beta}_{[m]} = \begin{bmatrix} \varrho' c^2 + u'_{v0} & 0 \\ 0 & p'_0\,\mathbf{U} \end{bmatrix}. \quad (261)$$

In terms of the susceptibility $\tilde{\chi}$ the field tensor reads:

$$T'^{\alpha\beta}_{[f]} = \begin{bmatrix} \frac{1}{2}\left(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}' + T'\frac{\partial\kappa}{\partial T'}\boldsymbol{E}'^2 + T'\frac{\partial\tilde{\chi}}{\partial T'}\boldsymbol{H}'^2\right) & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + \frac{1}{2}\left(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}' + v'\frac{\partial\kappa}{\partial v'}\boldsymbol{E}'^2 + v'\frac{\partial\tilde{\chi}}{\partial v'}\boldsymbol{H}'^2\right)\mathbf{U} \end{bmatrix}. \quad (262)$$

The expression (253) for the field energy–momentum tensor gets a simpler form if the properties of the dipole fluid are further specified. This will be done in the next subsection.

e. *Induced dipole and permanent dipole substances*

In the expressions (243), (247) and (253) for p', u'_v and $T^{\alpha\beta}_{[f]}$ derivatives of the susceptibilities κ and χ with respect to specific volume v' and temperature T' occur. These derivatives may be expressed in terms of κ and χ themselves if more is known about the properties of the dipole fluid (cf. chapter II, section 8*a* of the non-relativistic treatment).

Let us consider first *induced dipole* substances that satisfy Clausius–Mossotti laws of the type:

$$\frac{\kappa}{\kappa+3} \sim \frac{1}{v'}, \qquad \frac{\chi}{3-2\chi} \sim \frac{1}{v'}, \quad (263)$$

while κ and χ are independent of the temperature T'. With the help of these laws the partial derivatives of the susceptibilities may be evaluated; the expressions for p' and u'_v become:

$$p' = p'_0 - \tfrac{1}{6}P_\alpha P^\alpha + \tfrac{1}{3}M_\alpha M^\alpha, \quad (264)$$

$$u'_v = u'_{v0} + \tfrac{1}{2}E_\alpha P^\alpha - \tfrac{1}{2}B_\alpha M^\alpha. \quad (265)$$

The energy–momentum tensor $T^{\alpha\beta}_{[f]}$ gets the form:

$$T^{\alpha\beta}_{[f]} = -E^\alpha D^\beta - H^\alpha B^\beta + \Delta^{\alpha\beta}(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma - \tfrac{1}{6}P_\gamma P^\gamma + \tfrac{1}{3}M_\gamma M^\gamma)$$
$$+\tfrac{1}{2}c^{-2}U^\alpha U^\beta(E_\gamma D^\gamma + B_\gamma H^\gamma) - c^{-2}U^\alpha\varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta\varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta. \quad (266)$$

(The combination $\frac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma + \frac{1}{3}M_\gamma M^\gamma$ may be written alternatively as $\frac{1}{2}H_\gamma H^\gamma - \frac{1}{6}M_\gamma M^\gamma$.) The energy–momentum tensor (266) reads in three-dimensional notation and in the local permanent rest frame:

$$T'^{\alpha\beta}_{[\mathrm{f}]} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}') & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & \begin{matrix}-\boldsymbol{E}'\boldsymbol{D}'-\boldsymbol{H}'\boldsymbol{B}' \\ +(\frac{1}{2}\boldsymbol{E}'^2+\frac{1}{2}\boldsymbol{B}'^2-\boldsymbol{B}'\cdot\boldsymbol{M}'-\frac{1}{6}\boldsymbol{P}'^2+\frac{1}{3}\boldsymbol{M}'^2)\mathrm{U}\end{matrix} \end{bmatrix}. \quad (267)$$

The field energy density for induced dipole fluids is thus $\frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}')$, while the scalar part of the field pressure tensor contains $\frac{1}{2}\boldsymbol{E}'^2+\frac{1}{2}\boldsymbol{B}'^2-\boldsymbol{B}'\cdot\boldsymbol{M}' -\frac{1}{6}\boldsymbol{P}'^2+\frac{1}{3}\boldsymbol{M}'^2$.

The relativistic first law for induced dipole fluids in equilibrium may be written in terms of $u'_0 = v'u'_{v0}$ and p'_0. In fact, substituting (264) and (265) into equation (226), we obtain as the first law valid in equilibrium

$$\varrho' D u'_0 = -p'_0\,\varrho' D v' + \tfrac{1}{2}\varrho' E_{\mathrm{L}\alpha} D(v'P^\alpha) - \tfrac{1}{2}P_\alpha D E^\alpha_{\mathrm{L}} + \tfrac{1}{2}\varrho' B_{\mathrm{L}\alpha} D(v'M^\alpha) - \tfrac{1}{2}M_\alpha D B^\alpha_{\mathrm{L}}, \quad (268)$$

where we used the abbreviations

$$E^\alpha_{\mathrm{L}} = E^\alpha + \tfrac{1}{3}P^\alpha, \qquad B^\alpha_{\mathrm{L}} = B^\alpha - \tfrac{2}{3}M^\alpha \ (= H^\alpha + \tfrac{1}{3}M^\alpha). \quad (269)$$

The relativistic second law (228) gets the form:

$$T'Ds' = Du'_0 + p'_0 Dv' - \tfrac{1}{2}E_{\mathrm{L}\alpha} D(v'P^\alpha) + \tfrac{1}{2}v'P_\alpha D E^\alpha_{\mathrm{L}} - \tfrac{1}{2}B_{\mathrm{L}\alpha} D(v'M^\alpha) + \tfrac{1}{2}v'M_\alpha D B^\alpha_{\mathrm{L}}. \quad (270)$$

(From (268) and (270) one obtains $Ds' = 0$ in equilibrium.)

As a second case we consider fluids with *permanent dipoles* that satisfy Clausius–Mossotti laws of the type (263) and Langevin–Debye laws for the temperature dependence of the susceptibilities:

$$\frac{\kappa}{\kappa+3} \sim \frac{1}{T'}, \qquad \frac{\chi}{3-2\chi} \sim \frac{1}{T'}. \quad (271)$$

The expression for p' (243) gets the form (264), while the expression (247) for u'_v becomes:

$$u'_v = u'_{v0} - B_\alpha M^\alpha - \tfrac{1}{6}P_\alpha P^\alpha + \tfrac{1}{3}M_\alpha M^\alpha. \quad (272)$$

