CHAPTER III

Charged point particles

1 *Introduction*

Classical electrodynamics in the non-relativistic approximation formed the subject of the preceding two chapters. Since the field equations are covariant with respect to Lorentz transformations one wants to give the complete classical theory of electrodynamics in the framework of special relativity. The first step of this programme will be concerned with the study of the fields and equations of motion of charged point particles. The results will serve as a basis for the derivation of the laws of electrodynamics for composite particles and for matter in bulk, which will be treated in the following pair of chapters.

In this chapter expressions of the fields generated by charged point particles are derived. It will be useful to give them not only in their covariant form, but also as series expansions in powers of c^{-1}. Subsequently the equation of motion for charged particles with the inclusion of radiation damping terms will be discussed. An important ingredient for their derivation is the evaluation of the self-fields of the particles at their own position.

2 *The field equations*

a. *Covariant formulation*

The microscopic Lorentz equations for the electromagnetic fields $\boldsymbol{e}$ and $\boldsymbol{b}$, produced at a time t and a position $\boldsymbol{R}$ by a set of point particles with charges e_j ($j = 1, 2, \ldots$), positions $\boldsymbol{R}_j(t)$ and velocities $\mathrm{d}\boldsymbol{R}_j(t)/\mathrm{d}t \equiv c\boldsymbol{\beta}_j(t)$ read:

$$\begin{aligned} \nabla\cdot\boldsymbol{e} &= \sum_j e_j\,\delta(\boldsymbol{R}_j-\boldsymbol{R}),\\ -\partial_0\,\boldsymbol{e}+\nabla\wedge\boldsymbol{b} &= \sum_j e_j\,\boldsymbol{\beta}_j\,\delta(\boldsymbol{R}_j-\boldsymbol{R}),\\ \nabla\cdot\boldsymbol{b} &= 0,\\ \partial_0\,\boldsymbol{b}+\nabla\wedge\boldsymbol{e} &= 0, \end{aligned} \qquad (1)$$

where ∂_0 and ∇ indicate differentiations with respect to ct and $\boldsymbol{R}$. We may put these equations into a covariant form by introducing the notations R^α ($\alpha = 0, 1, 2, 3$) for $(ct, \boldsymbol{R})$, R_j^α for $(ct_j, \boldsymbol{R}_j)$, ∂_α for (∂_0, ∇), $f^{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2, 3$) for the antisymmetric field tensor with components $(f^{01}, f^{02}, f^{03}) = \boldsymbol{e}$ and $(f^{23}, f^{31}, f^{12}) = \boldsymbol{b}$. (We use the metric $g^{00} = -1$, $g^{ii} = 1$ if $i = 1, 2, 3$, $g^{\alpha\beta} = 0$ if $\alpha \neq \beta$. The inner product $a_\alpha b^\alpha$ of two four-vectors a^α and b^α will sometimes be denoted as $a \cdot b$.) In this way the first two equations of (1) are the cases $\alpha = 0$ and $\alpha = 1, 2, 3$ of

$$\partial_\beta f^{\alpha\beta} = \sum_j e_j \int \frac{\mathrm{d}R_j^\alpha}{\mathrm{d}R_j^0} \delta(\boldsymbol{R}_j - \boldsymbol{R}) \delta(R_j^0 - R^0) \mathrm{d}R_j^0. \tag{2}$$

Considering R_j^0 as a function $R_j^0(s_j)$ of an arbitrary parameter s_j for each particle j, we can write (2) as

$$\partial_\beta f^{\alpha\beta} = \sum_j e_j \int \frac{\mathrm{d}R_j^\alpha(s_j)}{\mathrm{d}s_j} \delta^{(4)}\{R_j(s_j) - R\} \mathrm{d}s_j, \tag{3}$$

where $\delta^{(4)}\{R_j(s_j) - R\}$ is the four-dimensional delta function. The parameters s_j, which are integration variables, may be chosen independently for each trajectory ($j = 1, 2, \ldots$). For convenience we shall choose for s_j a monotonically increasing function of the time $c^{-1}R_j^0$. We shall write equation (3) as

$$\partial_\beta f^{\alpha\beta} = c^{-1} j^\alpha, \tag{4}$$

with the four-current j^α given by:

$$c^{-1} j^\alpha(R) \equiv \sum_j e_j \int u_j^\alpha(s_j) \delta^{(4)}\{R_j(s_j) - R\} \mathrm{d}s_j, \tag{5}$$

where we introduced the abbreviation u_j^α for $\mathrm{d}R_j^\alpha(s_j)/\mathrm{d}s_j$; it represents the four-velocity if s_j is the proper time. The components $c^{-1}j^\alpha = (\rho, c^{-1}\boldsymbol{j})$ are the sources of the first two equations of (1). From this expression the conservation of charge

$$\partial_\alpha j^\alpha = 0 \tag{6}$$

follows immediately, since $u_j^\alpha \partial_\alpha$ acting on the delta function is equal to $-\mathrm{d}/\mathrm{d}s_j$ acting on it.

The last two field equations of (1) may be written in covariant form:

$$\partial_\alpha f_{\beta\gamma} + \partial_\beta f_{\gamma\alpha} + \partial_\gamma f_{\alpha\beta} = 0. \tag{7}$$

With the help of the completely antisymmetric Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\delta}$ (with $\varepsilon_{\alpha\beta\gamma\delta} = \pm 1$ if α, β, γ, δ is an even/odd permutation of 0, 1, 2, 3, $\varepsilon_{\alpha\beta\gamma\delta} = 0$ if two indices are equal), the dual field tensor $f_{\alpha\beta}^* = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} f^{\gamma\delta}$ can be introduced, with components $(f_{01}^*, f_{02}^*, f_{03}^*) = \boldsymbol{b}$ and $(f_{23}^*, f_{31}^*, f_{12}^*) = \boldsymbol{e}$. In terms of this field tensor we can write (7) as:

$$\partial_\beta f^{*\alpha\beta} = 0. \tag{8}$$

The covariant equations (4) and (7) give the fields as measured in the space-time reference frame $(ct, \boldsymbol{R})$.

b. *The solutions of the field equations*

In order to solve the equations (4) and (8), we note first that the general form of the solution of (8) is

$$f^{*\alpha\beta} = \varepsilon^{\alpha\beta\gamma\delta} \partial_\gamma a_\delta, \tag{9}$$

where a^α is an arbitrary four-vector[1]. The field tensor $f^{\alpha\beta} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} f_{\gamma\delta}^*$ becomes thus

$$f^{\alpha\beta} = \partial^\alpha a^\beta - \partial^\beta a^\alpha. \tag{10}$$

The vector a^α is called the four-potential, with components $(\varphi, \boldsymbol{a})$. Substitution into (4) gives:

$$\Box a^\alpha - \partial^\alpha \partial_\beta a^\beta = -c^{-1} j^\alpha, \tag{11}$$

where the d'Alembertian $\Box$ is the operator $\partial_\alpha \partial^\alpha$. From the form of (10) it follows that the potentials a^α are not uniquely determined by the fields $f^{\alpha\beta}$: a gauge transformation (I.5) $a'^\alpha = a^\alpha + \partial^\alpha \psi$ with arbitrary ψ yields the same fields. As a consequence one may choose the potentials such that the Lorentz condition (I.6)

$$\partial_\alpha a^\alpha = 0 \tag{12}$$

is fulfilled. Then one finds the wave equation for the four-potential

$$\Box a^\alpha = -c^{-1} j^\alpha, \tag{13}$$

of which the solutions will be needed in the following. The general solution of this inhomogeneous linear differential equation may be written as the sum of one (arbitrary) particular solution and the general solution of the corre-

[1] In the first instance one finds from (8) that $f^{*\alpha\beta}$ has the form $\partial_\gamma(\lambda^{\alpha\beta\gamma\delta}\mu_\delta)$ with $\lambda^{\alpha\beta\gamma\delta}$ a tensor, antisymmetric only in its first three indices, and μ_δ an arbitrary four-vector. The tensor $\lambda^{\alpha\beta\gamma\delta}\mu_\delta$ has four independent components. Instead one may introduce the four-vector $a^\alpha \equiv \frac{1}{6}\varepsilon^{\alpha\beta\gamma\delta}\lambda_{\beta\gamma\delta\zeta}\mu^\zeta$ with the inverse $\lambda^{\alpha\beta\gamma\delta}\mu_\delta = \varepsilon^{\alpha\beta\gamma\delta}a_\delta$. (Strictly spoken the quantity a^α in (9) does not have to be a four-vector, if only its non-covariant part is such that it drops out in $f^{*\alpha\beta}$. One may however confine oneself to a four-vector a^α without impairing the generality of $f^{*\alpha\beta}$.)

sponding homogeneous equation. (This general theorem remains valid if the subsidiary condition (12) is added, since the latter is linear and homogeneous.)

We may fix the arbitrary particular solution of (13) by imposing certain requirements. In the first place we want to confine ourselves to solutions linear in j^α (such linear solutions exist since (13) is linear itself). Then the general form of the particular solution sought is

$$a^\alpha(R) = c^{-1}\int G^{\alpha\beta}(R, R')j_\beta(R')\mathrm{d}^4R' \tag{14}$$

with the Green function $G^{\alpha\beta}(R, R')$, which must satisfy the equation

$$\Box\, G^{\alpha\beta}(R, R') = -g^{\alpha\beta}\delta^{(4)}(R-R') \tag{15}$$

with $g^{\alpha\beta}$ the metric tensor.

In the second place we require the invariance of the Green function with respect to Poincaré transformations without inversions, i.e. translations in space and time and Lorentz transformations without inversions. In other words we require for the Poincaré transformation

$$\hat{R}^\alpha = \Lambda^\alpha_{.\beta}R^\beta + c^\alpha \tag{16}$$

(with $\Lambda^{\alpha\beta}\Lambda_{\alpha\gamma} = g^\beta_\gamma$, as follows from the defining relation $\hat{R}^\alpha\hat{R}_\alpha = R^\alpha R_\alpha$ for $c^\alpha = 0$) that the Poincaré transform

$$\hat{a}^\alpha(\hat{R}) = \Lambda^\alpha_{.\beta}a^\beta(R) \tag{17}$$

of the four-potential follows from the Poincaré transformed four-current

$$\hat{j}^\alpha(\hat{R}) = \Lambda^\alpha_{.\beta}j^\beta(R) \tag{18}$$

by the relation

$$\hat{a}^\alpha(R) = c^{-1}\int G^{\alpha\beta}(R, R')\hat{j}_\beta(R')\mathrm{d}^4R' \tag{19}$$

with the *same* Green function as in (14). With $\hat{R}$ and $\hat{R}'$ instead of R and R' one has for (19) with (17) and (18)

$$\Lambda^\alpha_{.\beta}a^\beta(R) = c^{-1}\int G^{\alpha\beta}(\hat{R}, \hat{R}')\Lambda_{\beta\gamma}j^\gamma(R')\mathrm{d}^4\hat{R}'. \tag{20}$$

Inserting (14) in the left-hand side, and introducing the new integration variable R' instead of $\hat{R}'$ at the right-hand side (the Jacobian of this transformation is unity) gives, since j^α is arbitrary:

$$G^{\alpha\beta}(\hat{R}, \hat{R}') = \Lambda^\alpha_{.\gamma}\Lambda^\beta_{.\delta}G^{\gamma\delta}(R, R'), \tag{21}$$

where we used the property $\Lambda^{\alpha\beta}\Lambda_{\alpha\gamma} = g^\beta_\gamma$ or alternatively $\Lambda^{\alpha\beta} = (\Lambda^{-1})^{\beta\alpha}$. In particular for pure translations one finds

$$G^{\alpha\beta}(R+c, R'+c) = G^{\alpha\beta}(R, R'), \tag{22}$$

so that $G^{\alpha\beta}(R, R')$ depends only on the difference $R-R'$:

$$G^{\alpha\beta}(R, R') = G^{\alpha\beta}(R-R'). \tag{23}$$

For Lorentz transformations the condition (21), with (23), reads

$$G^{\alpha\beta}(\Lambda\cdot R) = \Lambda^\alpha_{.\gamma}\Lambda^\beta_{.\delta}G^{\gamma\delta}(R). \tag{24}$$

To solve equation (15) with (23) we take its Fourier transform

$$k^2\tilde{G}^{\alpha\beta}(k) = g^{\alpha\beta}, \tag{25}$$

where we employed the Fourier transforms $\tilde{G}^{\alpha\beta}(k)$ and 1 of the Green function and the delta function according to

$$G^{\alpha\beta}(R-R') = \frac{1}{(2\pi)^4}\int \tilde{G}^{\alpha\beta}(k)\mathrm{e}^{ik\cdot(R-R')}\mathrm{d}^4k, \tag{26}$$

$$\delta^{(4)}(R-R') = \frac{1}{(2\pi)^4}\int \mathrm{e}^{ik\cdot(R-R')}\mathrm{d}^4k. \tag{27}$$

The Fourier transform of the relation (24) is

$$\tilde{G}^{\alpha\beta}(\Lambda\cdot k) = \Lambda^\alpha_{.\gamma}\Lambda^\beta_{.\delta}\tilde{G}^{\gamma\delta}(k). \tag{28}$$

Hence the Fourier transform $\tilde{G}^{\alpha\beta}(k)$ is a tensor which depends only on the vector k^α, on invariant tensors (as $g^{\alpha\beta}$) and scalars. This means that the solution of (25) has the form

$$\tilde{G}^{\alpha\beta}(k) = \frac{1}{k^2}g^{\alpha\beta} + \{\lambda^+\theta(k)+\lambda^-\theta(-k)\}g^{\alpha\beta}\delta(k^2) + \{\mu^+\theta(k)+\mu^-\theta(-k)\}k^\alpha k^\beta\delta(k^2), \tag{29}$$

where $\theta(k) = 1$ if $k^0 \geqslant 0$ and 0 if $k^0 < 0$ and where λ^+, λ^-, μ^+ and μ^- are arbitrary constants. (Since the Lorentz transformation $\Lambda^{\alpha\beta}$ in (28) did not contain inversions we could not exclude the possibility that $\tilde{G}^{\alpha\beta}$ has different values on the positive and negative parts of the light cone in k-space ($k^2 = 0$).) The Green function $G^{\alpha\beta}(R-R')$ follows now by substitution of (29) into (26). The corresponding four-potential is then obtained with (14) and (23), and the field with (10). Then the contribution due to the 'longitudinal' term with $k^\alpha k^\beta$ in (29) drops out. For that reason it may be suppressed from now on, and we write the Green function as

$$G^{\alpha\beta}(R-R') = g^{\alpha\beta}G(R-R') \tag{30}$$

with the abbreviation

$$G(R) = \frac{1}{(2\pi)^4}\int \left[\frac{1}{k^2} + \{\lambda^+\theta(k)+\lambda^-\theta(-k)\}\delta(k^2)\right] e^{ik\cdot R}\mathrm{d}^4k. \tag{31}$$

With (23), (30) and (31) inserted into (14), we have obtained the general form of the particular solution that satisfies the requirements of linearity and covariance (without the longitudinal terms). It contains two arbitrary constants, which may be chosen at will.

