At this point, recourse was had to an IBM 7040 digital computer. The matrix M was evaluated as a function of p. A unitary matrix U was generated which diagonalized M by

$$\Lambda = U^{\dagger} M U . \qquad (10)$$

The elements of Λ are $\lambda_i \delta_{ij}$. The matrix $e^{-i\Lambda}$ was then formed. Its elements are

$$(e^{-i\Lambda})_{ij} = \exp(-i\lambda_i)\delta_{ij} . \qquad (11)$$

Finally e^{-iM} was obtained from

$$e^{-iM} = U e^{-i\Lambda} U^{\dagger} . \qquad (12)$$

$a(p)$ was then found from eq. (3), and numerically integrated according to eq. (5) to find the excitation cross sections. The transition turns out to be relatively weak: typically $|a_{2p_1}(p)|^2 < 0.1$. The largest partial cross section is that to the $2p_0$ state. Our results are shown in fig. 2 along with the calculation of Bell and Skinner [4], and the measurements of Stebbings et al. [5]. The excitation to the $2p_1$ and $2s_0$ states are also shown.

The 2p cross section is in good agreement with that calculated by Bell and Skinner, but is too large in relation to experiment by a factor of about 3. Part of this discrepancy may be due to the neglect of charge exchange. The cross section for resonant charge exchange is rather large compared to that for excitation. If this process could be included in the calculation, it would tend to diminish the calculated excitation cross section through the requirements of unitarity.

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THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN A DIELECTRIC

S. R. de GROOT and L. G. SUTTORP
Institute of Theoretical Physics, Amsterdam

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The relativistic energy-momentum tensor of the electromagnetic field is derived from electron theory and compared with the forms proposed by Abraham and Minkowski.

An unsolved problem of relativistic electrodynamics is the derivation of the energy-momentum tensor of the electromagnetic field in matter from microscopic electron theory. Several proposals for this tensor, expressed in the macroscopic Maxwell fields, were made since Minkowski and Abraham [1,2]. However, because no explicit specification of the energy-momentum tensor of matter was given, it was not possible to obtain a unique expression for the energy-momentum tensor of the field [3].

A complete derivation must start from the microscopic conservation laws of energy and momentum of a system of point particles (electrons and nuclei) in the presence of electromagnetic fields. The point particles are supposed to be grouped into atoms (or molecules, ions etc.), which - in the model adopted here - carry only electric dipoles (in their momentary rest frame [4]), but no other electromagnetic multipoles. Then conservation laws on the "atomic" level are obtained. Using a covariant way of statistical averaging [5] one gets finally the macroscopic conservation laws. They read for a solid in the approximation of small accelerations (which is the case the authors quoted [1-3] had in mind):

$$\partial_\beta(\rho U_\alpha U^\beta + T_\alpha^{\ \beta}(m) + T_\alpha^{\ \beta}(f)) = 0. \qquad (1)$$

Here is α, β = 0, 1, 2, 3 (with $x^\alpha = (ct, \mathbf{R})$, and metric $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = 1$, $g^{\alpha\beta} = 0$ if

$\alpha \neq \beta$), ρ the rest mass energy and bulk internal energy density in the rest frame and cU^{α} the bulk four-velocity of matter. The tensors $\rho U_{\alpha} U^{\beta}$ and $T_{\alpha}^{\cdot\beta(m)}$ form together the material energy-momentum tensor: the first term being due to bulk motion, the second containing only correlations and velocity fluctuations (in the rest frame). The space-space components of $T_{\alpha}^{\cdot\beta(m)}$ form in the rest frame the pressure-tensor, which in equilibrium agrees with the pressure defined in the statistical thermodynamics of polarized systems [6].