The expression (253) for $T^{\alpha\beta}_{[\mathrm{f}]}$ reads now:

$$T^{\alpha\beta}_{[\mathrm{f}]} = -E^\alpha D^\beta - H^\alpha B^\beta + \Delta^{\alpha\beta}(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma - \tfrac{1}{6}P_\gamma P^\gamma + \tfrac{1}{3}M_\gamma M^\gamma) + c^{-2}U^\alpha U^\beta(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma - \tfrac{1}{6}P_\gamma P^\gamma + \tfrac{1}{3}M_\gamma M^\gamma) - c^{-2}U^\alpha \varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta \varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta, \quad (273)$$

or in three-dimensional notation and in the local permanent rest frame:

$$T'^{\alpha\beta}_{[\mathrm{f}]} = \begin{bmatrix} \frac{1}{2}\boldsymbol{E}'^2+\frac{1}{2}\boldsymbol{B}'^2-\boldsymbol{B}'\cdot\boldsymbol{M}'-\frac{1}{6}\boldsymbol{P}'^2+\frac{1}{3}\boldsymbol{M}'^2 & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & \begin{matrix}-\boldsymbol{E}'\boldsymbol{D}'-\boldsymbol{H}'\boldsymbol{B}' \\ +(\frac{1}{2}\boldsymbol{E}'^2+\frac{1}{2}\boldsymbol{B}'^2-\boldsymbol{B}'\cdot\boldsymbol{M}'-\frac{1}{6}\boldsymbol{P}'^2+\frac{1}{3}\boldsymbol{M}'^2)\mathrm{U}\end{matrix} \end{bmatrix}. \quad (274)$$

The field energy density for permanent dipole fluids is hence $\frac{1}{2}\boldsymbol{E}'^2+\frac{1}{2}\boldsymbol{B}'^2 -\boldsymbol{B}'\cdot\boldsymbol{M}'-\frac{1}{6}\boldsymbol{P}'^2+\frac{1}{3}\boldsymbol{M}'^2$ in contrast with the expression $\frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}')$ for induced dipole fluids; no polarization energy of the form $\frac{1}{2}\boldsymbol{E}'_{\mathrm{L}}\cdot\boldsymbol{P}'+\frac{1}{2}\boldsymbol{B}'_{\mathrm{L}}\cdot\boldsymbol{M}'$ occurs here.

The relativistic first law for permanent dipole fluids in equilibrium reads in terms of u'_0 and p'_0:

$$\varrho' D u'_0 = -p'_0\,\varrho' D v' + \varrho' E_{\mathrm{L}\alpha} D(v'P^\alpha) + \varrho' B_{\mathrm{L}\alpha} D(v'M^\alpha), \quad (275)$$

while the relativistic second law is:

$$T'Ds' = Du'_0 + p'_0 Dv' - E_{\mathrm{L}\alpha} D(v'P^\alpha) - B_{\mathrm{L}\alpha} D(v'M^\alpha). \quad (276)$$

For *diluted* media terms quadratic in the susceptibilities may be neglected so that the expressions (266) and (273) may be further simplified. For a diluted fluid with induced dipoles we get:

$$T^{\alpha\beta}_{[\mathrm{f}]} = -E^\alpha D^\beta - H^\alpha B^\beta + \Delta^{\alpha\beta}(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma) + c^{-2}U^\alpha U^\beta(\tfrac{1}{2}E_\gamma D^\gamma + \tfrac{1}{2}B_\gamma H^\gamma) - c^{-2}U^\alpha \varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta \varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta, \quad (277)$$

whereas for a diluted fluid with permanent dipoles the result is:

$$T^{\alpha\beta}_{[\mathrm{f}]} = -E^\alpha D^\beta - H^\alpha B^\beta + \Delta^{\alpha\beta}(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma) + c^{-2}U^\alpha U^\beta(\tfrac{1}{2}E_\gamma E^\gamma + \tfrac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma) - c^{-2}U^\alpha \varepsilon^{\beta\gamma\zeta\eta}E_\gamma H_\zeta U_\eta - c^{-2}U^\beta \varepsilon^{\alpha\gamma\zeta\eta}E_\gamma H_\zeta U_\eta. \quad (278)$$

(In the approximation used here the scalar $\frac{1}{2}B_\gamma B^\gamma - B_\gamma M^\gamma$ is equal to $\frac{1}{2}H_\gamma H^\gamma$.)

The tensor (277) for diluted induced dipole fluids shows some similarity to the tensors proposed by Minkowski and Abraham[1]. To make comparison easier we write in the local permanent rest frame:

$$T'^{\alpha\beta}_{[\mathrm{f}]} = \begin{bmatrix} \tfrac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}') & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}'-\boldsymbol{H}'\boldsymbol{B}'+(\tfrac{1}{2}\boldsymbol{E}'^2+\tfrac{1}{2}\boldsymbol{B}'^2-\boldsymbol{B}'\cdot\boldsymbol{M}')\mathsf{U} \end{bmatrix}. \qquad (279)$$

The rest frame expressions of Minkowski's and Abraham's tensor contain the same energy density and energy flow. However Minkowski's momentum density $c^{-1}\boldsymbol{D}'\wedge\boldsymbol{B}'$ is not found and neither is the field pressure tensor $-\boldsymbol{E}'\boldsymbol{D}'-\boldsymbol{H}'\boldsymbol{B}'+\frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}')\mathsf{U}$ proposed by both these authors[2] (v. also the discussion at the end of this chapter).

f. *The generalized Helmholtz force density*

The energy–momentum conservation laws for a polarized fluid of neutral atoms in equilibrium may be written as:

$$\partial_\beta(T^{\alpha\beta}_{[\mathrm{f}]}+T^{\alpha\beta}_{[\mathrm{m}]}) = 0, \qquad (280)$$

where $T^{\alpha\beta}_{[\mathrm{f}]}$ is the field energy–momentum tensor (253) and $T^{\alpha\beta}_{[\mathrm{m}]}$ the material energy–momentum tensor (254), which has been defined as the energy–momentum tensor in the absence of fields. The conservation laws can be put into the form:

$$\partial_\beta T^{\alpha\beta}_{[\mathrm{m}]} = \mathscr{F}^\alpha, \qquad (281)$$

where a force density $\mathscr{F}^\alpha$ is introduced, which is given by

$$\mathscr{F}^\alpha \equiv -\partial_\beta T^{\alpha\beta}_{[\mathrm{f}]}. \qquad (282)$$

It corresponds to a material pressure and internal energy density defined at zero fields and is therefore equal to the difference of the force densities in the presence and in the absence of electromagnetic fields.

With the help of (253) an explicit expression for the force density $\mathscr{F}^\alpha$ may be obtained. If use is made of the Maxwell equations (64), the definitions (126), (140) and the relations (235), one gets:

[1] H. Minkowski, op. cit.: M. Abraham, op. cit.