Since the first term of the integrand of (31) has poles if $k^2 = 0$, we need a prescription for the treatment of these poles. Since the prescription must be invariant ($G(R)$ itself has to be invariant) all poles on the positive part of the light cone ($k^0 = |\boldsymbol{k}|$) should be treated in the same way, and likewise all poles on the negative part ($k^0 = -|\boldsymbol{k}|$). If various (invariant) prescriptions for the integration of the first term are used, one obtains results which differ from each other by a multiple of the residues. Exactly such a multiple of residues is obtained if one performs the integration over k^0 in the second and third parts of $G(R)$. For that reason taking all possible (invariant) prescriptions for the integration and omitting the λ^+- and λ^--terms is equivalent to taking one invariant prescription, but maintaining the λ^+- and λ^--terms with arbitrary values for these parameters. One may thus write instead of (31):

$$G(R) = \frac{1}{(2\pi)^4}\int_C \frac{1}{k^2}\, e^{ik\cdot R}\mathrm{d}^4k \tag{32}$$

with an arbitrary invariant integration contour C in the complex k-space.

A prescription (of which we shall show that it is indeed invariant) consists in replacing k^2 by $(k+i\varepsilon)^2$ or by $(k-i\varepsilon)^2$, where ε^α is a time-like infinitesimal four-vector with positive time-component ε^0, and integrating along the real axes k^0, k^1, k^2 and k^3. The resulting Green functions will be labelled by the indices r and a respectively:

$$G_{\mathrm{r,a}}(R) = \frac{1}{(2\pi)^4}\int \frac{1}{(k\pm i\varepsilon)^2}\, e^{ik\cdot R}\mathrm{d}^4k. \tag{33}$$

For both signs the integral is real (since it is seen to be equal to its complex conjugate).

The poles of the integrand lie at

$$k^0 = |\boldsymbol{k}| \mp i\left(\varepsilon^0 - \frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{k}}{|\boldsymbol{k}|}\right) \tag{34}$$

and at

$$k^0 = -|\boldsymbol{k}| \mp i\left(\varepsilon^0 + \frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{k}}{|\boldsymbol{k}|}\right). \tag{35}$$

Since ε^α is a time-like vector with $\varepsilon^0 > 0$, the factors between brackets are both infinitesimally positive, independent of the precise values of the components of ε^α and $\boldsymbol{k}$.

The integral (33) may now be calculated by performing first the integration over k^0. For the upper sign, which we shall consider first, the poles (34) and (35) lie just below the real k^0-axis in the neighbourhood of $|\boldsymbol{k}|$ and $-|\boldsymbol{k}|$. For $R^0 < 0$ one may close the contour by a semi-circle in the upper part of the complex k^0-plane. Since then no poles are surrounded by the contour the result is zero, i.e.

$$G_{\mathrm{r}}(R) = 0 \qquad (\text{if } R^0 < 0). \tag{36}$$

For $R^0 > 0$ one closes the contour by a semi-circle in the lower part of the complex k^0-plane. Cauchy's theorem then gives

$$G_{\mathrm{r}}(R) = -\frac{i}{2(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\, \frac{e^{i|\boldsymbol{k}|R^0} - e^{-i|\boldsymbol{k}|R^0}}{|\boldsymbol{k}|} \qquad (\text{if } R^0 > 0). \tag{37}$$

Performing the integration (first over the angles and then over the absolute value of $\boldsymbol{k}$) one gets thus

$$G_{\mathrm{r}}(R) = \frac{1}{4\pi|\boldsymbol{R}|}\,\delta(R^0 - |\boldsymbol{R}|) \qquad (\text{if } R^0 > 0). \tag{38}$$

Combining (36) and (38) we have as the Green function G_{r} for all R^0:

$$G_{\mathrm{r}}(R) = \frac{1}{4\pi|\boldsymbol{R}|}\,\delta(R^0 - |\boldsymbol{R}|). \tag{39}$$

From the derivation it is apparent that the precise position of the poles in the complex k^0-plane is irrelevant, provided that they lie below the real axis and infinitesimally near to $k^0 = +|\boldsymbol{k}|$ and $-|\boldsymbol{k}|$. From (34) and (35) it then follows that the Green function $G_{\mathrm{r}}(R)$ given by (33) has the same value for all infinitesimal time-like ε^α with $\varepsilon^0 > 0$. Hence the prescription for the integration was indeed an invariant prescription, since all such ε^α transform into each other under Lorentz transformations.

From (14) with (23) and (30) it follows that with the choice of the Green function (39) the potential only depends on the charge-current at time-space points which are earlier than the observer's time ($R'^0 \equiv ct' < ct \equiv R^0$). For this reason the Green function G_r in question is said to have retarded character.

The Green function $G_a(R)$ (33) is found in a similar way (lower signs in (33–35)):

$$G_a(R) = \frac{1}{4\pi|\boldsymbol{R}|}\,\delta(R^0+|\boldsymbol{R}|). \tag{40}$$

It has the property to lead to advanced potentials, since it vanishes if $R^0 > 0$.

From the property for delta functions

$$\delta\{f(x)\} = \sum_n \frac{1}{|\partial f/\partial x|}\,\delta(x-x_n), \tag{41}$$

with x_n the (simple) roots of $f(x) = 0$, it follows that

$$\delta(R^2) = \frac{1}{2|\boldsymbol{R}|}\{\delta(R^0-|\boldsymbol{R}|)+\delta(R^0+|\boldsymbol{R}|)\}, \tag{42}$$

and hence, with $\theta(R) = 1$ for $R^0 \geqslant 0$ and $\theta(R) = 0$ for $R^0 < 0$,

$$\delta(R^2)\theta(\pm R) = \frac{1}{2|\boldsymbol{R}|}\,\delta(R^0\mp|\boldsymbol{R}|). \tag{43}$$

Then the retarded and advanced Green functions (39) and (40) may be written in the form

$$G_{r,a}(R) = \frac{1}{2\pi}\,\delta(R^2)\theta(\pm R), \tag{44}$$

which shows explicitly their invariant character.

The integration prescription, contained in (33), is such that the contour passes either below both poles, or above both poles in the complex k^0-plane, as (34) and (35) show. The only other independent invariant integration prescription consists in letting the contour pass below one of the poles and above the other one. This is achieved if in (32) one writes $k^2-i\varepsilon$ for k^2:

$$G_f(R) = \frac{1}{(2\pi)^4}\int \frac{1}{k^2-i\varepsilon}\,e^{ik\cdot R}d^4k. \tag{45}$$

The choice of $+i\varepsilon$ in the denominator leads to a Green function

$$G_{af}(R) = \frac{1}{(2\pi)^4}\int \frac{1}{k^2+i\varepsilon}\,e^{ik\cdot R}d^4k, \tag{46}$$

which depends on the other three. In fact from (33), (45) and (46) it may be proved that the following linear relation between the four Green functions exists:

$$G_r+G_a = G_f+G_{af}. \tag{47}$$

(Inspection of the various contours shows immediately the validity of this connexion.)

The evaluation of the Green functions (45) and (46) proceeds in the same fashion as before, by closing the contours (distinguishing between $R^0 > 0$ and $R^0 < 0$) and applying Cauchy's theorem. This leads to the results

$$G_f(R) = \frac{i}{4\pi^2}\,\frac{1}{R^2+i\varepsilon}, \tag{48}$$

$$G_{af}(R) = -\frac{i}{4\pi^2}\,\frac{1}{R^2-i\varepsilon}. \tag{49}$$

The latter function is the complex conjugate of the former, as is also visible in (45) and (46).

The most general invariant Green function is a linear combination of the independent functions G_r, G_a and G_f, i.e.

$$G = \alpha G_r+\beta G_a+(1-\alpha-\beta)G_f, \tag{50}$$

where α and β are complex constants. The sum of the three coefficients has to be equal to 1 in order to get a solution of the inhomogeneous equation (15) with (23) and (30).

Since the vector potential and the four-current are both real functions, we want to confine ourselves from now on to invariant Green functions which are real. This imposes conditions on the complex constants in (50). In fact one finds from the reality condition $G = G^*$ and the relations $G_r^* = G_r$, $G_a^* = G_a$ and $G_f^* = G_{af} = G_r+G_a-G_f$ that

$$\operatorname{re}\alpha+\operatorname{re}\beta = 1, \qquad \operatorname{im}\alpha-\operatorname{im}\beta = 0. \tag{51}$$

The general form of the real invariant Green function is hence (with the notations $\xi \equiv \operatorname{re}\alpha$ and $\eta \equiv 2\operatorname{im}\alpha$)

$$G = G^* = \xi G_r+(1-\xi)G_a+\eta\operatorname{im}G_f, \tag{52}$$

where ξ and η are arbitrary real constants. The function im G_{f} follows from (48) with the identity (for real x)

$$\frac{1}{x \pm i\varepsilon} = \mathscr{P}\frac{1}{x} \mp i\pi\delta(x) \tag{53}$$

($\mathscr{P}$ indicates the principal value). One finds then

$$\text{im } G_{\mathrm{f}} = \frac{1}{4\pi^2}\mathscr{P}\frac{1}{R^2}. \tag{54}$$

With this expression and (44) the general form (52) of the real invariant Green function becomes

$$G(R) = G^*(R) = \frac{1}{2\pi}\delta(R^2)\{\xi\theta(R)+(1-\xi)\theta(-R)\}+\eta\mathscr{P}\frac{1}{R^2}, \tag{55}$$

(a factor $(4\pi^2)^{-1}$ has been absorbed into the coefficient η).

The particular solutions of the (inhomogeneous) wave equation (13) that follow from (14) with (23), (30) and (55) read

$$a^\alpha(R) = c^{-1}\int\left[\frac{1}{2\pi}\delta\{(R-R')^2\}\{\xi\theta(R-R')+(1-\xi)\theta(-R+R')\}\right.$$
$$\left.+\eta\mathscr{P}\frac{1}{(R-R')^2}\right]j^\alpha(R')\mathrm{d}^4R'. \tag{56}$$

The general solution of the wave equation may be obtained by adding the general solution of the homogeneous equation to this expression with an arbitrary but fixed choice for ξ and η. As to the expression (56), it represents, as we have seen, the general solution of the inhomogeneous equation subject to the conditions that it be linear in the source j^α and connected with the latter by means of an invariant, real Green function. In the following we shall be concerned with these solutions only. They still contain two parameters, which we shall now fix with the help of a further requirement.

In the three terms of (56) the space-time points R' of the source and R of the observer are related in three different ways as a consequence of the different properties of the three Green functions. In fact the first term contains the retarded Green function $G_{\mathrm{r}}(R-R')$ (44) which ensures that the signal from the source travels with the speed of light and reaches the observer at a later time. The second term, with the advanced Green function $G_{\mathrm{a}}(R-R')$ (44), gives a contribution to the potential at an observer's time earlier than the source term. The third term is an integral over the whole of four-space, except for the light cone; it contains even source points R' which are at a space-like distance from the observer's point R. If one wants to exclude the acausal effects described by the last two terms, one has to choose the parameters $\xi = 1$ and $\eta = 0$. Then one arrives at the retarded linear solution, with an invariant and real Green function, of the inhomogeneous wave equation:

$$a_{\mathrm{r}}^\alpha(R) = \frac{c^{-1}}{2\pi}\int\delta\{(R-R')^2\}\theta(R-R')j^\alpha(R')\mathrm{d}^4R'. \tag{57}$$

Inserting the expression (5) for the four-current we obtain

$$a_{\mathrm{r}}^\alpha(R) = \frac{1}{2\pi}\sum_j e_j\int u_j^\alpha(s_j)\delta[\{R-R_j(s_j)\}^2]\theta\{(R-R_j(s_j)\}\mathrm{d}s_j, \tag{58}$$

with the abbreviation $u_j^\alpha(s_j) \equiv \mathrm{d}R_j^\alpha(s_j)/\mathrm{d}s_j$.

Up to now the Lorentz condition has not been imposed explicitly on the solution. However one may verify that the solution (58) as it stands satisfies the Lorentz condition (12).

The retarded fields follow by insertion of (58) into (10):

$$f_{\mathrm{r}}^{\alpha\beta}(R) = \frac{1}{2\pi}\sum_j e_j\int\{u_j^\beta(s_j)\partial^\alpha-u_j^\alpha(s_j)\partial^\beta\}\delta[\{R-R_j(s_j)\}^2]\theta\{R-R_j(s_j)\}\mathrm{d}s_j. \tag{59}$$

For future use in calculations we shall also need the advanced potentials and fields. They follow from (57–59) by replacing $\theta(x)$ by $\theta(-x)$. Half the sum and half the difference of the retarded and advanced fields are conventionally called the 'plus' and 'minus' fields. (The former is a solution of the field equations with sources, whereas the latter is a solution of the source-free field equations.) They are obtained by replacing $\theta(x)$ in (57–59) by $\frac{1}{2}\{\theta(x)\pm\theta(-x)\}$, i.e. by $\frac{1}{2}$ and $\frac{1}{2}\varepsilon(x)$ respectively ($\varepsilon(x) = 1$ for $x^0 \geqslant 0$, $\varepsilon(x) = -1$ for $x^0 < 0$). One should note in this connexion that although we shall often treat the retarded and advanced fields on the same footing, only the first will represent the physical fields, while the latter (and also the plus and minus fields) will only serve as mathematical ancillaries.

c. *Expansion of the retarded and advanced potentials and fields into powers of c^{-1}*

The retarded potential (58) and the corresponding advanced one may be written in an alternative form, if we use the identity (43):

$$a_{\mathrm{r,a}}^\alpha(\boldsymbol{R}, t) = \sum_j e_j\int u_j^\alpha(s_j)\frac{\delta\{R^0-R_j^0(s_j)\mp|\boldsymbol{R}-\boldsymbol{R}_j(s_j)|\}}{4\pi|\boldsymbol{R}-\boldsymbol{R}_j(s_j)|}\mathrm{d}s_j \tag{60}$$

with $u_j^\alpha(s_j) \equiv \mathrm{d}R_j^\alpha/\mathrm{d}s_j$. Choosing in particular as the parametrization of the world lines their time $t_j \equiv c^{-1}R_j^0$, we obtain for the retarded and advanced scalar and vector potentials $(\varphi, \boldsymbol{a}) = a^\alpha$:

$$\begin{aligned}\varphi_{\mathrm{r,a}}(\boldsymbol{R}, t) &= \sum_j e_j \int \frac{\delta\{t-t_j \mp c^{-1}|\boldsymbol{R}-\boldsymbol{R}_j(t_j)|\}}{4\pi|\boldsymbol{R}-\boldsymbol{R}_j(t_j)|}\,\mathrm{d}t_j,\\ \boldsymbol{a}_{\mathrm{r,a}}(\boldsymbol{R}, t) &= \sum_j e_j \int \boldsymbol{\beta}_j(t_j)\frac{\delta\{t-t_j \mp c^{-1}|\boldsymbol{R}-\boldsymbol{R}_j(t_j)|\}}{4\pi|\boldsymbol{R}-\boldsymbol{R}_j(t_j)|}\,\mathrm{d}t_j\end{aligned} \tag{61}$$

due to a set of point sources. Here $c\boldsymbol{\beta}_j(t_j)$ stands for the velocity $\mathrm{d}\boldsymbol{R}_j(t_j)/\mathrm{d}t_j$.