The energy-momentum tensor of the electromagnetic field in the dielectric is found to be of the form:

$$T_{\alpha}^{\cdot\beta(f)} = - F_{\alpha\gamma} H^{\gamma\beta} - \tfrac{1}{4} F_{\gamma\delta} F^{\gamma\delta} \delta_{\alpha}^{\beta} + -U^{\beta}(F_{\alpha\gamma} H^{\gamma\delta} - H_{\alpha\gamma} F^{\gamma\delta}) U_{\delta} + +\tfrac{1}{2} U_{\alpha} U^{\beta} F_{\gamma\delta} (H^{\gamma\delta} - F^{\gamma\delta}) , \qquad (2)$$

where the antisymmetric field tensors $F^{\alpha\beta}$ and $H^{\alpha\beta}$ have components $(F^{01}, F^{02}, F^{03}) = \boldsymbol{E}$, $(F^{23}, F^{31}, F^{12}) = \boldsymbol{B}$, $(H^{01}, H^{02}, H^{03}) = \boldsymbol{D}$ and $(H^{23}, H^{31}, H^{12}) = \boldsymbol{H}$. This result agrees with Kluitenberg and de Groot's proposal [2 sect. 9] it may be compared with Minkowski's tensor [1]:

$$T_{\alpha}^{\cdot\beta(f)} = - F_{\alpha\gamma} H^{\gamma\beta} - \tfrac{1}{4} F_{\gamma\delta} H^{\gamma\delta} \delta_{\alpha}^{\beta} , \qquad (3)$$

and Abraham's tensor [1]:

$$T_{\alpha}^{\cdot\beta(f)} = - F_{\alpha\gamma} H^{\gamma\beta} - \tfrac{1}{4} F_{\gamma\delta} H^{\gamma\delta} \delta_{\alpha}^{\beta} + - U^{\beta} (F_{\alpha\gamma} H^{\gamma\delta} - H_{\alpha\gamma} F^{\gamma\delta}) U_{\delta} . \qquad (4)$$

Comparison is made easier by evaluating the tensor $T^{\alpha\beta(f)} = g^{\alpha\gamma} T_{\gamma}^{\cdot\beta(f)}$ in the rest frame. Using three-dimensional notation for the fields one obtains from (2) for the energy-density $T^{oo(f)}$, the Poynting vector $cT^{oi(f)}$, the momentum density $c^{-1}T^{io(f)}$ and the Maxwell stresses $-T^{ij(f)}$ (with $i, j = 1, 2, 3$):

$$T^{oo(f)} = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \quad T^{oi(f)} = (\boldsymbol{E} \times \boldsymbol{B})^i , \quad T^{io(f)} = (\boldsymbol{E}\times\boldsymbol{B})^i , \quad T^{ij(f)} = -E^i D^j - B^i B^j + \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) g^{ij} . \qquad (5)$$

Correspondingly one finds from Minkowski's tensor (3):

$$T^{oo(f)} = \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}^2), \quad T^{oi(f)} = (\boldsymbol{E} \times \boldsymbol{B})^i , \quad T^{io(f)} = = (\boldsymbol{D}\times\boldsymbol{B})^i, \quad T^{ij(f)} = - E^i D^j - B^i B^j + \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}^2) g^{ij} , \qquad (6)$$

and from Abraham's tensor [4]:

$$T^{oo(f)} = \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}^2) , \quad T^{oi(f)} = (\boldsymbol{E}\times\boldsymbol{B})^i , \quad T^{io(f)} = (\boldsymbol{E}\times\boldsymbol{B})^i , \quad T^{ij(f)} = -\tfrac{1}{2}(E^i D^j + D^i E^j) + -B^i B^j + \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}^2) g^{ij} . \qquad (7)$$

(Note that in the rest frame $\boldsymbol{H} = \boldsymbol{B}$ for the dielectric.)

The tensor (2), or (5), appears to be symmetric for an isotropic dielectric (where $\boldsymbol{D}$ and $\boldsymbol{E}$ are parallel in the rest frame); it is not symmetric for an anisotropic medium. Minkowski's tensor (3), or (6), is not symmetric even for isotropic materials, whereas Abraham's tensor (4), or (7), is always symmetric.

As known the four-vector $\partial_{\beta} T_{\alpha}^{\cdot\beta(f)}$ represents the ponderomotive four-force in the medium. A non-relativistic statistical treatment of the ponderomotive force in the electrostatic case yields a result which agrees with the present theory [8].

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