[2] As shown already in chapter II (v. equation (II.371)) a special case exists in which this (Maxwell–Heaviside) field pressure tensor shows up, namely if one considers a body immersed in an incompressible liquid.

$$\begin{aligned}\mathscr{F}^\alpha = {} & -\tfrac{1}{2}E_\gamma E^\gamma\partial^\alpha\kappa-\tfrac{1}{2}B_\gamma B^\gamma\partial^\alpha\chi \\ & -\tfrac{1}{2}\partial_\beta\left\{\Delta^{\alpha\beta}\left(v'\frac{\partial\kappa}{\partial v'}E_\gamma E^\gamma+v'\frac{\partial\chi}{\partial v'}B_\gamma B^\gamma\right)\right. \\ & \left.+c^{-2}U^\alpha U^\beta\left(T'\frac{\partial\kappa}{\partial T'}E_\gamma E^\gamma+T'\frac{\partial\chi}{\partial T'}B_\gamma B^\gamma\right)\right\} \\ & -c^{-2}\varepsilon^{\alpha\beta\gamma\zeta}\varrho' D\{v'(\kappa+\chi)E_\beta B_\gamma U_\zeta\}-c^{-2}(\partial^\alpha U^\beta)\varepsilon_{\beta\gamma\zeta\eta}(\kappa+\chi)E^\gamma B^\zeta U^\eta. \qquad (283)\end{aligned}$$

In the local rest frame the components $\alpha = 0$ and $\alpha = 1, 2, 3$ of the force density $\mathscr{F}^\alpha$ read for a substance of constant and uniform velocity:

$$\mathscr{F}'^0 = \tfrac{1}{2}\boldsymbol{E}'^2\partial'_0\kappa-\tfrac{1}{2}\partial'_0\left(T'\frac{\partial\kappa}{\partial T'}\boldsymbol{E}'^2\right)+\tfrac{1}{2}\boldsymbol{B}'^2\partial'_0\chi-\tfrac{1}{2}\partial'_0\left(T'\frac{\partial\chi}{\partial T'}\boldsymbol{B}'^2\right), \qquad (284)$$

$$\begin{aligned}\boldsymbol{\mathscr{F}}' = {} & -\tfrac{1}{2}\boldsymbol{E}'^2\nabla'\kappa-\tfrac{1}{2}\nabla'\left(v'\frac{\partial\kappa}{\partial v'}\boldsymbol{E}'^2\right)-\tfrac{1}{2}\boldsymbol{B}'^2\nabla'\chi-\tfrac{1}{2}\nabla'\left(v'\frac{\partial\chi}{\partial v'}\boldsymbol{B}'^2\right) \\ & +\partial'_0\{(\kappa+\chi)\boldsymbol{E}'\wedge\boldsymbol{B}'\}. \qquad (285)\end{aligned}$$

If we introduce the susceptibility $\tilde{\chi}$ as in (256) these expressions for $\mathscr{F}'^0$ and $\boldsymbol{\mathscr{F}}'$ become:

$$\mathscr{F}'^0 = \tfrac{1}{2}\boldsymbol{E}'^2\partial'_0\kappa-\tfrac{1}{2}\partial'_0\left(T'\frac{\partial\kappa}{\partial T'}\boldsymbol{E}'^2\right)+\tfrac{1}{2}\boldsymbol{H}'^2\partial'_0\tilde{\chi}-\tfrac{1}{2}\partial'_0\left(T'\frac{\partial\tilde{\chi}}{\partial T'}\boldsymbol{H}'^2\right), \qquad (286)$$

$$\begin{aligned}\boldsymbol{\mathscr{F}}' = {} & -\tfrac{1}{2}\boldsymbol{E}'^2\nabla'\kappa-\tfrac{1}{2}\nabla'\left(v'\frac{\partial\kappa}{\partial v'}\boldsymbol{E}'^2\right)-\tfrac{1}{2}\boldsymbol{H}'^2\nabla'\tilde{\chi}-\tfrac{1}{2}\nabla'\left(v'\frac{\partial\tilde{\chi}}{\partial v'}\boldsymbol{H}'^2\right) \\ & +\partial'_0\{(\kappa+\tilde{\chi}+\kappa\tilde{\chi})(\boldsymbol{E}'\wedge\boldsymbol{H}')\}. \qquad (287)\end{aligned}$$

The expression (285) or (287) may be called the relativistic Helmholtz ponderomotive force in view of its analogy with the non-relativistic expression (II.350). Comparison with this expression (for constant and uniform macroscopic velocity and equilibrium polarizations, i.e. the same physical situation as studied here) shows that the only difference consists in the appearance in (285) or (287) of a term $-\partial'_0(\boldsymbol{M}'\wedge\boldsymbol{E}')$ on a par with $\partial'_0(\boldsymbol{P}'\wedge\boldsymbol{B}')$ that figures already in the non-relativistic theory.

In this section the laws of thermodynamics for polarized systems have been obtained. From the second law expressions could be derived for the difference between the material pressure and the material energy density in the presence of fields and those in the absence of fields. The field and material part of the relativistic energy–momentum tensor could then be defined in such a way that the material tensor contains a pressure and an energy density without fields.

7 *On the uniqueness of the energy–momentum tensor*

The derivation of the energy–momentum laws in section 4 showed that they may be formulated in terms of a macroscopic energy–momentum tensor. Furthermore complete statistical expressions in terms of atomic quantities have been obtained for the components of this tensor.

Since only the four-divergence of the energy–momentum tensor occurs in the conservation laws

$$\partial_\beta T^{\alpha\beta} = 0, \tag{288}$$

the energy–momentum tensor is not uniquely determined: one may add a divergenceless tensor $\hat{T}^{\alpha\beta}$ without changing the contents of the laws. In the inner angular momentum law (section 5)

$$\partial_\gamma(S^{\alpha\beta}U^\gamma) = -\partial_\gamma J^{\alpha\beta\gamma} + T^{\alpha\beta} - T^{\beta\alpha} \tag{289}$$

such a change of $T^{\alpha\beta}$ into $T^{\alpha\beta}+\hat{T}^{\alpha\beta}$ may be compensated by a corresponding change of the inner angular momentum flow $J^{\alpha\beta\gamma}$ to $J^{\alpha\beta\gamma}+\hat{J}^{\alpha\beta\gamma}$ with a quantity $\hat{J}^{\alpha\beta\gamma}$ of the form $-R^\alpha\hat{T}^{\beta\gamma}+R^\beta\hat{T}^{\alpha\gamma}$. Then the right-hand side of (289) remains invariant. Thus such a change of the energy–momentum tensor and of the inner angular momentum flow does not alter the physical description by means of (288) and (289). The particular forms of $T^{\alpha\beta}$ and $J^{\alpha\beta\gamma}$ given in sections 4 and 5 have been adopted since the statistical expressions allow an interpretation which is analogous to that of the corresponding non-relativistic quantities, given in chapter II.

In the course of the treatment of section 4 it turned out to be convenient to call a certain part of the energy–momentum tensor its 'field part', and the remaining term its 'material part'. This nomenclature, which arose from the form of the various expressions, has of course no influence on the physical contents of the laws. Such a splitting of the energy–momentum tensor could be performed in different ways, which each have their particular advantages as shown in section 6. Exactly the same situation was encountered already in the non-relativistic theory, where the material pressure could be defined in different ways: Kelvin's and Helmholtz's, each with its own force density. It appeared there that both could be utilized to describe the physical phenomena.