Let us consider in the following the potentials due to a single particle, of which for convenience we denote the charge as e instead of e_j and the velocity as $c\boldsymbol{\beta}$ instead of $c\boldsymbol{\beta}_j$. If we furthermore employ the abbreviations $\boldsymbol{r} = \boldsymbol{R}-\boldsymbol{R}_j$ and $r = |\boldsymbol{r}|$ and write t' instead of t_j for the integration variable, we have the potentials

$$\begin{aligned}\varphi_{\mathrm{r,a}}(\boldsymbol{R}, t) &= e\int \frac{1}{4\pi r(t')}\,\delta\left\{t-t' \mp \frac{r(t')}{c}\right\}\mathrm{d}t',\\ \boldsymbol{a}_{\mathrm{r,a}}(\boldsymbol{R}, t) &= e\int \frac{\boldsymbol{\beta}(t')}{4\pi r(t')}\,\delta\left\{t-t' \mp \frac{r(t')}{c}\right\}\mathrm{d}t'.\end{aligned} \tag{62}$$

The delta function may be expanded in a Taylor series

$$\delta\left\{t-t' \mp \frac{r(t')}{c}\right\} = \sum_{n=0}^{\infty} \frac{\{\pm r(t')\}^n}{c^n n!}\frac{\partial^n \delta(t-t')}{\partial t'^n}. \tag{63}$$

Substitution into (62) and partial integration over t' yield the expansions:

$$\varphi(\boldsymbol{R}, t) = \sum_{n=0}^{\infty} \varphi^{(n)}(\boldsymbol{R}, t), \qquad \boldsymbol{a}(\boldsymbol{R}, t) = \sum_{n=1}^{\infty} \boldsymbol{a}^{(n)}(\boldsymbol{R}, t), \tag{64}$$

with the partial potentials $\varphi^{(n)}$ and $\boldsymbol{a}^{(n)}$ of order c^{-n} given by:

$$\begin{aligned}\varphi_{\mathrm{r,a}}^{(n)}(\boldsymbol{R}, t) &= \frac{e}{4\pi}\frac{(\mp 1)^n}{c^n n!}\frac{\partial^n r^{n-1}}{\partial t^n}, \qquad (n = 0, 1, \ldots),\\ \boldsymbol{a}_{\mathrm{r,a}}^{(n+1)}(\boldsymbol{R}, t) &= \frac{e}{4\pi}\frac{(\mp 1)^n}{c^n n!}\frac{\partial^n (r^{n-1}\boldsymbol{\beta})}{\partial t^n}, \qquad (n = 0, 1, \ldots).\end{aligned} \tag{65}$$

In this way 'synchronous' expressions for the potentials in the form of power series in c^{-1} have been obtained. From these expressions (in which $\partial/\partial t$ denotes the sum of an explicit time derivative and an implicit one of the form $-c\boldsymbol{\beta}\cdot\nabla$) it follows that the power series in c^{-1} are in fact expansions with respect to a set of dimensionless parameters, namely the 'retardation time' $r/c \equiv t_{\mathrm{r}}$ multiplied by $c\beta/r$, by $\dot{\beta}/\beta$, by $\ddot{\beta}/\beta$, etc. These parameters are of two types: the first is simply β and thus independent of the retardation time, while the others have the general form of the retardation time divided by a characteristic time of the (accelerated) motion of the source. Thus the series may be broken off if the velocity of the source and moreover the distance between source and observer are not too large. (Sometimes it may be useful to consider separately expansions with respect to one of the parameters only, then the others need not be limited in magnitude.)

The potentials satisfy the Lorentz gauge condition, which reads for the partial potentials

$$\partial_0 \varphi^{(n)} + \nabla\cdot\boldsymbol{a}^{(n+1)} = 0, \qquad (n = 0, 1, \ldots) \tag{66}$$

and for the total potentials

$$\partial_0 \varphi + \nabla\cdot\boldsymbol{a} = 0. \tag{67}$$

If the lowest orders of (65) are evaluated one finds the expressions:

$$\begin{aligned}\varphi_{\mathrm{r,a}}^{(0)} &= \frac{e}{4\pi r},\\ \varphi_{\mathrm{r,a}}^{(1)} &= 0,\\ \varphi_{\mathrm{r,a}}^{(2)} &= \frac{e}{4\pi r}\left\{\tfrac{1}{2}\beta^2 - \tfrac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{\beta})^2 - \frac{r}{2c}(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})\right\},\\ \varphi_{\mathrm{r,a}}^{(3)} &= \pm\frac{e}{4\pi r}\left\{-\frac{r}{c}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) + \frac{r^2}{3c^2}(\boldsymbol{n}\cdot\ddot{\boldsymbol{\beta}})\right\},\\ \boldsymbol{a}_{\mathrm{r,a}}^{(1)} &= \frac{e\boldsymbol{\beta}}{4\pi r},\\ \boldsymbol{a}_{\mathrm{r,a}}^{(2)} &= \mp\frac{e\dot{\boldsymbol{\beta}}}{4\pi c},\\ \boldsymbol{a}_{\mathrm{r,a}}^{(3)} &= \frac{e\boldsymbol{\beta}}{4\pi r}\left\{\tfrac{1}{2}\beta^2 - \tfrac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{\beta})^2 - \frac{r}{2c}(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})\right\} - \frac{e\dot{\boldsymbol{\beta}}(\boldsymbol{n}\cdot\boldsymbol{\beta})}{4\pi c} + \frac{er\ddot{\boldsymbol{\beta}}}{8\pi c^2}.\end{aligned} \tag{68}$$

The fields follow from these potentials. We write the result as

$$\boldsymbol{e}(\boldsymbol{R}, t) = \sum_{n=0}^{\infty} \boldsymbol{e}^{(n)}(\boldsymbol{R}, t), \qquad \boldsymbol{b}(\boldsymbol{R}, t) = \sum_{n=0}^{\infty} \boldsymbol{b}^{(n)}(\boldsymbol{R}, t) \tag{69}$$

with partial fields $\boldsymbol{e}^{(n)}$ and $\boldsymbol{b}^{(n)}$ of order c^{-n} given by the following connexions with the partial potentials:

$$e^{(n)} = -\nabla\varphi^{(n)} - \partial_0 a^{(n-1)}, \qquad b^{(n)} = \nabla \wedge a^{(n)}, \qquad (n = 0, 1, \ldots), \tag{70}$$

where, by definition, $a^{(-1)}$ and $a^{(0)}$ vanish. They read with (65) inserted (for $n = 0, 1, \ldots$)

$$\begin{aligned} e^{(n)}_{\mathrm{r,a}} &= -\frac{e}{4\pi}\frac{(\mp 1)^n}{c^n n!}\nabla\frac{\partial^n r^{n-1}}{\partial t^n} - \frac{e}{4\pi}\frac{(\mp 1)^n}{c^{n-1}(n-2)!}\frac{\partial^{n-2}(r^{n-3}\boldsymbol{\beta})}{\partial t^{n-2}}, \\ b^{(n)}_{\mathrm{r,a}} &= \frac{e}{4\pi}\frac{(\mp 1)^{n-1}}{c^{n-1}(n-1)!}\nabla\wedge\frac{\partial^{n-1}(r^{n-2}\boldsymbol{\beta})}{\partial t^{n-1}}. \end{aligned} \tag{71}$$

The lowest orders are explicitly

$$\begin{aligned} e^{(0)}_{\mathrm{r,a}} &= \frac{en}{4\pi r^2}, \\ e^{(1)}_{\mathrm{r,a}} &= 0, \\ e^{(2)}_{\mathrm{r,a}} &= \frac{en\beta^2}{8\pi r^2} - \frac{3en(n\cdot\beta)^2}{8\pi r^2} - \frac{en(n\cdot\dot{\beta})}{8\pi rc} - \frac{e\dot{\beta}}{8\pi rc}, \\ e^{(3)}_{\mathrm{r,a}} &= \pm\frac{e\ddot{\beta}}{6\pi c^2}, \\ b^{(0)}_{\mathrm{r,a}} &= 0, \\ b^{(1)}_{\mathrm{r,a}} &= \frac{e\beta\wedge n}{4\pi r^2}, \\ b^{(2)}_{\mathrm{r,a}} &= 0, \\ b^{(3)}_{\mathrm{r,a}} &= \frac{e\beta\wedge n}{8\pi r^2}\left\{\beta^2 - 3(n\cdot\beta)^2 - \frac{r}{c}(n\cdot\dot{\beta})\right\} - \frac{e\dot{\beta}\wedge n(n\cdot\beta)}{4\pi rc} + \frac{e\dot{\beta}\wedge\beta}{8\pi rc} - \frac{e\ddot{\beta}\wedge n}{8\pi c^2}. \end{aligned} \tag{72}$$

The partial advanced fields are related to the partial retarded fields by:

$$e^{(n)}_{\mathrm{a}} = (-1)^n e^{(n)}_{\mathrm{r}}, \qquad b^{(n)}_{\mathrm{a}} = (-1)^{n+1} b^{(n)}_{\mathrm{r}}. \tag{73}$$

Since $e^{(1)}$, $b^{(0)}$ and $b^{(2)}$ vanish, the lowest orders in which retarded and advanced fields differ are $n = 3$ for the electric field and $n = 4$ for the magnetic field.

The plus and minus fields, defined as half the sum and difference of the retarded and advanced fields, may also be written as series in c^{-1}:

$$e_{\pm} = \sum_{n=0}^{\infty} e^{(n)}_{\pm}, \qquad b_{\pm} = \sum_{n=0}^{\infty} b^{(n)}_{\pm} \tag{74}$$

with the partial fields

$$\begin{aligned} e^{(n)}_{\pm} &= \tfrac{1}{2}\{1 \pm (-1)^n\} e^{(n)}_{\mathrm{r}}, \\ b^{(n)}_{\pm} &= \tfrac{1}{2}\{1 \pm (-1)^{n+1}\} b^{(n)}_{\mathrm{r}}. \end{aligned} \tag{75}$$

In particular it follows thus that the lowest order minus fields different from zero are $n = 3$ for the electric field and $n = 4$ for the magnetic field.

In the non-relativistic theory, when only terms up to order c^{-1} are taken into account, only the partial fields $e^{(0)}$ and $b^{(1)}$ occur. Then no retardation effects are included, as is also manifest from the fact that the retarded and advanced fields are the same in this order (or, in other words, the minus field vanishes then).

In the preceding we employed potentials which satisfy the Lorentz condition (66). The field may be found alternatively from potentials φ' and a' in a different gauge. The partial potentials are then related to the Lorentz partial potentials (65) as

$$\begin{aligned} \varphi'^{(n)} &= \varphi^{(n)} - \partial_0 \psi^{(n-1)}, \\ a'^{(n)} &= a^{(n)} + \nabla\psi^{(n)}, \end{aligned} \tag{76}$$

with an arbitrary gauge function

$$\psi = \sum_n \psi^{(n)}. \tag{77}$$

The potentials (76) give the same fields as before, as follows from (70).

In particular one may require for the vector potential: $\nabla\cdot a' = 0$, or

$$\nabla\cdot a'^{(n)} = 0. \tag{78}$$

This defines the 'Coulomb gauge'. Quantities in this gauge will be indicated by an index (C). From the second line of (76) and (78) we get the differential equations

$$\Delta\psi^{(n)} = -\nabla\cdot a^{(n)} \tag{79}$$

or with the partial potentials (65) inserted:

$$\Delta\psi^{(n)}_{\mathrm{r,a}} = \frac{e}{4\pi}\frac{(\mp 1)^{n+1}}{c^n(n-1)!}\frac{\partial^n r^{n-2}}{\partial t^n}. \tag{80}$$

This equation has as a solution

$$\psi^{(n)}_{\mathrm{r,a}} = \frac{e}{4\pi}\frac{(\mp 1)^{n+1}}{c^n(n+1)!}\frac{\partial^n r^n}{\partial t^n}, \qquad (n = 0, 1, \ldots) \tag{81}$$

(for negative values of n the function vanishes by definition). The Coulomb gauge potentials now follow by insertion of (65) and (81) into (76):

$$\varphi^{(0)}_{(\mathrm{C})\mathrm{r,a}} = \frac{e}{4\pi r}, \qquad \varphi^{(n)}_{(\mathrm{C})\mathrm{r,a}} = 0, \qquad (n = 1, 2, \ldots),$$

$$\boldsymbol{a}^{(n+1)}_{(\mathrm{C})\mathrm{r,a}} = \frac{e}{4\pi}\left\{\frac{(\mp 1)^n}{c^n n!}\frac{\partial^n(r^{n-1}\boldsymbol{\beta})}{\partial t^n} + \nabla\frac{(\mp 1)^n}{c^{n+1}(n+2)!}\frac{\partial^{n+1}r^{n+1}}{\partial t^{n+1}}\right\}, \quad (n = 0, 1, 2, \ldots). \tag{82}$$

The partial potentials of lowest order read explicitly

$$\begin{aligned}
\varphi^{(0)}_{(\mathrm{C})\mathrm{r,a}} &= \frac{e}{4\pi r},\\
\varphi^{(n)}_{(\mathrm{C})\mathrm{r,a}} &= 0, \qquad (n = 1, 2, \ldots),\\
\boldsymbol{a}^{(1)}_{(\mathrm{C})\mathrm{r,a}} &= \frac{e\{\boldsymbol{\beta}+\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\beta})\}}{8\pi r},\\
\boldsymbol{a}^{(2)}_{(\mathrm{C})\mathrm{r,a}} &= \mp\frac{e\dot{\boldsymbol{\beta}}}{6\pi c},\\
\boldsymbol{a}^{(3)}_{(\mathrm{C})\mathrm{r,a}} &= -\frac{3}{8}\frac{e\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\beta})^3}{4\pi r} - \frac{1}{8}\frac{e\boldsymbol{\beta}(\boldsymbol{n}\cdot\boldsymbol{\beta})^2}{4\pi r} + \frac{3}{8}\frac{e\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\beta})\beta^2}{4\pi r} + \frac{1}{8}\frac{e\boldsymbol{\beta}\beta^2}{4\pi r}\\
&\quad -\frac{3}{8}\frac{e\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\beta})(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})}{4\pi c} - \frac{1}{8}\frac{e\boldsymbol{\beta}(\boldsymbol{n}\cdot\boldsymbol{\beta})}{4\pi c} + \frac{3}{8}\frac{e\boldsymbol{n}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})}{4\pi c} - \frac{1}{8}\frac{e\boldsymbol{n}(\boldsymbol{n}\cdot\ddot{\boldsymbol{\beta}})r}{4\pi c^2}\\
&\quad -\frac{5}{8}\frac{e\dot{\boldsymbol{\beta}}(\boldsymbol{n}\cdot\boldsymbol{\beta})}{4\pi c} + \frac{3}{8}\frac{e\ddot{\boldsymbol{\beta}}r}{4\pi c^2}.
\end{aligned} \tag{83}$$

The potentials used in the well-known *Darwin Lagrangian* are the same as $\varphi^{(0)}_{(\mathrm{C})}$ and $\boldsymbol{a}^{(1)}_{(\mathrm{C})}$ given above. From (82) it follows now that, since $\varphi^{(1)}_{(\mathrm{C})}$ and $\varphi^{(2)}_{(\mathrm{C})}$ vanish and $\boldsymbol{a}^{(2)}_{(\mathrm{C})}$ is independent of $\boldsymbol{R}$, one finds from the 'Darwin potentials' the fields with $n = 0, 1, 2$. In other words *the fields* $\boldsymbol{e}$ *and* $\boldsymbol{b}$ *which play a role in the Darwin approximation are correct up to order* c^{-2} (v. problem 6).

d. *The Liénard–Wiechert potentials and fields*

The retarded and advanced four-potentials of a single particle with charge e, four-position $R^\alpha_1(s)$ (with s an arbitrary parameter along the world line of the particle, which increases as a monotonic function with time) and the derivative $u^\alpha(s) = \mathrm{d}R^\alpha_1/\mathrm{d}s$ are (cf. 58)):

$$a^\alpha_{\mathrm{r,a}}(R) = \frac{1}{2\pi}e\int u^\alpha(s)\delta[\{R-R_1(s)\}^2]\theta[\pm\{R-R_1(s)\}]\mathrm{d}s. \tag{84}$$

This integral may be calculated if the delta function is written in an alternative form by making use of the property (41). Denoting the two roots of the light

cone equation

$$\{R-R_1(s)\}^2 = 0 \tag{85}$$

by s_r and s_a (where $R^0-R^0_1(s_{\mathrm{r,a}}) \gtrless 0$), we have for $R \neq R_1(s_{\mathrm{r,a}})$:

$$\delta[\{R-R_1(s)\}^2] = \frac{1}{2|u(s_\mathrm{r})\cdot\{R-R_1(s_\mathrm{r})\}|}\delta(s-s_\mathrm{r}) + \frac{1}{2|u(s_\mathrm{a})\cdot\{R-R_1(s_\mathrm{a})\}|}\delta(s-s_\mathrm{a}). \tag{86}$$

Hence we have

$$\delta[\{R-R_1(s)\}^2]\theta[\pm\{R-R_1(s)\}] = \frac{1}{2|u(s_{\mathrm{r,a}})\cdot\{R-R_1(s_{\mathrm{r,a}})\}|}\delta(s-s_{\mathrm{r,a}}). \tag{87}$$

The potentials (84) then get the form

$$a^\alpha_{\mathrm{r,a}}(R) = \frac{eu^\alpha(s_{\mathrm{r,a}})}{4\pi|u(s_{\mathrm{r,a}})\cdot\{R-R_1(s_{\mathrm{r,a}})\}|}. \tag{88}$$

The expression between the bars is negative and positive for the retarded and advanced solution respectively, since u^α is a time-like vector with a positive zero-component. Therefore we may write (88) alternatively, using moreover the abbreviation $r^\alpha = R^\alpha - R^\alpha_1$, as:

$$a^\alpha_{\mathrm{r,a}} = \mp\left.\frac{eu^\alpha}{4\pi u\cdot r}\right|_{\mathrm{r,a}}, \tag{89}$$

where the bar with the suffixes r, a indicates that one should take the dynamical quantities r^α and u^α at $s = s_\mathrm{r}$ and $s = s_\mathrm{a}$ respectively.

The formula (89) shows again, explicitly, that the parametrization of the world line may be arbitrarily chosen without changing the result, since both the numerator and the denominator contain one differentiation with respect to s. If one chooses in particular the time R^0_1/c of the particle as the parameter one has

$$u^\alpha = c(1, \boldsymbol{\beta}), \qquad (\boldsymbol{\beta} \equiv \mathrm{d}\boldsymbol{R}_1/\mathrm{d}R^0_1) \tag{90}$$

and, since $r^0_{\mathrm{r,a}} = \pm|\boldsymbol{r}| \equiv \pm r$ according to (85),

$$u\cdot r = \mp c\kappa_{\mathrm{r,a}}r, \qquad (\kappa_{\mathrm{r,a}} \equiv 1\mp\boldsymbol{\beta}\cdot\boldsymbol{n}, \quad \boldsymbol{n} \equiv \boldsymbol{r}/r). \tag{91}$$

Then one obtains for the scalar and vector potentials $(\varphi, \boldsymbol{a}) \equiv a^\alpha$:

$$\varphi_{\mathrm{r,a}} = \left.\frac{e}{4\pi\kappa_{\mathrm{r,a}}r}\right|_{\mathrm{r,a}}, \qquad \boldsymbol{a}_{\mathrm{r,a}} = \left.\frac{e\boldsymbol{\beta}}{4\pi\kappa_{\mathrm{r,a}}r}\right|_{\mathrm{r,a}}, \tag{92}$$

which are the expressions of Liénard and Wiechert[1].

[1] A. Liénard, L'éclairage électrique **16**(1898)5, 53, 106; E. Wiechert, Archives néerlandaises **5**(1900)549.

The retarded and advanced fields follow by differentiation of the potentials. From (89) one has

$$\partial^\alpha a^\beta_{\mathrm{r,a}} = \left[\pm \frac{e u^\alpha u^\beta}{4\pi(u\cdot r)^2} \mp \frac{e}{4\pi} \left\{ \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{u^\beta}{u\cdot r} \right) \right\} \partial^\alpha s \right] \Bigg|_{\mathrm{r,a}}. \tag{93}$$

Here the partial derivative of the parameter s follows from differentiation of the light cone equation (85):

$$\partial^\alpha s = \frac{r^\alpha}{u\cdot r}. \tag{94}$$

With (93) and (94) we find for the fields (10):

$$f^{\alpha\beta}_{\mathrm{r,a}} = \mp \frac{e}{4\pi u\cdot r} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{r^\alpha u^\beta - r^\beta u^\alpha}{u\cdot r} \right) \Bigg|_{\mathrm{r,a}}, \tag{95}$$

or, if the differentiation is carried out,

$$f^{\alpha\beta}_{\mathrm{r,a}} = \left\{ \pm \frac{e}{4\pi(u\cdot r)^3} (a\cdot r - u^2)(r^\alpha u^\beta - r^\beta u^\alpha) \mp \frac{e}{4\pi(u\cdot r)^2} (r^\alpha a^\beta - r^\beta a^\alpha) \right\} \Bigg|_{\mathrm{r,a}}, \tag{96}$$

where $a^\alpha \equiv \mathrm{d}u^\alpha/\mathrm{d}s$ (not to be confused with the four-potential).

Choosing for the parameter s the time component R^0_1/c one obtains, with (90), (91) and

$$a^\alpha = c(0, \dot{\boldsymbol{\beta}}), \qquad (\dot{\boldsymbol{\beta}} \equiv c\,\mathrm{d}^2\boldsymbol{R}_1/\mathrm{d}R^{0\,2}_1), \tag{97}$$

for the fields:

$$\begin{aligned} \boldsymbol{e}_{\mathrm{r,a}} &= \frac{e}{4\pi} \left\{ \frac{\boldsymbol{n}\mp\boldsymbol{\beta}}{\gamma^2 \kappa^3_{\mathrm{r,a}} r^2} + \frac{(\boldsymbol{n}\mp\boldsymbol{\beta})\dot{\boldsymbol{\beta}}\cdot\boldsymbol{n}}{c\kappa^3_{\mathrm{r,a}} r} - \frac{\dot{\boldsymbol{\beta}}}{c\kappa^2_{\mathrm{r,a}} r} \right\} \Bigg|_{\mathrm{r,a}}, \\ \boldsymbol{b}_{\mathrm{r,a}} &= \frac{e}{4\pi} \left(\frac{\boldsymbol{\beta}\wedge\boldsymbol{n}}{\gamma^2 \kappa^3_{\mathrm{r,a}} r^2} + \frac{\boldsymbol{\beta}\wedge\boldsymbol{n}\,\dot{\boldsymbol{\beta}}\cdot\boldsymbol{n}}{c\kappa^3_{\mathrm{r,a}} r} \pm \frac{\dot{\boldsymbol{\beta}}\wedge\boldsymbol{n}}{c\kappa^2_{\mathrm{r,a}} r} \right) \Bigg|_{\mathrm{r,a}}, \end{aligned} \tag{98}$$

where $\gamma \equiv (1-\boldsymbol{\beta}^2)^{-\frac{1}{2}}$. These expressions show that the fields consist of two types of terms: one without the acceleration, proportional to $|\boldsymbol{r}|^{-2}$, and another with acceleration, proportional to $|\boldsymbol{r}|^{-1}$ (all taken at the retarded or the advanced times).

e. *The self-field of a charged particle*

In the following we shall need the field due to a point particle with charge e at the position of the particle itself[1]. To that purpose we shall start from expression (96) for the retarded and the advanced fields, which reads

[1] P. A. M. Dirac, Proc. Roy. Soc. A **167**(1938)148.

$$f^{\alpha\beta}_{\mathrm{r,a}} = \left\{ \pm \frac{e}{4\pi(u\cdot r)^3} (a\cdot r + c^2)(r^\alpha u^\beta - r^\beta u^\alpha) \mp \frac{e}{4\pi(u\cdot r)^2} (r^\alpha a^\beta - r^\beta a^\alpha) \right\} \Bigg|_{\mathrm{r,a}}, \tag{99}$$

where we have chosen for the arbitrary parameter s of (96) the proper time along the world line. Furthermore we have $r^\alpha(s) \equiv R^\alpha - R^\alpha_1(s)$, with $R^\alpha_1(s)$ the four-position of the particle and R^α the four-position of the observer, $u^\alpha = \mathrm{d}R^\alpha_1(s)/\mathrm{d}s$ the four-velocity ($u^2 = -c^2$) and $a^\alpha = \mathrm{d}u^\alpha(s)/\mathrm{d}s$ the four-acceleration. The indices r and a denote that one has to take for s the retarded and advanced values s_r and s_a respectively, which satisfy

$$\{R - R_1(s_{\mathrm{r,a}})\}^2 = 0, \qquad R^0 - R^0_1(s_{\mathrm{r,a}}) \gtrless 0. \tag{100}$$

The field (99) will be considered here for positions of the observer

$$R^\alpha = R^\alpha_1(s_1) + \varepsilon n^\alpha, \tag{101}$$

with fixed s_1 and space-like unit vector n^α orthogonal to $u^\alpha(s_1)$:

$$n^2 = 1, \qquad u(s_1)\cdot n = 0. \tag{102}$$

If the parameter ε (> 0) tends to zero, one gets the expression for the field at the position $R^\alpha_1(s_1)$ of the particle. If (101) is substituted into (100) one finds upon a Taylor expansion of $R^\alpha_1(s_{\mathrm{r,a}})$ around $R^\alpha_1(s_1)$ that s_r and s_a are the two roots of the following equation in s:

$$\{\varepsilon n^\alpha - (s-s_1)u^\alpha(s_1) - \tfrac{1}{2}(s-s_1)^2 a^\alpha(s_1) - \tfrac{1}{6}(s-s_1)^3 \dot{a}^\alpha(s_1) + \ldots\}^2 = 0. \tag{103}$$

(The dot indicates a differentiation with respect to s_1.) Solving for s in terms of ε we find the roots s_r and s_a:

$$\begin{aligned} s_{\mathrm{r,a}} - s_1 = \mp c^{-1}\varepsilon[1 - \tfrac{1}{2}c^{-2} a\cdot n\varepsilon + \{\tfrac{3}{8}c^{-4}(a\cdot n)^2 \\ - \tfrac{1}{24}c^{-4}a^2 \pm \tfrac{1}{6}c^{-3}\dot{a}\cdot n\}\varepsilon^2 + \ldots], \end{aligned} \tag{104}$$

where at the right-hand side the quantities a and $\dot{a}$ depend on s_1. By Taylor expansions of those quantities in (99) that depend on s_r or s_a around their values at s_1 and introduction of (104) we get expansions in powers of the parameter ε. In this way one finds, using also (101), the auxiliary formulae

$$\begin{aligned} u(s)\cdot r(s) = \mp c\varepsilon[1 + \tfrac{1}{2}c^{-2} a\cdot n\varepsilon \\ + \{-\tfrac{1}{8}c^{-4}(a\cdot n)^2 + \tfrac{1}{8}c^{-4}a^2 \mp \tfrac{1}{3}c^{-3}\dot{a}\cdot n\}\varepsilon^2 + \ldots], \end{aligned} \tag{105}$$

$$a(s)\cdot r(s) = \varepsilon\{a\cdot n \mp c^{-1}(\dot{a}\cdot n \mp \tfrac{1}{2}c^{-1}a^2)\varepsilon + \ldots\}, \tag{106}$$

$$r^\alpha(s)u^\beta(s)-r^\beta(s)u^\alpha(s) = \varepsilon\{n^\alpha u^\beta+(\mp c^{-1}n^\alpha a^\beta-\tfrac{1}{2}c^{-2}u^\alpha a^\beta)\varepsilon$$
$$+(\pm\tfrac{1}{2}c^{-3}a{\cdot}nn^\alpha a^\beta+\tfrac{1}{2}c^{-2}n^\alpha\dot{a}^\beta+\tfrac{1}{2}c^{-4}a{\cdot}nu^\alpha a^\beta\pm\tfrac{1}{3}c^{-3}u^\alpha\dot{a}^\beta)\varepsilon^2+\dots\}$$
$$-(\alpha,\beta), \quad (107)$$

$$r^\alpha(s)a^\beta(s)-r^\beta(s)a^\alpha(s) = \varepsilon\{n^\alpha a^\beta\pm c^{-1}u^\alpha a^\beta$$
$$+(\mp c^{-1}n^\alpha\dot{a}^\beta\mp\tfrac{1}{2}c^{-3}a{\cdot}nu^\alpha a^\beta-c^{-2}u^\alpha\dot{a}^\beta)\varepsilon+\dots\}-(\alpha,\beta), \quad (108)$$

where (α, β) stands for the preceding terms with α and β interchanged.

With the use of these expressions we obtain for the field (99) up to order ε^0:

$$f^{\alpha\beta}_{\mathrm{r,a}} = -\frac{e}{4\pi}c^{-1}\varepsilon^{-2}[n^\alpha u^\beta+\tfrac{1}{2}c^{-2}(u^\alpha a^\beta-a{\cdot}nn^\alpha u^\beta)\varepsilon$$
$$+c^{-2}\{\tfrac{1}{8}c^{-2}a^2n^\alpha u^\beta-\tfrac{3}{4}c^{-2}a{\cdot}nu^\alpha a^\beta+\tfrac{3}{8}c^{-2}(a{\cdot}n)^2n^\alpha u^\beta$$
$$-\tfrac{1}{2}n^\alpha\dot{a}^\beta\mp\tfrac{2}{3}c^{-1}u^\alpha\dot{a}^\beta\}\varepsilon^2+\dots]-(\alpha,\beta). \quad (109)$$

It is useful to write also separately half the sum and half the difference of the retarded and the advanced fields, i.e. the plus and minus fields. From (109) one has then immediately

$$f^{\alpha\beta}_{+} = -\frac{e}{4\pi}c^{-1}\varepsilon^{-2}[n^\alpha u^\beta+\tfrac{1}{2}c^{-2}(u^\alpha a^\beta-a{\cdot}nn^\alpha u^\beta)\varepsilon$$
$$+c^{-2}\{\tfrac{1}{8}c^{-2}a^2n^\alpha u^\beta-\tfrac{3}{4}c^{-2}a{\cdot}nu^\alpha a^\beta+\tfrac{3}{8}c^{-2}(a{\cdot}n)^2n^\alpha u^\beta$$
$$-\tfrac{1}{2}n^\alpha\dot{a}^\beta\}\varepsilon^2+\dots]-(\alpha,\beta), \quad (110)$$

$$f^{\alpha\beta}_{-} = \frac{e}{4\pi}\tfrac{2}{3}c^{-4}(u^\alpha\dot{a}^\beta-u^\beta\dot{a}^\alpha)+\dots. \quad (111)$$

While the plus field diverges at the world line, the minus field is finite in the neighbourhood of the world line. The minus field at the world line is the part of the self-field that will give rise to the radiation damping force in the equation of motion.