Just as it is fruitless to discuss – in non-relativistic theory – the relative merits of the Kelvin and the Helmholtz forces without considering the ensuing difference in the corresponding material pressure tensors, a dispute on the correct form of the field energy–momentum tensor – in relativistic theory – is useless if one does not bring into the argument the form of the material energy–momentum tensor: the problem would then remain undetermined.

In the present statistical theory each choice of the field tensor determines the form of the corresponding material tensor explicitly, so that no ambiguity can arise. If the distribution functions are given, one may in principle calculate both the material and the field tensor. (In practice this requires a theory from which one may derive these distribution functions. A well-known example is the relativistic generalization of Boltzmann's kinetic theory.)

The history of the discussions on the energy–momentum tensor for polarized media goes back to the beginning of this century. Often only the field part of the total tensor was considered. As a consequence various authors could arrive at altogether different expressions: a manifestation of the inherent ambiguity which results, as explained, if one forgets about the material part.

After Lorentz's[1] original non-relativistic considerations on the electromagnetic forces in a polarized medium of electric dipoles at rest (v. chapter II) Einstein and Laub[2] were the first to try and give a relativistic expression for the force density in a polarized medium of electric and magnetic dipoles. By taking the same electric dipole terms as Lorentz and by postulating an analogy between electric and magnetic effects they arrived at an expression for the force density in a medium at rest which had a form as (167) apart from the second term, where they wrote $(\nabla \boldsymbol{H})\cdot\boldsymbol{M}$. The material part was not considered at all, so that their treatment suffers from the ambiguity mentioned above.

In the same year Minkowski[3] put forward an expression for the field energy–momentum tensor on the purely formal grounds that it should be form-invariant in all Lorentz frames. This implies that the field energy–momentum tensor should depend on the fields $F^{\alpha\beta}$ and $H^{\alpha\beta}$, but not on the four-velocity U^α of the polarized medium with respect to the observer. Furthermore the expressions for the field energy density, the field energy flow and the field pressure due to Maxwell, Poynting and Heaviside were

[1] H. A. Lorentz, Enc. Math. Wiss. V 2, fasc. 1 (Teubner, Leipzig 1904) 200.
[2] A. Einstein and J. Laub, Ann. Physik **26**(1908)541.
[3] H. Minkowski, Nachr. Ges. Wiss. Göttingen (1908)53; Math. Ann. **68**(1910)472.

taken over. In this way he arrived at the field energy–momentum tensor

$$T^{\alpha\beta}_{(f)M} = F^{\alpha\gamma}H^{\beta}_{\cdot\gamma} - \tfrac{1}{4}F_{\gamma\varepsilon}H^{\gamma\varepsilon}g^{\alpha\beta}, \qquad (290)$$

which in three-dimensional notation has the form

$$T^{\alpha\beta}_{(f)M} = \begin{pmatrix} \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H}) & \boldsymbol{E}\wedge\boldsymbol{H} \\ \boldsymbol{D}\wedge\boldsymbol{B} & -\boldsymbol{ED}-\boldsymbol{HB}+\frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})\mathrm{U} \end{pmatrix}. \qquad (291)$$

The material tensor was again not considered. Moreover the principle of form invariance represents a mathematical requirement, which is foreign to the theory.

Abraham[1] abandoned the principle of form invariance but instead assumed that the field pressure tensor is symmetric in all Lorentz frames, even for anisotropic media. Maxwell's and Poynting's expressions were adopted as the field energy density and the field energy flow in the rest frame, whereas the field pressure tensor in the rest frame was taken to be represented by Hertz's symmetrized form. In this way a completely symmetric field energy–momentum tensor was obtained; it reads in covariant notation:

$$\begin{aligned} T^{\alpha\beta}_{(f)A} = {} & \tfrac{1}{2}(F^{\alpha\gamma}H^{\beta}_{\cdot\gamma}+F^{\beta\gamma}H^{\alpha}_{\cdot\gamma}) - \tfrac{1}{4}F_{\gamma\varepsilon}H^{\gamma\varepsilon}g^{\alpha\beta} \\ & + \tfrac{1}{2}c^{-2}\{U^{\beta}(F^{\alpha\gamma}M_{\gamma\varepsilon}-M^{\alpha\gamma}F_{\gamma\varepsilon})+U^{\alpha}(F^{\beta\gamma}M_{\gamma\varepsilon}-M^{\beta\gamma}F_{\gamma\varepsilon})\}U^{\varepsilon}, \end{aligned} \qquad (292)$$

while the rest frame expression in three-dimensional notation is:

$$T'^{\alpha\beta}_{(f)A} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}') & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ & -\frac{1}{2}(\boldsymbol{E}'\boldsymbol{D}'+\boldsymbol{D}'\boldsymbol{E}'+\boldsymbol{H}'\boldsymbol{B}'+\boldsymbol{B}'\boldsymbol{H}') \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & +\frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}'+\boldsymbol{B}'\cdot\boldsymbol{H}')\mathrm{U} \end{bmatrix}. \qquad (293)$$

In a subsequent paper Abraham[2] remarks that the expression for the field energy–momentum tensor should be derived from electron-theoretical considerations, but he limits himself to a discussion of several possible approaches to carry out this programme. As an argument in favour of the symmetry of the field energy–momentum tensor he mentions the fact that the microscopic energy–momentum tensor is symmetric. However this argument ensures only that the system of polarized matter and fields is closed, so that one expects macroscopic conservation of total angular momentum and hence the possibility to symmetrize[3] the total energy–momentum tensor.

[1] M. Abraham, R. C. Circ. Mat. Palermo **28**(1909)1, **30**(1910)33; Theorie der Elektrizität II (Teubner, Leipzig 1923) 300.

[2] M. Abraham, Ann. Physik **44**(1914)537.

[3] F. J. Belinfante, Physica **6**(1939)887; L. Rosenfeld, Mém. Acad. Roy. Belg. (Cl. Sc.) **18**(1940)6.

About the symmetry of the field part nothing can be found on these grounds.

Abraham's field energy–momentum tensor contains a field momentum density which (in the rest frame) is equal to the field energy flow (apart from a factor c^{-2}). Hence the field tensor has a property formulated as early as 1908 by Planck[1]; it is called sometimes the 'inertia law of energy'. This circumstance has often been considered as a strong argument in favour of Abraham's tensor[2]. However, Planck's law is valid only for a closed system[3], so that it may be applied to the total energy–momentum tensor only: the latter may be written in symmetrical form.

Soon after Minkowski's and Abraham's papers Dällenbach[4] tried to give a treatment of the energy–momentum laws on the basis of microscopic considerations valid for the electrostatic case only. These considerations are then generalized without further justification, with Minkowski's field tensor as a result. To explain its asymmetry he rightly remarks that only the sum of the field and the material energy–momentum tensor ought to be symmetrical. However the material tensor is not considered any further.

In Frenkel's[5] treatment microscopic concepts are employed together with macroscopic arguments. By a consideration of the forces exerted on surface charges and currents he obtains as the field energy–momentum tensor for a stationary medium an expression which is near to the result (148). However, as he is convinced that covariance should imply form invariance he rejects this tensor since it does not possess this property. Owing to this difficulty a definite conclusion on the field tensor is not reached. By postulating the symmetry of the field energy–momentum tensor with respect to time–space and space–time components in the rest frame, and employing a reasoning similar to Frenkel's, Rancoita[6] arrives at a field energy–momentum tensor of the form (148).

A much discussed argument in favour of the asymmetric Minkowski tensor was put forward in 1950 by Von Laue[7] following an old idea of

[1] M. Planck, Phys. Z. **9**(1908)828.

[2] M. von Laue, Die Relativitätstheorie I (Vieweg, Braunschweig 1919) 185; G. Marx and G. Györgyi, Acta Phys. Acad. Sci. Hung. **3**(1953)213; N. L. Balázs, Phys. Rev. **91**(1953) 408; G. Györgyi, Acta Phys. Acad. Sci. Hung. **4**(1954)121; G. Marx and G. Györgyi, Ann. Physik **16**(1955)241; J. Agudín, Phys. Letters **24A**(1967)761.

[3] C. Møller, Theory of relativity (Clarendon Press, Oxford 1952) 164, 189; cf. F. Beck, Naturwiss. **39**(1952)254; Z. Physik **134**(1953)136.

[4] W. Dällenbach, Ann. Physik **58**(1919)523.

[5] J. Frenkel, Lehrbuch der Elektrodynamik II (Springer-Verlag, Berlin 1928) 48–94.

[6] G. M. Rancoita, Suppl. N. Cim. **11**(1959)183.

[7] M. von Laue, Z. Physik **128**(1950)387; Die Relativitätstheorie I (Vieweg, Braunschweig 1952) 139.

Scheye[1]. According to this argument the energy transport velocity, which is the quotient of the energy flow and the energy density, should transform in such a way that the addition theorem for velocities is obeyed. Since the Minkowski field tensor satisfies this criterion, a number of authors[2] have advocated it. However Schöpf[3] remarked that Von Laue's criterion does not lead exclusively to Minkowski's tensor since it is satisfied by tensors of a different form as well. Furthermore Tang and Meixner[4] showed that even if Minkowski's tensor is adopted as the field part of the total energy–momentum tensor an explicit evaluation of the material contributions to the total energy density and flow leads to the conclusion that the total energy transport velocity does not satisfy Von Laue's criterion. Hence this criterion cannot be considered as a physical requirement to be imposed on an energy–momentum tensor.

Often reasonings which start from macroscopic variational principles are considered as derivations of the form of the energy–momentum tensor. In this way some authors try to derive the field energy–momentum tensor and arrive at Minkowski's[5] or Abraham's[6] tensor or still different tensors[7], whereas others[8] obtain expressions for the *total* energy–momentum tensor, which according to them is the only one that can be deduced from a variational principle. However against all such treatments the same objection may be raised: at the outset a macroscopic Lagrangian (or Hamiltonian) is postulated, not derived from first principles. Therefore arguments of this kind do not lead to a solution of the problem.

In the course of the discussions various *ad hoc* arguments of a macroscopic nature have been put forward in favour of one or the other of the field

[1] A. Scheye, Ann. Physik **30**(1909)805.

[2] H. Ott, Ann. Physik **11**(1952)33; F. Beck, loc. cit.; C. Møller, loc. cit. 206–211; E. Schmutzer, Ann. Physik **18**(1956)171; J. I. Horváth, Bull. Acad. Polon. Sci. **4**(1956)447; W. Pauli, Theory of relativity (Pergamon Press, London 1958) 216, note 11.

[3] H. G. Schöpf, Z. Physik **148**(1957)417.

[4] C. L. Tang and J. Meixner, Phys. Fluids **4**(1961)148.

[5] J. Ishiwara, Ann. Physik **42**(1913)986; W. Dällenbach, Ann. Physik **59**(1919)28; E. Schmutzer, Ann. Physik **20**(1957)349; U. E. Schröder, Z. Naturf. **24A**(1969)1356.

[6] E. Henschke, Ann. Physik **40**(1913)887; K. F. Novobátzky, Hung. Acta Phys. **1**(1949) fasc. 5; G. Marx, Acta Phys. Acad. Sci. Hung. **2**(1952)67; **3**(1953)75; G. Marx and G. Györgyi, Ann. Physik **16**(1955)241; H. G. Schöpf, Ann. Physik **13**(1964)41.

[7] K. Furutsu, Phys. Rev. **185**(1969)257.

[8] H. G. Schöpf, Ann. Physik **9**(1962)301: P. Penfield Jr. and H. A. Haus, Phys. Fluids **9**(1966)1195.

energy–momentum tensors[1]. An argument which was thought[2] to be in favour of Minkowski's tensor is the fact that the corresponding force density vanishes for a neutral, current-free and homogeneous medium with linear constitutive relations, if it moves with a uniform velocity. An argument which was claimed[3] to be in favour of Abraham's tensor is the fact that the field energy density of this tensor is positive for all values of the macroscopic velocity. As a consequence, if one performs a quantization of the macroscopic electromagnetic fields, one finds photons with positive energy[4]. However again the material energy–momentum was left out of consideration. Sometimes it was thought[5] that radiation pressure experiments can throw light on the correct form of the field momentum density. However, as shown in chapter II, the explanation of the experimental results is independent of the expression for the momentum density, since terms with time derivatives drop out from the equations[6]. (The results of chapter II are not altered in relativity theory since the extra term in the force density (167), as compared with (II.106), is also a time derivative.) Still other postulates[7] have been put forward in order to justify the choice of a particular form of the field

[1] A review of these and the other arguments in favour of one or the other field energy–momentum tensor was given by I. Brevik, Mat. Fys. Medd. Vid. Selsk. **37**(1970)no. 11, 13. In his first paper he seems to adopt Minkowski's tensor on the basis of *ad hoc* postulates, while in his second one he rightly remarks that only the total energy–momentum tensor has physical meaning. Yet he thinks that only Minkowski's and Abraham's field tensors do not run into conflict with experimental evidence. His argument to reject (122) with (92) as a useful splitting is that according to him only Helmholtz-type material pressures are in agreement with experiment. However, as explained in chapter II, the use of both the Kelvin and the Helmholtz pressures and forces is allowed, provided one employs them consistently.

[2] H. Ott, op. cit.; F. Beck, op. cit.; C. Møller, op. cit. p. 206–211; I. Brevik, op. cit.