An alternative way to obtain the minus part of the self-field starts from the expressions (69) with (71) for the fields developed in powers of c^{-1}. In the series of problems 7–12 it is shown how the expression (111) may be obtained in this way.

3 *The equation of motion*

a. *A single particle in a field*

If a particle with charge e and mass m is moving in an external field $F^{\alpha\beta}$, it will be subject to a Lorentz force. As a direct generalization of the non-

relativistic law one might write

$$ma^\alpha = c^{-1}eF^{\alpha\beta}u_\beta, \quad (112)$$

where u^α and a^α are the four-velocity $\mathrm{d}R_1^\alpha(s)/\mathrm{d}s$ and the four-acceleration $\mathrm{d}^2R_1^\alpha(s)/\mathrm{d}s^2$ with $R_1^\alpha(s)$ the four-position and s the proper time. The field has to be taken at the four-position $R_1^\alpha(s)$ of the particle.

The equation of motion (112) is certainly not complete, since it does not contain the damping force due to the fact that the particle emits radiation and hence is subject to a recoil force. Such a recoil force may be added *ad hoc*. It is however more illuminating to obtain it starting from the equation

$$m_0a^\alpha = c^{-1}e(F^{\alpha\beta}+f_{\mathrm{r}}^{\alpha\beta})u_\beta. \quad (113)$$

At the left-hand side we have written a constant m_0 – the 'bare mass'. Its relation to the experimental mass m, which has been written in (112), will become apparent from the following. Furthermore at the right-hand side figures the total field which is the sum of the external field $F^{\alpha\beta}$ and the retarded field $f_{\mathrm{r}}^{\alpha\beta}$, generated by the particle itself. The fields have to be taken at the four-position of the particle. One should thus have to substitute for $f_{\mathrm{r}}^{\alpha\beta}$ the expression (109), or the sum of (110) and (111), taken with $\varepsilon\to 0$. The minus field $f_{-}^{\alpha\beta}$ presents no difficulties but the plus field $f_{+}^{\alpha\beta}$ diverges at the world line. Therefore the equation has to be handled with caution.

To begin with, (113) will be written in the form of a conservation law of energy and momentum. To that purpose we first write it in the form of a local equation by multiplying it by a four-dimensional delta function and adding an integration over proper times:

$$cm_0\int a^\alpha(s)\delta^{(4)}\{R_1(s)-R\}\mathrm{d}s$$
$$=\int e[F^{\alpha\beta}\{R_1(s)\}+f_{\mathrm{r}}^{\alpha\beta}\{R_1(s)\}]u_\beta(s)\delta^{(4)}\{R_1(s)-R\}\mathrm{d}s. \quad (114)$$

The right-hand side of (114) may be transformed with the help of the field equation (4) with (5) for a single particle. In the left-hand side we may perform a partial integration. In this way the equation becomes

$$c\partial_\beta m_0\int u^\alpha u^\beta\delta^{(4)}\{R_1(s)-R\}\mathrm{d}s = \{F^{\alpha\beta}(R)+f_{\mathrm{r}}^{\alpha\beta}(R)\}\partial_\gamma f_{\mathrm{r}\beta}^{\cdot\gamma}(R). \quad (115)$$

With the use of the homogeneous equations (7) for $f_{\mathrm{r}}^{\alpha\beta}$ and the equations of the form (4) and (7), but without sources, for the external field $F^{\alpha\beta}$ one may cast the right-hand side in the form of a divergence. Thus (115) is then indeed a conservation law of the form

$$\partial_\beta t^{\alpha\beta}_{\text{tot}} = 0 \tag{116}$$

with an energy–momentum tensor defined as

$$t^{\alpha\beta}_{\text{tot}}(R) = cm_0 \int u^\alpha u^\beta \delta^{(4)}\{R_1(s)-R\}\mathrm{d}s + F^{\alpha\gamma}(R)f^{\beta}_{\text{r}.\gamma}(R) + f^{\alpha\gamma}_{\text{r}}(R)F^{\beta}_{.\gamma}(R) + f^{\alpha\gamma}_{\text{r}}(R)f^{\beta}_{\text{r}.\gamma}(R) - \{\tfrac{1}{2}f^{\gamma\varepsilon}_{\text{r}}(R)F_{\gamma\varepsilon}(R) + \tfrac{1}{4}f^{\gamma\varepsilon}_{\text{r}}(R)f_{\text{r}\gamma\varepsilon}(R)\}g^{\alpha\beta}. \tag{117}$$

We note that instead of (117) one might alternatively use in (116) a tensor with a field part of the Maxwell–Heaviside type

$$-(F+f_{\text{r}})^{\alpha\gamma}(F+f_{\text{r}})_{\gamma}^{\cdot\beta} - \tfrac{1}{4}(F+f_{\text{r}})^{\gamma\varepsilon}(F+f_{\text{r}})_{\gamma\varepsilon}g^{\alpha\beta}, \tag{118}$$

since it differs from the field part in (117) only by a divergence-free contribution.

Let us, following Dirac[1], integrate (116) over a narrow tube around the world line. For each value of the proper time s the tube section is chosen to be spherical with constant radius ε in that Lorentz frame in which the particle is momentarily at rest. Furthermore the tube extends from proper time s_1 to proper time s_2 and is closed by plane surfaces through $R_1^\alpha(s_1)$ and $R_1^\alpha(s_2)$ with normals $c^{-1}u^\alpha(s_1)$ and $c^{-1}u^\alpha(s_2)$ respectively. A convenient starting point for this integration is the following hybrid of equations (114) and (116–117), viz.

$$cm_0 \int a^\alpha \delta^{(4)}(R_1 - R)\mathrm{d}s = \int e(F^{\alpha\beta} + f_-^{\alpha\beta})u_\beta \delta^{(4)}(R_1 - R)\mathrm{d}s + \partial_\beta(f_+^{\alpha\gamma}f_{+\gamma}^{\cdot\beta} + \tfrac{1}{4}f_+^{\gamma\varepsilon}f_{+\gamma\varepsilon}g^{\alpha\beta}), \tag{119}$$

where the retarded field has been split into a plus and a minus part according to

$$f_{\text{r}}^{\alpha\beta} = \tfrac{1}{2}(f_{\text{r}}^{\alpha\beta} + f_{\text{a}}^{\alpha\beta}) + \tfrac{1}{2}(f_{\text{r}}^{\alpha\beta} - f_{\text{a}}^{\alpha\beta}) = f_+^{\alpha\beta} + f_-^{\alpha\beta} \tag{120}$$

with $f_{\text{a}}^{\alpha\beta}$ the advanced field. The reason for a different treatment of the plus and minus fields is that the minus field, in contrast to the plus field, is finite on the world line[2]. Integration of (119) over the tube and application of

[1] P. A. M. Dirac, op. cit.

[2] Alternatively one may refrain from splitting the retarded field, or split it in a different way, v. C. Teitelboim, Phys. Rev. **D 1**(1970)1572, **D 3**(1971)297, to obtain the same final equation of motion.

Gauss's theorem give now

$$cm_0 \int_{s_1}^{s_2} a^\alpha \mathrm{d}s = \int_{s_1}^{s_2} e(F^{\alpha\beta} + f_-^{\alpha\beta})u_\beta \,\mathrm{d}s + \int_{\Sigma_{\text{lat}}} (f_+^{\alpha\gamma}f_{+\gamma}^{\cdot\beta} + \tfrac{1}{4}f_+^{\gamma\varepsilon}f_{+\gamma\varepsilon}g^{\alpha\beta})n_\beta\,\mathrm{d}^3\Sigma + \Phi^\alpha(s_2) - \Phi^\alpha(s_1). \tag{121}$$

Here Σ_{lat} indicates the lateral part of the surface of the tube, while $\Phi^\alpha(s_1)$ and $\Phi^\alpha(s_2)$ are integrals over the closing plane surfaces of the tube; the definition of $\Phi^\alpha(s)$ is:

$$\Phi^\alpha(s) \equiv -c^{-1} \int_{\Sigma_{\text{sect}}} (f_+^{\alpha\gamma}f_{+\gamma}^{\cdot\beta} + \tfrac{1}{4}f_+^{\gamma\varepsilon}f_{+\gamma\varepsilon}g^{\alpha\beta})u_\beta\,\mathrm{d}^3\Sigma. \tag{122}$$

The index at the integration sign means that the integral is to be extended over that part of the plane surface $\Sigma(s)$ (which passes through $R_1^\alpha(s)$ and has normal $c^{-1}u^\alpha(s)$) that lies within the tube, i.e. over a sphere with radius ε around the world line in the plane surface $\Sigma(s)$.

In the first term at the right-hand side of (121) one may substitute (111) for the minus field and in the second term (110) for the plus field. Then the integrand of the first term becomes:

$$eF^{\alpha\beta}u_\beta + \frac{e^2c^{-2}}{6\pi}\Delta^\alpha_\beta(u)\dot{a}^\beta, \tag{123}$$

where the tensor $\Delta^\alpha_\beta(u)$ is defined as $\delta^\alpha_\beta + c^{-2}u^\alpha u_\beta$. The integrand of the second term becomes:

$$(f_+^{\alpha\gamma}f_{+\gamma}^{\cdot\beta} + \tfrac{1}{4}f_+^{\gamma\varepsilon}f_{+\gamma\varepsilon}g^{\alpha\beta})n_\beta = \frac{e^2}{16\pi^2}\varepsilon^{-4}[\tfrac{1}{2}n^\alpha - \tfrac{1}{2}c^{-2}(a^\alpha + n^\alpha a\cdot n)\varepsilon + \{\tfrac{1}{2}c^{-4}(a\cdot n)^2 n^\alpha - \tfrac{1}{2}c^{-4}a^2n^\alpha + \tfrac{5}{4}c^{-4}a\cdot n a^\alpha\}\varepsilon^2 + \ldots]. \tag{124}$$

This expression has to be integrated over the lateral part Σ_{lat} of the surface of the tube. We now need an expression for the surface element $\mathrm{d}^3\Sigma$. Let us consider to that end a surface element at the position

$$R^\alpha = R_1^\alpha(s) + \varepsilon n^\alpha(s), \tag{125}$$

namely a (three-dimensional) strip with edges parallel to the velocity $u^\alpha(s)$, with basis situated in $\Sigma(s)$ and top situated in $\Sigma(s+\mathrm{d}s)$. The height h of the strip is given by the condition that $R^\alpha + hu^\alpha/c$ lie in the plane $\Sigma(s+\mathrm{d}s)$, i.e. by the condition

$$\{R^\alpha + hu^\alpha(s)/c - R_1^\alpha(s+\mathrm{d}s)\}\cdot u_\alpha(s+\mathrm{d}s) = 0. \tag{126}$$

After expansion in powers of $\mathrm{d}s$, insertion of (125) and the use of the relation (v. (102)):

$$n(s)\cdot u(s) = 0 \tag{127}$$

one finds

$$h = c\,\mathrm{d}s(1+c^{-2}\varepsilon n\cdot a). \tag{128}$$

The surface element $\mathrm{d}^3\Sigma$ of Σ_{lat} at the position R^α may now be written as

$$\mathrm{d}^3\Sigma = c\varepsilon^2\mathrm{d}^2\Omega\,\mathrm{d}s(1+c^{-2}\varepsilon n\cdot a), \tag{129}$$

where $\mathrm{d}^2\Omega$ is the differential of the solid angle which parametrizes the direction of the unit vector n^α in the frame in which $u^\alpha = (c, 0, 0, 0)$ [1].

If one inserts (124) and (129) into the second term at the right-hand side of (121), one finds, up to terms of order ε^0

$$\frac{e^2c}{16\pi^2}\,\varepsilon^{-2}\int_{s_1}^{s_2}\{\tfrac{1}{2}n^\alpha-\tfrac{1}{2}c^{-2}a^\alpha\varepsilon+(\tfrac{3}{4}c^{-4}a\cdot na^\alpha-\tfrac{1}{2}c^{-4}a^2n^\alpha)\varepsilon^2\}\mathrm{d}^2\Omega\,\mathrm{d}s. \tag{130}$$

After integration over the solid angle we obtain

$$-\frac{e^2c^{-1}}{8\pi\varepsilon}\int_{s_1}^{s_2}a^\alpha\mathrm{d}s, \tag{131}$$

since the integration of the unit vector n^α yields a vanishing result.

With (123) and (131) we obtain from (121), since s_1 and s_2 are arbitrary,

$$m_0\,a^\alpha = c^{-1}eF^{\alpha\beta}u_\beta+\frac{e^2}{6\pi}\,c^{-3}\Delta^\alpha_\beta(u)\dot{a}^\beta-\frac{e^2c^{-2}a^\alpha}{8\pi\varepsilon}+c^{-1}\frac{\mathrm{d}\Phi^\alpha}{\mathrm{d}s}. \tag{132}$$

We are left with the task to calculate the four-vector Φ^α (122). In the frame in which u^α is $(c, 0, 0, 0)$ the components of Φ^α are

$$\begin{aligned}\Phi^0 &= -\tfrac{1}{2}\int_0^\varepsilon r^2\mathrm{d}r\int^{4\pi}\mathrm{d}^2\Omega(e_+^2+b_+^2),\\ \boldsymbol{\Phi} &= -\int_0^\varepsilon r^2\mathrm{d}r\int^{4\pi}\mathrm{d}^2\Omega(\boldsymbol{e}_+\wedge\boldsymbol{b}_+),\end{aligned} \tag{133}$$

where we have written the integration element $\mathrm{d}^3\Sigma$ as $r^2\,\mathrm{d}r\,\mathrm{d}^2\Omega$. We may insert here the expressions (74) with (75) and (71) for $\boldsymbol{e}_+$ and $\boldsymbol{b}_+$. They show that the partial fields $\boldsymbol{e}_+^{(n)}$ (and $\boldsymbol{b}_+^{(n)}$) are given by (71) for n even (odd) and equal

[1] From (129) one finds an expression for the volume of a parallelepiped with basis $\mathrm{d}^3\Sigma$ in $\Sigma(s)$ at the position R^α, with edges parallel to u^α and top in $\Sigma(s+\mathrm{d}s)$, namely $\mathrm{d}^4V = c\mathrm{d}^3\Sigma\mathrm{d}s[1+c^{-2}\{R-R_1(s)\}\cdot a(s)]$. An expression of this type will be needed in section 3c of the following chapter.

to zero for n odd (even). In the expression (71) the time derivative stands for the sum of an explicit time derivative (which we shall denote here as $\partial/\partial t$) and an implicit time derivative $-c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}$. In this way we have

$$\begin{aligned}\boldsymbol{e}_+^{(n)} &= \frac{-e}{4\pi c^n n!}\,\boldsymbol{\nabla}\left(\frac{\partial}{\partial t}-c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\right)^n r^{n-1}-\frac{e}{4\pi c^{n-1}(n-2)!}\left(\frac{\partial}{\partial t}-c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\right)^{n-2}(r^{n-3}\boldsymbol{\beta}), \quad (n\text{ even}),\\ \boldsymbol{e}_+^{(n)} &= 0, \qquad (n\text{ odd}),\\ \boldsymbol{b}_+^{(n)} &= \frac{e}{4\pi c^{n-1}(n-1)!}\,\boldsymbol{\nabla}\wedge\left(\frac{\partial}{\partial t}-c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\right)^{n-1}(r^{n-2}\boldsymbol{\beta}), \qquad (n\text{ odd}),\\ \boldsymbol{b}_+^{(n)} &= 0, \qquad (n\text{ even}).\end{aligned} \tag{134}$$