[3] K. F. Novobátzky, op. cit.; M. von Laue, op. cit.; F. Beck, op. cit.; E. Schmutzer, Ann. Physik **18**(1956)171.

[4] K. Nagy, Acta Phys. Acad. Sci. Hung. **5**(1955)95. Minkowski's tensor leads to the possibility of negative energy densities and photons of negative energy: J. M. Jauch and K. M. Watson, Phys. Rev. **74**(1948)950, 1485; I. Brevik and B. Lautrup, Mat. Fys. Medd. Dan. Vid. Selsk. **38**(1970)no. 1.

[5] R. V. Jones and J. C. S. Richards, Proc. Roy. Soc. **221A**(1954)480; I. Brevik, op. cit., uses the experimental result of Jones and Richards as an argument pro Minkowski's momentum density (no. 11, p. 29), but elsewhere remarks (no. 11, p. 5) that it cannot exclude other forms of the field energy–momentum tensor.

[6] G. Marx and G. Györgyi, Acta Phys. Acad. Sci. Hung. **3**(1953)213.

[7] A. Rubinowicz, Acta Phys. Polon. **14**(1955)209. 225; G. Marx and K. Nagy, Bull. Acad. Polon. Sci. **4**(1956)79; J. I. Horváth, N. Cim. **7**(1958)628; O. Costa de Beauregard, C. R. Acad. Sci. Paris **260**(1965)6546; **263B**(1966)1007, 1279; N. Cim. **48B**(1967)293; W. Shockley, Proc. Nat. Acad. Sci. U.S.A. **60**(1968)807.

energy–momentum tensor, but again they are of an *ad hoc* and macroscopic character.

Arguments based on the propagation of light waves in connexion with the refraction of light, the Cherenkov effect, the Sagnac effect and Fizeau's experiments do not lead to a decision on the correct form of the field energy–momentum tensor[1], since these phenomena can be explained on the basis of the field equations alone.

A special class of theories is based on thermodynamical considerations. In the framework of the treatment of Kluitenberg and de Groot[2] a relativistic Gibbs relation and the symmetric character of the material energy–momentum tensor were postulated. As a result a field energy–momentum tensor was obtained which comes very near to that given in (92). The hypothesis about the symmetry of the material tensor is rather essential; if it is dropped different forms for the field energy–momentum tensor (for instance Minkowski's tensor) are justifiable from a thermodynamical point of view, as has been shown by Schmutzer[3]. De Sá[4] and Meixner[5] discuss various possibilities for the splitting of the total energy–momentum tensor into a material and a field part. They rightly conclude that thermodynamical considerations do not allow to specify the material part sufficiently well; the field part remains then undetermined. Chu, Haus and Penfield[6] postulate a form for the first law of thermodynamics together with the symmetrical character of the material tensor. Since this starting point is equivalent to Kluitenberg and de Groot's, their resulting field energy–momentum tensor is also the same apart from some diagonal terms.

In general it may be stated that the solution of the problem of deriving the energy–momentum and angular momentum laws for polarized media cannot be solved as long as macroscopic arguments are utilized. The problem is even undetermined if only the field energy–momentum tensor is considered without giving expressions for the material energy–momentum tensor. The complete programme can only be carried out if one starts from the microscopic laws. Then statistical expressions for the total energy–momentum tensor and

[1] G. Marx and G. Györgyi, Ann. Physik **16**(1955)241; G. Györgyi, Am. J. Phys. **28**(1960) 85; cf. however I. Brevik, op. cit.

[2] G. A. Kluitenberg and S. R. de Groot, Physica **20**(1954)199; **21**(1955)148, 169.

[3] E. Schmutzer, Ann. Physik **14**(1964)56; cf. G. Neugebauer, Wiss. Z. Friedrich Schiller Univ. Jena **13**(1964)209.

[4] B. de Sá, thesis, Aachen (1960).

[5] J. Meixner, Univ. Michigan Report, RL-184(1961); Z. Physik **229**(1969)352.

[6] L. J. Chu, H. A. Haus and P. Penfield jr., Proc. I.E.E.E. **54**(1966)920; P. Penfield jr. and H. A. Haus, Electrodynamics of moving media (M.I.T. Press, Cambridge, Mass. 1967).

the angular momentum density and flow can be derived, as shown in this chapter. These are then unambiguously defined (apart from terms which drop out from the conservation laws, as discussed in the beginning of this section): the splitting into a field and material part is a question of nomenclature only.

PROBLEMS

1. Prove that the average $A(R)$ of the microscopic quantity

$$a(R) = \sum_i \int \alpha(i)\delta^{(4)}(R-R_i)\mathrm{d}s_i$$

may be written either as

$$A(R) = c^{-1}\int \alpha(1)\delta^{(4)}(R-R_1)f_1(1)\mathrm{d}1$$

or as

$$A(R) = -\int \frac{\alpha(1)}{n\cdot R_1^{[1]}}\delta^{(3)}(n;R-R_1)f_1^{\mathrm{syn}}(1;n,-c^{-1}n\cdot R)\mathrm{d}1,$$

where the three-dimensional delta function has been defined in (2).

Note that the first form shows that the second is independent of the unit four-vector n^α. Show that the second expression may be written as

$$A(R) = c^{-1}\int \alpha(1)\gamma_1^{-1}f_1^{\mathrm{syn}}(\boldsymbol{R},\boldsymbol{\beta}_1,\partial_0\boldsymbol{\beta}_1,\ldots;t)\mathrm{d}\boldsymbol{\beta}_1\,\mathrm{d}\partial_0\boldsymbol{\beta}_1\ldots,$$

as follows by employing $\mathring{n}^\alpha = (1,0,0,0)$ and (55). In spite of its appearance the derivation shows that this form of $A(R)$ has covariant character. (A particular case of physical importance is obtained by choosing $\alpha(i) = ce_iu_i^\alpha$ with u_i^α the four-velocity and e_i the charge; then one gets the expressions (72).)

2. Calculate the Jacobian occurring in (55) for the velocities, i.e.,

$$\frac{\partial(\boldsymbol{R}_1^{[1]})}{\partial(\boldsymbol{\beta}_1)},$$

and for the velocities with accelerations, i.e.,

$$\frac{\partial(\boldsymbol{R}_1^{[1]},\boldsymbol{R}_1^{[2]})}{\partial(\boldsymbol{\beta}_1,\partial_0\boldsymbol{\beta}_1)}.$$

3. Prove from (20) that one has for synchronous averages the identity

$$\partial_\mu\int \alpha(1;R)f_1^{\mathrm{syn}}(1;n,-c^{-1}n\cdot R)\mathrm{d}1 = \int \mathrm{d}_\mu^{\mathrm{syn}}\alpha(1;R)f_1^{\mathrm{syn}}(1;n,-c^{-1}n\cdot R)\mathrm{d}1,$$

where the notation (28) has been used. To prove this relation one should substitute a quantity of the type (21) (with $\tau = -c^{-1}n\cdot R$) into (20) and then employ the definition (30) of an average, using the transformation which leads from (23) to (26).

4. Prove (27) and (37) by differentiating the ancillary conditions, which procedure leads to

$$\partial_\mu s_i = \frac{n_\mu}{n\cdot R_i^{[1]}}$$

and

$$\partial_\mu s_i = \frac{(R-R_i)_\mu}{(R-R_i)\cdot R_i^{[1]}},$$

respectively.

5. Prove from the definition (31) that synchronous distribution functions characterized by a different normal unit vector n^α and n'^α are related as

$$f_1^{\mathrm{syn}}(1;n,\tau) = -\frac{n\cdot R_1^{[1]}}{n\cdot n'n'\cdot R_1^{[1]}}f_1^{\mathrm{syn}}\left(1;n',-\frac{c^{-1}n\cdot\Delta'\cdot R_1+\tau}{n\cdot n'}\right),$$

where τ is a constant (independent of R_1^α) and where $\Delta'^{\alpha\beta}$ is the tensor $g^{\alpha\beta}+n'^\alpha n'^\beta$. To prove this relation one should use the property (48).

Note that the relation is not symmetric in the distribution functions for n^α and n'^α. This is due to the fact that the distribution function at the left-hand side contains as an argument a quantity τ independent of R_1^α, while the distribution function at the right-hand side contains a third argument which does depend on R_1^α. For that reason one may be interested in a relation which contains also at the left-hand side a distribution function of which the third argument is linear in R_1^α. This relation, which may be proved along the same lines, reads

$$f_1^{\mathrm{syn}}(1;n,\tau_0+\tau_1\cdot R_1)$$
$$= \frac{n\cdot R_1^{[1]}}{|n\cdot n'+cn'\cdot\tau_1|n'\cdot R_1^{[1]}}f_1^{\mathrm{syn}}\left(1;n',-\frac{c^{-1}n\cdot\Delta'\cdot R_1+\tau_1\cdot\Delta'\cdot R_1+\tau_0}{n\cdot n'+cn'\cdot\tau_1}\right)$$

for arbitrary constants τ_0 and τ_1^α. One may check that a repeated use of this formula leads to an identity. A symmetrical relation is obtained by choosing in particular

$$\tau_1^\alpha = c^{-1}\frac{n^\alpha n\cdot n'+n'^\alpha}{1-n\cdot n'}.$$

Then one gets:

$$f_1^{\text{syn}}\left(1;\, n,\, \tau_0 + c^{-1}\frac{n\cdot R_1\, n\cdot n' + n'\cdot R_1}{1 - n\cdot n'}\right)$$
$$= \frac{n\cdot R_1^{[1]}}{n'\cdot R_1^{[1]}} f_1^{\text{syn}}\left(1;\, n',\, \tau_0 + c^{-1}\frac{n'\cdot R_1\, n\cdot n' + n\cdot R_1}{1 - n\cdot n'}\right).$$

6. Prove from (159) that the inner product of the four-velocity U^α and the ponderomotive force F^α is given by

$$U_\alpha F^\alpha = \tfrac{1}{2}(DF^{\alpha\beta})M_{\alpha\beta} + c^{-2}(DU_\alpha)(F^{\alpha\beta}M_{\beta\gamma} - M^{\alpha\beta}F_{\beta\gamma})U^\gamma$$
$$- c^{-2}\varrho' D(v' U^\alpha F_{\alpha\beta} M^{\beta\gamma} U_\gamma).$$

Show, by introducing the splitting of $M^{\alpha\beta}$ into two parts, defined as

$$M^{(1)\alpha\beta} \equiv -c^{-2}(U^\alpha U_\gamma M^{\gamma\beta} + U^\beta U_\gamma M^{\alpha\gamma}), \qquad M^{(2)\alpha\beta} \equiv \Delta^\alpha_\gamma \Delta^\beta_\varepsilon M^{\gamma\varepsilon},$$

(so that in the rest frame $M^{(1)\alpha\beta}$ and $M^{(2)\alpha\beta}$ represent the electric and magnetic polarization respectively) that one may write this expression as

$$U_\alpha F^\alpha = -\tfrac{1}{2}F'_{\alpha\beta}\,\varrho' D(v' M^{(1)'\alpha\beta}) + \tfrac{1}{2}M^{(2)'}_{\alpha\beta} DF'^{\alpha\beta},$$

where the primes indicate quantities in the permanent local rest frame. The proof follows from the mathematical identity

$$A'_{\alpha\beta} DB'^{\alpha\beta} = A_{\alpha\beta} DB^{\alpha\beta} + 2c^{-2} U^\alpha (A_{\alpha\beta} B^{\beta\gamma} - B_{\alpha\beta} A^{\beta\gamma}) DU_\gamma,$$

which itself is obtained by considering the Lorentz transformations from the permanent local rest frame to the observer's frame.

7. Show that the components of the kinetic contribution $T^{\alpha\beta}_{(\text{m})\text{I}}$ (108) to the material part of the energy–momentum tensor for a neutral plasma may be written as (cf. problem 1)

$$T^{00}_{(\text{m})\text{I}} = c^2 \sum_a \int m_a \gamma_1\, f_1^{\text{syn},a}(\boldsymbol{R}, \boldsymbol{\beta}_1; t)\,\mathrm{d}\boldsymbol{\beta}_1,$$

$$T^{0i}_{(\text{m})\text{I}} = T^{i0}_{(\text{m})\text{I}} = c^2 \sum_a \int m_a \gamma_1\, \beta_1^i\, f_1^{\text{syn},a}(\boldsymbol{R}, \boldsymbol{\beta}_1; t)\,\mathrm{d}\boldsymbol{\beta}_1,$$

$$T^{ij}_{(\text{m})\text{I}} = c^2 \sum_a \int m_a \gamma_1\, \beta_1^i\, \beta_1^j\, f_1^{\text{syn},a}(\boldsymbol{R}, \boldsymbol{\beta}_1; t)\,\mathrm{d}\boldsymbol{\beta}_1.$$

Prove that in the non-relativistic limit (i.e. up to order c^{-1}) the energy density, the energy flow, the momentum density and the momentum flow reduce to

$$T^{00}_{(\text{m})\text{I}} = \varrho c^2 + \tfrac{1}{2}\varrho \boldsymbol{v}^2 + \varrho u^{\text{K}},$$
$$cT^{0i}_{(\text{m})\text{I}} = (\varrho c^2 \boldsymbol{v} + \boldsymbol{J}_q^{\text{K}} + \varrho u^{\text{K}} \boldsymbol{v} + \mathbf{P}^{\text{K}}\cdot\boldsymbol{v})^i,$$
$$c^{-1}T^{i0}_{(\text{m})\text{I}} = \varrho v^i,$$
$$T^{ij}_{(\text{m})\text{I}} = \varrho v^i v^j + P^{\text{K}ji}$$

with non-relativistic quantities given by (II.119, 120, 122, 131, 132).