We used the frame in which $u^\alpha = (c, 0, 0, 0)$ and hence $\boldsymbol{\beta} = 0$. To find the consequences for the expressions given, one has to work out the powers of the operator $\partial/\partial t-c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}$. This gives in the first term of the expression for $\boldsymbol{e}_+^{(n)}$ (for n even) a sum of terms which contains p times the operator $\partial/\partial t$ and $n-p$ times the operator $-c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}$ in all possible arrangements with $p = 0, \ldots, n$. Since $\boldsymbol{\beta} = 0$ the contribution to $\boldsymbol{e}_+^{(n)}$ will vanish if the number (p) of times that the operator $\partial/\partial t$ occurs is smaller than the number $(n-p)$ of times that the velocity $c\boldsymbol{\beta}$ is present. Hence only the terms with $p \geqslant \frac{1}{2}n$ contribute. These terms contain less than $\frac{1}{2}n+1$ times the operator $\boldsymbol{\nabla}$; since they operate on r^{n-1} the lowest power of r which occurs in the first term of $\boldsymbol{e}_+^{(n)}$ for n even is $r^{\frac{1}{2}n-2}$. With an analogous reasoning one finds that the lowest power in the second term of $\boldsymbol{e}_+^{(n)}$ for n even ($\geqslant 4$) has the exponent $\frac{1}{2}n-1$ and the lowest power in $\boldsymbol{b}_+^{(n)}$ for n odd ($\geqslant 3$) has exponent $\frac{1}{2}n-\frac{3}{2}$. In other words the partial electric and magnetic fields for $\boldsymbol{\beta} = 0$ have the form

$$\begin{aligned}\boldsymbol{e}_+^{(n)} &= \mathrm{O}(r^{\frac{1}{2}n-2}), && (n = 0, 2, 4, \ldots),\\ \boldsymbol{e}_+^{(n)} &= 0, && (n = 1, 3, 5, \ldots),\\ \boldsymbol{b}_+^{(n)} &= \mathrm{O}(r^{\frac{1}{2}n-\frac{3}{2}}), && (n = 3, 5, 7, \ldots),\\ \boldsymbol{b}_+^{(n)} &= 0, && (n = 1;\ n = 0, 2, 4, \ldots).\end{aligned} \tag{135}$$

The expressions (133) for Φ^0 and $\boldsymbol{\Phi}$ may be written in terms of the partial fields:

$$\begin{aligned}\Phi^0 = &-\sum_{n=0(\text{even})}^{\infty}\ \sum_{m=0(\text{even})}^{n}\frac{1}{2}\int_0^\varepsilon r^2\mathrm{d}r\int^{4\pi}\mathrm{d}^2\Omega\,\boldsymbol{e}_+^{(m)}\cdot\boldsymbol{e}_+^{(n-m)}\\ &-\sum_{n=6(\text{even})}^{\infty}\ \sum_{m=3(\text{odd})}^{n-3}\frac{1}{2}\int_0^\varepsilon r^2\mathrm{d}r\int^{4\pi}\mathrm{d}^2\Omega\,\boldsymbol{b}_+^{(m)}\cdot\boldsymbol{b}_+^{(n-m)},\\ \boldsymbol{\Phi} = &-\sum_{n=3(\text{odd})}^{\infty}\ \sum_{m=0(\text{even})}^{n-3}\int_0^\varepsilon r^2\mathrm{d}r\int^{4\pi}\mathrm{d}^2\Omega\,\boldsymbol{e}_+^{(m)}\wedge\boldsymbol{b}_+^{(n-m)}.\end{aligned} \tag{136}$$

From (135) it follows that

$$e_+^{(m)} \wedge b_+^{(n-m)} = O(r^{n-\frac{7}{2}}), \qquad (n = 3, 5, \ldots; m = 0, 2, \ldots, n-3). \tag{137}$$

This means that the integrand for the expression Φ contains powers of r with exponents greater than 0, so that the integrand vanishes in the limit in which ε tends to zero:

$$\Phi = 0. \tag{138}$$

Furthermore it follows from (135) that

$$\begin{aligned} e_+^{(m)} \cdot e_+^{(n-m)} &= O(r^{\frac{1}{2}n-4}), \qquad (n = 0, 2, \ldots; m = 0, 2, \ldots, n), \\ b_+^{(m)} \cdot b_+^{(n-m)} &= O(r^{\frac{1}{2}n-3}), \qquad (n = 6, 8, \ldots; m = 3, 5, \ldots, n-3). \end{aligned} \tag{139}$$

Hence one finds that in the limit $\varepsilon \to 0$ the expression for Φ^0 becomes

$$\Phi^0 = -\int_0^\varepsilon r^2 \mathrm{d}r \int^{4\pi} \mathrm{d}^2\Omega (\tfrac{1}{2} e_+^{(0)2} + e_+^{(0)} \cdot e_+^{(2)}). \tag{140}$$

With the expressions (72) (with $\beta = 0$) we get

$$\Phi^0 = -\int_0^\varepsilon \mathrm{d}r \int^{4\pi} \mathrm{d}^2\Omega \, \frac{e^2}{32\pi^2 r} \left(\frac{1}{r} - 2 \frac{\beta \cdot n}{c} \right) \tag{141}$$

and thus, after integration over the angles,

$$\Phi^0 = -\frac{e^2}{8\pi} \int_0^\varepsilon \frac{1}{r^2} \mathrm{d}r. \tag{142}$$

The expressions (138) and (142), valid in the rest frame, show that in an arbitrary frame the expression for the four-vector Φ^α is, up to terms of order ε^0,

$$\Phi^\alpha = -\frac{e^2 u^\alpha}{8\pi c} \int_0^\varepsilon \frac{1}{r^2} \mathrm{d}r. \tag{143}$$

Substituting this result into the equation of motion (132) and using the identity

$$\frac{1}{\varepsilon} = \int_\varepsilon^\infty \frac{1}{r^2} \mathrm{d}r, \tag{144}$$

one finds an equation of motion, which is independent of ε:

$$\left(m_0 + c^{-2} \int_0^\infty \frac{e^2}{8\pi r^2} \mathrm{d}r \right) a^\alpha = c^{-1} e F^{\alpha\beta} u_\beta + \frac{e^2}{6\pi} c^{-3} \Delta_\beta^\alpha(u) \dot{a}^\beta. \tag{145}$$

This equation has a pathological character in so far that it contains a

divergent integral at the left-hand side, which represents the Coulomb energy of the charged point particle. This difficulty may be veiled by means of a 'renormalization' procedure, namely by writing

$$m_0 + c^{-2} \int_0^\infty \frac{e^2}{8\pi r^2} \mathrm{d}r = m \tag{146}$$

and taking m to be the finite (experimental) mass of the point particle. (m_0 turns out to be negative infinite.) With this artifice the equation of motion (145) gets its final form:

$$m a^\alpha = c^{-1} e F^{\alpha\beta} u_\beta + \frac{e^2}{6\pi} c^{-3} \Delta_\beta^\alpha(u) \dot{a}^\beta. \tag{147}$$

This is Dirac's equation of motion. Dirac did not calculate an expression for Φ^α, but instead assumed that it is proportional to the four-velocity u^α.

The general form (147) of the equation of motion may be cast into the form of a local law, in a way analogous to that leading from (113) to (116) with (117). First we write it as

$$m a^\alpha = c^{-1} e (F^{\alpha\beta} + f_-^{\alpha\beta}) u_\beta, \tag{148}$$

where (111) has been used. We then multiply by the four-dimensional delta function $\delta^{(4)}\{R_1(s) - R\}$ (with $R_1(s)$ the four-position, depending on the proper time s) and integrate over s (cf. (114)):

$$m \int a^\alpha \delta^{(4)}\{R_1(s) - R\} \mathrm{d}s = c^{-1} e \int (F^{\alpha\beta} + f_-^{\alpha\beta}) u_\beta \delta^{(4)}\{R_1(s) - R\} \mathrm{d}s. \tag{149}$$

This equation may be written as a conservation law or as a balance equation. An equation of the latter type is obtained by bringing the left-hand side in the form of a divergence by means of a partial integration and by using at the right-hand side the field equation for the plus and minus parts of the field $f^{\alpha\beta}$:

$$f^{\alpha\beta} \equiv f_+^{\alpha\beta} + f_-^{\alpha\beta}. \tag{150}$$

These equations, which follow from (4) and (7) with (5), read

$$\begin{aligned} \partial_\beta f_+^{\alpha\beta} &= e \int u^\alpha(s) \delta^{(4)}\{R_1(s) - R\} \mathrm{d}s, \\ \partial_\beta f_-^{\alpha\beta} &= 0, \\ \partial^\alpha f_\pm^{\beta\gamma} + \partial^\beta f_\pm^{\gamma\alpha} + \partial^\gamma f_\pm^{\alpha\beta} &= 0. \end{aligned} \tag{151}$$

Then one obtains for (149) the balance equation:

$$\partial_\beta t^{\alpha\beta} = f^\alpha, \tag{152}$$

where we introduced the energy–momentum tensor of the charged particle (cf. (117)):

$$t^{\alpha\beta}(R) = cm \int u^\alpha u^\beta \delta^{(4)}\{R_1(s)-R\}\mathrm{d}s + f_+^{\alpha\gamma}(R) f_{-\cdot\gamma}^{\beta}(R) + f_-^{\alpha\gamma}(R) f_{+\cdot\gamma}^{\beta}(R) - \tfrac{1}{2} f_+^{\gamma\varepsilon}(R) f_{-\gamma\varepsilon}(R) g^{\alpha\beta} \tag{153}$$

(which is seen to be symmetric) and the Lorentz force density

$$f^\alpha(R) = c^{-1} F^{\alpha\beta}(R) j_\beta(R). \tag{154}$$

Here $j^\alpha(R)$ is the four-current, given by (5):

$$c^{-1} j^\alpha(R) = e \int u^\alpha(s) \delta^{(4)}\{R_1(s)-R\}\mathrm{d}s. \tag{155}$$

The balance equation (152) with (153–154) represents the local law, corresponding to the equation of motion (147) or (148).

The energy–momentum tensor contains as components t^{00} the energy density, ct^{0i} the energy flow, $c^{-1}t^{i0}$ the momentum density and t^{ij} the momentum flow. The total energy and the total momentum follow by integrating t^{00} and $c^{-1}t^{i0}$ respectively over the whole of space for a fixed time t. From the expressions (110) and (111) for the plus and minus fields one notices that these integrals converge in the neighbourhood of the world line. However for large distances from the world line it follows from the expressions (96) or (98) that the fields diminish inversely proportionally to the distance if the particle suffers accelerations in the remote past and future. As a consequence the integrals for the total energy and the total momentum of the charged particle diverge in that case. However if one imposes the subsidiary condition that in the remote past and future the particle is not accelerated, one is left – according to (96) or (98) – with fields that diminish inversely proportionally to the square of the distance. Thus the integrals mentioned converge under those circumstances. This means that one is led to the conclusion that only those solutions of the equation of motion (147) or (152) make sense, for which the subsidiary condition about the asymptotic behaviour of the particle is satisfied[1]. This is indeed a necessary condition since the equation (147) as it stands would allow runaway solutions of self-accelerating particles.

[1] R. Haag, Z. Naturf. **10A**(1955)752; F. Rohrlich, Ann. Physics **13**(1961)93.

Often the derivative $\dot{a}^\alpha$ of the acceleration which appears in the last term of (147) is much smaller than the acceleration itself divided by the characteristic time e^2/mc^3. Then one may limit oneself to the truncated equation (112). In non-relativistic approximation the equations (112) and (147) reduce both to the form that has been used in the previous chapters.

b. *A set of particles in a field*

A particle i of a set moves in the combined field due to the other particles j ($\neq i$) and to sources outside the system considered:

$$\sum_{j(\neq i)} f_j^{\alpha\beta}(R) + F^{\alpha\beta}(R). \tag{156}$$

Hence the equation of motion of particle i becomes (cf. (147))

$$m_i a_i^\alpha = c^{-1} e_i \Big(\sum_{j(\neq i)} f_j^{\alpha\beta} + F^{\alpha\beta} \Big) u_{i\beta} + \frac{e_i^2}{6\pi} c^{-3} \Delta_\beta^\alpha(u_i) \dot{a}_i^\beta, \tag{157}$$

where m_i is the mass of the particle, a_i^α its four-acceleration, e_i its charge and u_i^α its four-velocity. The fields depend on the four-position R_i^α.

A local law follows from (157) in a similar way as discussed for a single particle. One then finds the balance

$$\partial_\beta t^{\alpha\beta} = f^\alpha, \tag{158}$$

with the (symmetric) energy–momentum tensor of the set of charged particles defined as:

$$\begin{aligned} t^{\alpha\beta}(R) = {} & c \sum_i m_i \int u_i^\alpha(s_i) u_i^\beta(s_i) \delta^{(4)}\{R_i(s_i)-R\}\mathrm{d}s_i \\ & + \sum_{i,j(i\neq j)} \{ f_i^{\alpha\gamma}(R) f_{j\cdot\gamma}^{\beta}(R) - \tfrac{1}{4} f_i^{\gamma\varepsilon}(R) f_{j\gamma\varepsilon}(R) g^{\alpha\beta} \} \\ & + \sum_i \{ f_{+i}^{\alpha\gamma}(R) f_{-i\cdot\gamma}^{\beta}(R) + f_{-i}^{\alpha\gamma}(R) f_{+i\cdot\gamma}^{\beta}(R) - \tfrac{1}{2} f_{+i}^{\gamma\varepsilon}(R) f_{-i\gamma\varepsilon}(R) g^{\alpha\beta} \} \end{aligned} \tag{159}$$

and the Lorentz force density

$$f^\alpha(R) = c^{-1} F^{\alpha\beta}(R) j_\beta(R), \tag{160}$$

that contains the four-current

$$c^{-1} j^\alpha(R) = \sum_i e_i \int u_i^\alpha(s_i) \delta^{(4)}\{R_i(s_i)-R\}\mathrm{d}s_i. \tag{161}$$

The balance equation (158) has the property that it reduces to a conservation law if no external electromagnetic fields $F^{\alpha\beta}$ are present, as (160) shows. This is a reflection of the fact that (159) is the complete energy–momentum tensor of the set of charged particles and the fields generated by them.

In the preceding subsection it has been remarked that often the self-force term in the equation of motion may be neglected. Then one must also omit the last sum of the energy–momentum tensor (159).

The balance equation (158) with (159) and (160) is not the only way to write the equation of motion (157) in a local form. An alternative way is discussed in the appendix. In the next chapter it will turn out that the form given here has certain advantages over the alternative.

In this survey of the classical covariant theory of charged particles and their fields we considered the field equations with their solutions and the equations of motion for the particles in each other's fields. These results of the microscopic theory will form the basis for the theory of atoms, considered as groups of point particles, and of matter in bulk.

APPENDIX

On an energy-momentum tensor with 'local' character

In the main text we have written the equation of motion for a set of charged particles in an external field in the form of a balance equation. The way to arrive at such a balance equation is by no means unique. It depends on the choice of what is called the force density acting on the particles and what is called the energy–momentum tensor of the particles. An alternative way will be discussed in this appendix.