Prove that the non-relativistic limits of $T^{\alpha\beta}_{(\text{m})\text{II}}$ (116) and $T^{\alpha\beta}_{(\text{m})\text{III}}$ (118) have the forms

$$T^{00}_{(\text{m})\text{II}+\text{III}} = \varrho u^{\text{C}},$$
$$cT^{0i}_{(\text{m})\text{II}+\text{III}} = (\boldsymbol{J}_q^{\text{C}} + \varrho u^{\text{C}} \boldsymbol{v} + \mathbf{P}^{\text{C}}\cdot\boldsymbol{v})^i,$$
$$c^{-1}T^{i0}_{(\text{m})\text{II}+\text{III}} = 0,$$
$$T^{ij}_{(\text{m})\text{II}+\text{III}} = P^{\text{C}ji}$$

with the non-relativistic quantities (II.127, 138, 139). Use in the proof that up to order c^{-2} one has $\delta(s^2) = \delta(s^0)/|\boldsymbol{s}|$, where one should take into account that s^0 is of order c.

Show finally that the non-relativistic limits of the components of $T^{\alpha\beta}_{(\text{m})\text{IV}}$ (120) vanish, as follows by considering the expressions for the minus fields, given in (III.75) with (III.72).

8. Show in a way analogous to that of the preceding problem, that the non-relativistic limits of the components of the material energy–momentum tensor for a dipole fluid are such that $T^{\alpha\beta}_{(\text{m})\text{I}}$ (87) leads to the kinetic contributions K, given in (II.63, 78, 81) (together with mass terms), $T^{\alpha\beta}_{(\text{m})\text{II}}$ (90) to the field dependent terms F (II.73, 89), $T^{\alpha\beta}_{(\text{m})\text{IV}+\text{V}}$ (98, 100) to the short range terms S (II.94, 96, 97, 100) and $T^{\alpha\beta}_{(\text{m})\text{VIII}}$ described below (102) to the correlation terms C (II.104, 111, 112). The terms $T^{\alpha\beta}_{(\text{m})\text{III}}$, $T^{\alpha\beta}_{(\text{m})\text{VI}}$ and $T^{\alpha\beta}_{(\text{m})\text{VII}}$ give no contributions in the non-relativistic limit.

9. Show that in the case of long range correlations the generalized Irving–Kirkwood procedure may be written as

$$c_2(R, 1, R-s, 2) = c_2(R + \tfrac{1}{2}s, 1, R - \tfrac{1}{2}s, 2) - \tfrac{1}{2}s^\alpha \partial_\alpha \tilde{c}_2(R + \tfrac{1}{2}s, 1, R - \tfrac{1}{2}s, 2)$$

with the mean correlation function (cf. II.149)

$$\tilde{c}_2(R + \tfrac{1}{2}s, 1, R - \tfrac{1}{2}s, 2) \equiv \int_{-1}^{0} c_2\{R + \tfrac{1}{2}(\lambda+1)s, 1, R + \tfrac{1}{2}(\lambda-1)s, 2\}\,\mathrm{d}\lambda.$$

Apply this relation to the correlation terms of the energy–momentum law for a plasma and show that one obtains again an equation of the form (121) with a second contribution to the energy–momentum tensor, given by an expression like (116), but with $\tilde{c}_2^{ab}$ instead of c_2^{ab}. (A similar extension may be given for dipole substances.)

10. Show that the non-relativistic limits of the space–space components of the macroscopic inner angular momentum density $S^{\alpha\beta}$ (175) are given by (II.166) of the non-relativistic treatment. Write to that end $S^{\alpha\beta}$ with synchronous distribution functions of the type (55), namely as

$$S^{\alpha\beta} = \int \gamma_1^{-1} s_1^{\alpha\beta} f_1^{\text{syn}}(\boldsymbol{R}, s_1^{\gamma\varepsilon}; t) \mathrm{d}^6 s_1^{\gamma\varepsilon}$$

(cf. problem 1).

11. Show that the antisymmetric part of the energy–momentum tensor for a neutral plasma may be written in the form (200) with the quantity $J^{\alpha\beta\gamma}$ given as

$$J^{\alpha\beta\gamma} = c^{-2} \sum_{a,b} \frac{e_a e_b}{16\pi} \int (s^\alpha u_2^\beta - s^\beta u_2^\alpha) u_1^\gamma \, \delta(s^2) c_2^{ab}(R+\tfrac{1}{2}s, 1, R-\tfrac{1}{2}s, 2) \mathrm{d}1 \, \mathrm{d}2 \, \mathrm{d}s.$$

Prove that in the non-relativistic limit both $c^{-1} J^{ij0}$ and J^{ijk} with $i, j, k = 1, 2, 3$ vanish.

12. Prove the statements on the non-relativistic limit of the inner angular momentum flow $J^{\alpha\beta\gamma}$ for a dipole fluid that are given below (199).

13. The energy–momentum tensor (122) for a neutral plasma is not symmetric since $T^{\alpha\beta}_{(\mathrm{m})\mathrm{II}}$ (116) contains a part that is asymmetric. By adding a divergenceless part to $T^{\alpha\beta}_{(\mathrm{m})\mathrm{II}}$ one may bring it into a symmetric form. Show that one may choose for this divergenceless term:

$$-c^{-2} \sum_{a,b} \frac{e_a e_b}{16\pi} \partial_\gamma \int s^\alpha u_2^\beta u_1^\gamma \, \delta(s^2) c_2^{ab}(R+\tfrac{1}{2}s, 1, R-\tfrac{1}{2}s, 2) \mathrm{d}1 \, \mathrm{d}2 \, \mathrm{d}s,$$

so that the second contribution to the energy–momentum tensor gets the symmetric form

$$\tfrac{1}{2}(T^{\alpha\beta}_{(\mathrm{m})\mathrm{II}} + T^{\beta\alpha}_{(\mathrm{m})\mathrm{II}}) - c^{-2} \sum_{a,b} \frac{e_a e_b}{32\pi} \partial_\gamma \int (s^\alpha u_2^\beta + s^\beta u_2^\alpha) \, u_1^\gamma \, \delta(s^2) c_2^{ab}(R+\tfrac{1}{2}s, 1, R-\tfrac{1}{2}s, 2) \mathrm{d}1 \, \mathrm{d}2 \, \mathrm{d}s.$$

Show that in the non-relativistic limit the two tensors discussed coincide.

14. Derive a form of the first law for a polarized fluid of neutral atoms in local equilibrium by starting from the 'Helmholtz' splitting (251) of the total energy–momentum tensor and using the force density (282).

Hint: calculate first the inner product $U^\alpha \mathscr{F}_\alpha$.