The energy–momentum tensor (159) that figures in the balance equation (158) contains the fields generated by all particles of the system. These fields may be split into plus and minus fields that have a different behaviour in the neighbourhood of the world line, as has been shown in section 2*e*. A special different form of the balance equation may now be obtained by writing the contributions of the minus fields generated by all particles at the right-hand side of (158), so that one gets

$$\partial_\beta t_+^{\alpha\beta} = f^{*\alpha} \tag{A1}$$

with the energy–momentum tensor

$$t_+^{\alpha\beta}(R) = c\sum_i m_i \int u_i^\alpha(s_i) u_i^\beta(s_i) \delta^{(4)}\{R_i(s_i)-R\}\mathrm{d}s_i + \sum_{i,j(i\neq j)} \{f_{+i}^{\alpha\gamma}(R) f_{+j,\gamma}^{\beta}(R) - \tfrac{1}{4} f_{+i}^{\gamma\varepsilon}(R) f_{+j\gamma\varepsilon}(R) g^{\alpha\beta}\}, \tag{A2}$$

(which is also symmetric) and the force density

$$f^{*\alpha}(R) = f^\alpha(R) + \sum_{i,j} e_i \int f_{-j}^{\alpha\beta}(R) u_{i\beta}\, \delta^{(4)}\{R_i(s_i)-R\}\mathrm{d}s_i, \tag{A3}$$

where $f^\alpha(R)$ has been given in (160–161).

The equation (A1) is not modified if a divergenceless tensor is added to $t_+^{\alpha\beta}$. This freedom will be used to introduce instead of $t_+^{\alpha\beta}$ a different tensor $t^{*\alpha\beta}$, which is 'local' in a sense that will be explained below.

To obtain such a tensor we shall start from the expression which Wheeler and Feynman[1] have given for the total energy–momentum of a set of charged

[1] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**(1949)425; J. W. Dettmann and A. Schild, Phys. Rev. **95**(1954)1057; cf. footnote on page 160.

particles which interact via plus fields. This total momentum $p^\alpha(\Sigma)$ associated to a plane space-like surface Σ with normal n^α (and thus given by the equation

$$n\cdot R + c\tau = 0 \tag{A4}$$

with fixed τ) reads as follows

$$p^\alpha(\Sigma) = \sum_i m_i u_i^\alpha(s_i^0)$$
$$- \frac{1}{4\pi c} \sum_{i,j(i\neq j)} e_i e_j \int \{u_i\cdot u_j(R_i-R_j)^\alpha - (R_i-R_j)\cdot u_i u_j^\alpha - (R_i-R_j)\cdot u_j u_i^\alpha\}$$
$$\{\theta(n\cdot R_i+c\tau)\theta(-n\cdot R_j-c\tau)-\theta(-n\cdot R_i-c\tau)\theta(n\cdot R_j+c\tau)\}$$
$$\delta'\{(R_i-R_j)^2\}\mathrm{d}s_i\,\mathrm{d}s_j, \tag{A5}$$

where s_i^0 is the proper time of the intersection of world line i with Σ, i.e. given by:

$$n\cdot R_i(s_i^0)+c\tau = 0. \tag{A6}$$

The first term of $p^\alpha(\Sigma)$ is the contribution of the material momentum of the particles, while the second is the momentum carried by the plus fields. The unit step functions θ ensure that in (A5) only those parts of the world lines of i and j contribute for which R_i^α is on one side of the surface Σ and R_j^α on the other. One may write the difference of products of step functions alternatively as

$$\theta(-s_i+s_i^0)\theta(s_j-s_j^0)-\theta(s_i-s_i^0)\theta(-s_j+s_j^0). \tag{A7}$$

The prime at the delta function in (A5) indicates a differentiation with respect to its argument. From the form of $p^\alpha(\Sigma)$ it is apparent that only finite parts of the world lines of the particles i contribute to the field momentum, namely those parts that lie outside the light cones having their top at the intersections of the world lines j ($\neq i$) with the plane Σ.

From the expression (A5) for $p^\alpha(\Sigma)$ one may infer the energy–momentum tensor $t_{\mp}^{\alpha\beta}$ which gives back $p^\alpha(\Sigma)$ by integration over the plane Σ, i.e.

$$-c^{-1}\int_\Sigma t_{\mp}^{\alpha\beta}\mathrm{d}^3\Sigma_\beta = p^\alpha(\Sigma). \tag{A8}$$

(The tensor $t_{\mp}^{\alpha\beta}$ will lead to the tensor $t^{*\alpha\beta}$, which we are trying to find.) To that end we write a tensor with the plus field momentum localized on the line that joins the positions R_i^α and R_j^α and with the particle momentum localized – of course – at the positions R_i^α themselves

$$t_{\mp}^{\alpha\beta}(R) = c\sum_i m_i \int u_i^\alpha u_i^\beta \delta^{(4)}(R_i-R)\mathrm{d}s_i$$
$$+\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \iint\int_{\lambda=0}^{1} \{u_i\cdot u_j(R_i-R_j)^\alpha-(R_i-R_j)\cdot u_i u_j^\alpha-(R_i-R_j)\cdot u_j u_i^\alpha\}$$
$$(R_i-R_j)^\beta\delta'\{(R_i-R_j)^2\}\delta^{(4)}\{R_i+\lambda(R_j-R_i)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \tag{A9}$$

We have to show whether this tensor satisfies (A8) and moreover whether it differs from $t_{+}^{\alpha\beta}$ (A2) by a divergenceless part so that

$$\partial_\beta t_{\mp}^{\alpha\beta} = \partial_\beta t_{+}^{\alpha\beta}. \tag{A10}$$

To begin with we check the validity of (A8) by integrating $t_{\mp}^{\alpha\beta}$ over a space-like plane Σ. One has then to employ the identity

$$|(R_i-R_j)\cdot n|\int_\Sigma\int_{\lambda=0}^{1}\delta^{(4)}\{R_i+\lambda(R_j-R_i)-R\}\mathrm{d}\lambda\,\mathrm{d}^3\Sigma$$
$$=\theta(n\cdot R_i+c\tau)\theta(-n\cdot R_j-c\tau)+\theta(-n\cdot R_i-c\tau)\theta(n\cdot R_j+c\tau), \tag{A11}$$

which may be proved by evaluating the left-hand side in a Lorentz frame in which Σ is purely space-like (i.e. in which $n^\alpha = (1, 0, 0, 0)$) and using the relation (41). Then one obtains for the integral of $t_{\mp}^{\alpha\beta}$:

$$-c^{-1}\int_\Sigma t_{\mp}^{\alpha\beta}\mathrm{d}\Sigma_\beta = -m_i\int u_i^\alpha u_i\cdot n\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}^3\Sigma$$
$$-\frac{1}{4\pi c}\sum_{i,j(i\neq j)} e_i e_j\int\{u_i\cdot u_j(R_i-R_j)^\alpha-(R_i-R_j)\cdot u_i u_j^\alpha-(R_i-R_j)\cdot u_j u_i^\alpha\}$$
$$\varepsilon\{(R_i-R_j)\cdot n\}\{\theta(n\cdot R_i+c\tau)\theta(-n\cdot R_j-c\tau)+\theta(-n\cdot R_i-c\tau)\theta(n\cdot R_j+c\tau)\}$$
$$\delta'\{(R_i-R_j)^2\}\mathrm{d}s_i\,\mathrm{d}s_j. \tag{A12}$$

In the first term at the right-hand side the integrations over s_i and Σ may be performed if use is made of (41). As a result one finds the first term at the right-hand side of (A5). In the second term at the right-hand side of (A12) the product of the ε-function and the sum of products of the θ-functions is equal to the difference of the products of the θ-functions so that also the second term at the right-hand side of (A5) is recovered.

The use of $t_{\mp}^{\alpha\beta}$ instead of $t_{+}^{\alpha\beta}$ in (A1) is justified if we succeed in proving (A10). With the explicit form (A9) of $t_{\mp}^{\alpha\beta}$ one finds, by performing a partial integration in the first term and using the chain rule of differentiation, for its divergence

$$\partial_\beta t_{\mp}^{\alpha\beta} = c\sum_i m_i \int a_i^\alpha \delta^{(4)}(R_i-R)\mathrm{d}s_i$$
$$+\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} \{u_i\cdot u_j(R_i-R_j)^\alpha-(R_i-R_j)\cdot u_i u_j^\alpha-(R_i-R_j)\cdot u_j u_i^\alpha\}$$
$$\delta'\{(R_i-R_j)^2\}\frac{\partial}{\partial\lambda}\delta^{(4)}\{R_i+\lambda(R_j-R_i)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad (A13)$$

The integration over λ may be performed so that the difference of the two delta functions $\delta^{(4)}(R_j-R)$ and $\delta^{(4)}(R_i-R)$ arises. By using the antisymmetry – in i and j – of the remaining part of the integrand the last term of (A13) becomes

$$-\frac{1}{2\pi}\sum_{i,j(i\neq j)} e_i e_j \int \{u_i\cdot u_j(R_i-R_j)^\alpha-(R_i-R_j)\cdot u_i u_j^\alpha-(R_i-R_j)\cdot u_j u_i^\alpha\}$$
$$\delta'\{(R_i-R_j)^2\}\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}s_j. \quad (A14)$$

The last term between the brackets gives a vanishing contribution since it leads to an integrand which contains the factor $(\mathrm{d}/\mathrm{d}s_j)\delta\{(R_i-R_j)^2\}$. The integral of this term over s_j may then be performed, with a vanishing result. The other terms may be handled similarly, with the result

$$-\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \int [(\partial_i^\alpha u_j^\beta-\partial_i^\beta u_j^\alpha)u_{i\beta}\,\delta\{(R_i-R_j)^2\}]\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}s_j. \quad (A15)$$

In this expression one recognizes the plus field due to particle j at the position i:

$$f_{+j}^{\alpha\beta}(R_i) = \frac{1}{4\pi} e_j \int (u_j^\beta \partial_i^\alpha - u_j^\alpha \partial_i^\beta)\delta\{(R_i-R_j)^2\}\mathrm{d}s_j, \quad (A16)$$

as follows from (59) and the subsequent remarks. Collecting the results (A13) with (A15) we have obtained

$$\partial_\beta t_{\mp}^{\alpha\beta} = c\sum_i \int \{m_i a_i^\alpha - c^{-1} e_i \sum_{j(\neq i)} f_{+j}^{\alpha\beta}(R_i) u_{i\beta}\}\delta^{(4)}(R_i-R)\mathrm{d}s_i. \quad (A17)$$

From the expression (A2) for $t_+^{\alpha\beta}$ it now follows that we have found here the relation (A10), which shows that the tensor $t_{\mp}^{\alpha\beta}$ (A9) may be used as well as $t_+^{\alpha\beta}$ (A2) in (A1)[1].

[1] The reader may note that the Wheeler–Feynman expression (A5) has played a heuristic role, since it was only used to infer the tensor $t_{\mp}^{\alpha\beta}$ (A9) from it. The main result is the equivalence of $t_{\mp}^{\alpha\beta}$ and $t_+^{\alpha\beta}$ in the sense of (A10). The relation (A8) clarifies the connexion between $t_{\mp}^{\alpha\beta}$ and the expression (A5).

In contrast to $t_+^{\alpha\beta}$ the tensor $t_{\mp}^{\alpha\beta}$ has 'local' character in so far that it vanishes for positions outside the 'envelope' of the world lines.

The tensor $t_{\mp}^{\alpha\beta}$ is asymmetric with respect to the indices α and β. The symmetry may be restored, without loosing the locality, by adding another divergenceless tensor, as we shall now show. The first two terms of the right-hand side of (A9) are already symmetric in α and β so that we shall focus our attention on the third and fourth term. In the third term we may employ the identity

$$(R_i-R_j)\cdot u_i\,\delta'\{(R_i-R_j)^2\} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s_i}\delta\{(R_i-R_j)^2\} \quad (A18)$$

and perform a partial integration with respect to s_i. If the fourth term is treated in a similar way and use is made of the symmetry in i and j the third and fourth terms of $t_{\mp}^{\alpha\beta}$ become

$$\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} \{u_i^\alpha u_j^\beta + \lambda u_i^\alpha (R_i-R_j)^\beta u_j^\gamma \partial_\gamma\}$$
$$\delta\{(R_i-R_j)^2\}\delta^{(4)}\{R_i+\lambda(R_j-R_i)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad (A19)$$

By adding a divergenceless term to this expression one obtains

$$\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} \{u_i^\alpha u_j^\beta + \lambda u_i^\alpha u_j^\beta (R_i-R_j)^\gamma \partial_\gamma\}$$
$$\delta\{(R_i-R_j)^2\}\delta^{(4)}\{R_i+\lambda(R_j-R_i)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad (A20)$$

If we use once more the chain rule of differentiation we may write this as

$$\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} \frac{\partial}{\partial\lambda}[\lambda u_i^\alpha u_j^\beta \delta\{(R_i-R_j)^2\}$$
$$\delta^{(4)}\{R_i+\lambda(R_j-R_i)-R\}]\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda, \quad (A21)$$

so that the integration over λ may be performed. Only the value at the boundary $\lambda = 1$ contributes:

$$\frac{1}{4\pi}\sum_{i,j(i\neq j)} e_i e_j \int u_i^\alpha u_j^\beta \delta\{(R_i-R_j)^2\}\delta^{(4)}(R_j-R)\mathrm{d}s_i\,\mathrm{d}s_j. \quad (A22)$$

This is still not symmetric, since j alone occurs in the second delta function. Let us write (A22) as half the sum plus half the difference of that expression and its transposed:

$$\frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int (u_i^\alpha u_j^\beta + u_i^\beta u_j^\alpha)\delta\{(R_i-R_j)^2\}\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}s_j$$
$$+\frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} u_i^\alpha u_j^\beta (R_i-R_j)^\gamma \partial_\gamma \delta\{(R_i-R_j)^2\}$$
$$\delta^{(4)}\{R_j+\lambda(R_i-R_j)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad (A23)$$

Adding a divergenceless part to the second term, we get

$$\frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int (u_i^\alpha u_j^\beta + u_i^\beta u_j^\alpha)\delta\{(R_i-R_j)^2\}\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}s_j$$

$$+\frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} \{u_i^\alpha(R_i-R_j)^\beta + u_i^\beta(R_i-R_j)^\alpha\}u_j^\gamma\partial_\gamma\delta\{(R_i-R_j)^2\}$$

$$\delta^{(4)}\{R_j+\lambda(R_i-R_j)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad \text{(A24)}$$

Indeed the second part of the second term has vanishing divergence in view of the antisymmetry in i and j (together with $\lambda \leftrightarrow 1-\lambda$); the first part of the second term has the same divergence as the second term of (A23).

Since the expression (A24) is symmetric with respect to an interchange of α and β, we have found now a symmetric energy–momentum tensor with the same divergence as $t_{\mp}^{\alpha\beta}$ (and hence as $t_{+}^{\alpha\beta}$). It is found by adding (A24) to the first two terms at the right-hand side of (A9):

$$t^{*\alpha\beta} = c\sum_i m_i \int u_i^\alpha u_i^\beta \delta^{(4)}(R_i-R)\mathrm{d}s_i$$

$$+\frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int (u_i^\alpha u_j^\beta + u_i^\beta u_j^\alpha)\delta\{(R_i-R_j)^2\}\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}s_j$$

$$+\frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} [2u_i\cdot u_j(R_i-R_j)^\alpha(R_i-R_j)^\beta\delta'\{(R_i-R_j)^2\}$$

$$+\{u_i^\alpha(R_i-R_j)^\beta + u_i^\beta(R_i-R_j)^\alpha\}u_j^\gamma\partial_\gamma\delta\{(R_i-R_j)^2\}]$$

$$\delta^{(4)}\{R_j+\lambda(R_i-R_j)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad \text{(A25)}$$

This symmetric tensor[1], which still has 'local' character, satisfies the energy–momentum balance

$$\partial_\beta t^{*\alpha\beta} = f^{*\alpha} \quad \text{(A26)}$$

with the force density (A3).

The two balance equations (158) and (A26) are both equivalent with the equation (157). Each of them may be used for the description of the system of charged particles. As remarked in the main text the typical feature of the balance equation (158) with (159–160) is that it reduces to a conservation law if no external fields are present. The typical feature of (A26) with (A3) and (A25) is that the contributions of the minus fields (which, just as the external fields, are finite at the world lines of the particles) are written at the right-hand side just as has been done in (148). The energy–momentum tensor (A25) has local character in the sense that it vanishes outside the region in which the world lines are situated, in contrast to the energy–momentum tensor (159) which also diminishes with increasing distances in space-like directions, but which does not possess a finite support in such directions.

Often the effect of the minus fields on the equation of motion is small (as may be discussed if they are explicitly evaluated: see the appendix of the following chapter). If they are neglected the force density (A3) reduces to (160).

In the next chapter we shall employ (158) extensively, but we shall also have occasion to show, in an appendix, the consequences of the use of (A26).

[1] An energy–momentum tensor of the same type has been given by S. Emid and J. Vlieger (Physica **52**(1971)329) in the form of a series expansion, valid for a continuous charge–current distribution. If that formula is applied to a set of charged point particles, one finds a result that is nearly the same as the expression that is obtained from (A25) if the integrand of the last term there is developed into powers of λ and the integration over λ is performed – the sole difference being that the summations in (A25) exclude the contributions $i=j$, thus avoiding infinite self energy contributions.

PROBLEMS

1. Show that the expressions (44), (48) and (49) for the Green functions G_{r}, G_{a}, G_{f} and G_{af} satisfy the relation (47).
Hint: use the identity (53).

2. Prove that the retarded and advanced potentials due to a source $j^{\alpha}(\boldsymbol{R}, t)$ may be written as

$$a^{\alpha}_{\mathrm{r,a}}(\boldsymbol{R}, t) = \frac{c^{-1}}{4\pi}\int j^{\alpha}\left(\boldsymbol{R}', t \mp \frac{|\boldsymbol{R}-\boldsymbol{R}'|}{c}\right)\frac{1}{|\boldsymbol{R}-\boldsymbol{R}'|}\,\mathrm{d}\boldsymbol{R}'.$$

This may be proved from (14) with (23), (30), (39) and (40), or alternatively from (57) with (43).

3. Prove – by insertion – that the potentials (61) satisfy the equations

$$\Box\, \varphi(\boldsymbol{R}, t) = -\rho^{\mathrm{e}}(\boldsymbol{R}, t) = -\sum_j e_j\delta\{\boldsymbol{R}-\boldsymbol{R}_j(t)\},$$

$$\Box\, \boldsymbol{a}(\boldsymbol{R}, t) = -c^{-1}\boldsymbol{j}(\boldsymbol{R}, t) = -\sum_j e_j\boldsymbol{\beta}_j(t)\delta\{\boldsymbol{R}-\boldsymbol{R}_j(t)\},$$

which are the equations (13) with (5) in three-dimensional notation (check that first).

4. Show that the expressions (61) for the potentials may be written alternatively as the following integrals over the space-coordinates

$$\varphi_{\mathrm{r,a}}(\boldsymbol{R}, t) = \sum_j e_j\int\frac{\delta\{\boldsymbol{R}_j(t\mp c^{-1}|\boldsymbol{R}-\boldsymbol{R}'|)-\boldsymbol{R}\}}{4\pi|\boldsymbol{R}-\boldsymbol{R}'|}\,\mathrm{d}\boldsymbol{R}',$$

$$\boldsymbol{a}_{\mathrm{r,a}}(\boldsymbol{R}, t) = \sum_j e_j\int\boldsymbol{\beta}_j(t\mp c^{-1}|\boldsymbol{R}-\boldsymbol{R}'|)\frac{\delta\{\boldsymbol{R}_j(t\mp c^{-1}|\boldsymbol{R}-\boldsymbol{R}'|)-\boldsymbol{R}\}}{4\pi|\boldsymbol{R}-\boldsymbol{R}'|}\,\mathrm{d}\boldsymbol{R}'.$$

Hint: add a factor $\delta\{\boldsymbol{R}_j(t_j)-\boldsymbol{R}'\}$ and an integration over $\boldsymbol{R}'$ to (61). Then replace in the integrand $\boldsymbol{R}_j(t_j)$ by $\boldsymbol{R}'$ and integrate over t_j.

5. Prove that for the field $f^{\alpha\beta}$ (95) or (96) one has the identity

$$\varepsilon_{\alpha\beta\gamma\delta}f^{\alpha\beta}r^{\gamma}|_{\mathrm{r,a}} = 0,$$

where $f^{\alpha\beta}$ has to be understood as the quantity 'inside' the bar symbol, as explained after (89).
Show that this identity reads in three-dimensional notation

$$\boldsymbol{b}|_{\mathrm{r,a}} = \pm(\boldsymbol{n}\wedge\boldsymbol{e})|_{\mathrm{r,a}}$$

and

$$\boldsymbol{b}\cdot\boldsymbol{n}|_{\mathrm{r,a}} = 0$$

with the same convention for the fields $\boldsymbol{e}$ and $\boldsymbol{b}$.

6. Prove that the Hamilton equations that follow from the so-called Darwin Hamiltonian (C. G. Darwin, Phil. Mag. **39**(1920)537) for two particles with charges e_1, e_2, masses m_1, m_2, positions $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, momenta $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, moving in each others fields

$$H(\boldsymbol{P}_1, \boldsymbol{R}_1, \boldsymbol{P}_2, \boldsymbol{R}_2) = \sum_{i=1}^{2}\left(\frac{\boldsymbol{P}_i^2}{2m_i} - \frac{\boldsymbol{P}_i^4}{8m_i^3c^2}\right) + \left\{1-\frac{\boldsymbol{P}_1\cdot\mathsf{T}(\boldsymbol{R}_1-\boldsymbol{R}_2)\cdot\boldsymbol{P}_2}{2m_1m_2c^2}\right\}\frac{e_1e_2}{4\pi|\boldsymbol{R}_1-\boldsymbol{R}_2|},$$

with $\mathsf{T}(\boldsymbol{R}_1-\boldsymbol{R}_2) \equiv \mathsf{U}+(\boldsymbol{R}_1-\boldsymbol{R}_2)(\boldsymbol{R}_1-\boldsymbol{R}_2)/|\boldsymbol{R}_1-\boldsymbol{R}_2|^2$, lead to equations of motion for the particles that are, up to order c^{-2}, equal to the c^{-2}-approximations of the complete relativistic equations (157).
Hint: Use the relevant expressions from (72) for the retarded fields.

7. In the following series of problems the minus part (111) of the self-field at the world line of the particle will be obtained from the expressions (69) with (71) for the fields in terms of series in powers of c^{-1}.

Prove that in the limit $r\to 0$ the expressions (75) with (71) for the partial minus fields of order n (i.e. n odd for the electric and n even for the magnetic field) lead to:

$$\begin{aligned}\boldsymbol{e}_-^{(n)} = &-\frac{ec^{-2}}{4\pi n!}\nabla\sum_{\substack{k,l=0\\k+l\leqq n-2}}^{n-2}(n-2-k-l)\{\dot{\boldsymbol{\beta}}\cdot\nabla(\boldsymbol{\beta}\cdot\nabla)^{n-3}\\&\qquad+(n-3-k)(\dot{\boldsymbol{\beta}}\cdot\nabla)^2(\boldsymbol{\beta}\cdot\nabla)^{n-4}\}r^{n-1}\\&+\frac{ec^{-2}}{4\pi(n-2)!}\sum_{\substack{k,l=0\\k+l\leqq n-3}}^{n-3}[\ddot{\boldsymbol{\beta}}(\boldsymbol{\beta}\cdot\nabla)^{n-3}+(n-3-k)\dot{\boldsymbol{\beta}}\dot{\boldsymbol{\beta}}\cdot\nabla(\boldsymbol{\beta}\cdot\nabla)^{n-4}\\&+(n-3-k-l)\{\boldsymbol{\beta}\dot{\boldsymbol{\beta}}\cdot\nabla(\boldsymbol{\beta}\cdot\nabla)^{n-4}+\boldsymbol{\beta}\ddot{\boldsymbol{\beta}}\cdot\nabla(\boldsymbol{\beta}\cdot\nabla)^{n-4}\\&\qquad+(n-4-k)\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\nabla)^2(\boldsymbol{\beta}\cdot\nabla)^{n-5}\}]r^{n-3},\end{aligned}$$

and to a similar expression for $\boldsymbol{b}_-^{(n)}$. (The dots indicate time derivatives.)

Hints: remember that $\partial/\partial t$ in (71) contains an explicit and an implicit differentiation, i.e. $\partial/\partial t = (\partial/\partial t)_{\text{expl}} - c\boldsymbol{\beta}\cdot\boldsymbol{\nabla}$. Furthermore note that in the limit $r \to 0$ only terms with $\boldsymbol{\nabla}^m r^m$, or in other words with two explicit time differentiations, subsist.

8. Prove the relations

$$\sum_{k,l=0(k+l\leqq n)}^{n} (n-k) = \tfrac{1}{3}n(n+1)(n+2),$$

$$\sum_{k,l=0(k+l\leqq n)}^{n} (n-k-l) = \tfrac{1}{6}n(n+1)(n+2),$$

$$\sum_{k,l=0(k+l\leqq n)}^{n} (n-k-l)(n-1-k) = \tfrac{1}{8}(n-1)n(n+1)(n+2).$$

Hint: use the sums $\sum_{k=0}^{n} k = \frac{1}{2}n(n+1)$, $\sum_{k=0}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ and $\sum_{k=0}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$.

9. Prove the relations (n odd)

$$(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})^{n-2} r^{n-1} = (n-1)!\beta^{n-3}\boldsymbol{\beta}\cdot\boldsymbol{r},$$

$$(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})^{n-3} r^{n-1} = \tfrac{1}{2}(n-1)(n-3)!\beta^{n-3}r^2 + \tfrac{1}{2}(n-1)(n-3)(n-3)!\beta^{n-5}(\boldsymbol{\beta}\cdot\boldsymbol{r})^2,$$

$$(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})^{n-4} r^{n-1} = \tfrac{1}{2}(n-1)(n-3)(n-4)!\beta^{n-5}\boldsymbol{\beta}\cdot\boldsymbol{r}r^2 + \tfrac{1}{6}(n-1)(n-3)(n-5)(n-4)!\beta^{n-7}(\boldsymbol{\beta}\cdot\boldsymbol{r})^3.$$

10. Prove from the results of the preceding three problems that

$$\boldsymbol{e}_-^{(n)}(\boldsymbol{r}=0) = \frac{e}{4\pi} c^{-2}\{\tfrac{1}{3}(n-1)\ddot{\boldsymbol{\beta}}\beta^{n-3} + \tfrac{1}{4}(n-1)(n-3)\dot{\boldsymbol{\beta}}\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\beta^{n-5}\}, \qquad (n \text{ odd}),$$

$$\boldsymbol{b}_-^{(n)}(\boldsymbol{r}=0) = \frac{e}{4\pi} c^{-2}\{\tfrac{1}{3}(n-2)\boldsymbol{\beta}\wedge\ddot{\boldsymbol{\beta}}\beta^{n-4} + \tfrac{1}{4}(n-2)(n-4)\boldsymbol{\beta}\wedge\dot{\boldsymbol{\beta}}\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\beta^{n-6}\}, \qquad (n \text{ even}).$$

11. Prove from the preceding result that the total minus field at the position of the particle is given by

$$\boldsymbol{e}_-(\boldsymbol{r}=0) = \frac{e}{4\pi} c^{-2}(\tfrac{2}{3}\gamma^4\ddot{\boldsymbol{\beta}} + 2\gamma^6\dot{\boldsymbol{\beta}}\boldsymbol{\beta}\cdot\boldsymbol{\beta}),$$

$$\boldsymbol{b}_-(\boldsymbol{r}=0) = \frac{e}{4\pi} c^{-2}(\tfrac{2}{3}\gamma^4\boldsymbol{\beta}\wedge\ddot{\boldsymbol{\beta}} + 2\gamma^6\boldsymbol{\beta}\wedge\dot{\boldsymbol{\beta}}\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}).$$

12. Show that the results given in the preceding problem are the components of the tensor (111) (at the world line):

$$f^{\alpha\beta} = \frac{e}{6\pi} c^{-4}(u^\alpha \dot{a}^\beta - u^\beta \dot{a}^\alpha),$$

where u^α and a^α are the four-velocity and four-acceleration and the dot now stands for a differentiation with respect to proper time.

13. Show by use of (A18) and a partial integration of the second and third terms in the first bracket expression of the integrand in (A5) that one may write this formula alternatively as

$$p^\alpha(\Sigma) = \sum_i m_i u_i^\alpha(s_i^0) - \frac{1}{4\pi c}\sum_{i,j(i\neq j)} e_i e_j \int u_i\cdot u_j (R_i - R_j)^\alpha$$
$$\{\theta(n\cdot R_i + c\tau)\theta(-n\cdot R_j - c\tau) - \theta(-n\cdot R_i - c\tau)\theta(n\cdot R_j + c\tau)\}\delta'\{(R_i-R_j)^2\}\mathrm{d}s_i\,\mathrm{d}s_j$$
$$-\frac{1}{4\pi c}\sum_{i,j(i\neq j)} e_i e_j \int u_j^\alpha n\cdot u_i\,\delta(n\cdot R_i + c\tau)\delta\{(R_i - R_j)^2\}\mathrm{d}s_i\,\mathrm{d}s_j.$$

Prove the ancillary formula (cf. (A11))

$$(R_i - R_j)\cdot n \int_\Sigma \int_{\lambda=0}^{1} u_j\cdot\partial\delta^{(4)}\{R_j + \lambda(R_i - R_j) - R\}\mathrm{d}^3\Sigma\,\mathrm{d}\lambda = -u_j\cdot n\{\delta(n\cdot R_i + c\tau) - \delta(n\cdot R_j + c\tau)\}.$$

Hint: evaluate the left-hand side in a Lorentz frame in which Σ is purely space-like; then one may apply Gauss's theorem to prove that only the time differentiation (i.e. $u_j^0\partial_0$) gives a contribution.

Show with the help of the latter formula that the total momentum (in a plane Σ) which corresponds to the tensor $t^{*\alpha\beta}$ (A25) is equal to that which corresponds to $t_{\#}^{\alpha\beta}$ (A9), i.e. that

$$p^{*\alpha}(\Sigma) = -c^{-1}\int t^{*\alpha\beta}\mathrm{d}^3\Sigma_\beta = -c^{-1}\int t_{\#}^{\alpha\beta}\mathrm{d}^3\Sigma_\beta = p^\alpha(\Sigma),$$

where the last member is given in the first formula of this problem (v. also (A8)).