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THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

VII. DISCUSSION OF THE RESULTS IN CONNEXION WITH PREVIOUS WORK*)

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Synopsis

The literature on the relativistic energy-momentum tensor in polarized media falls apart in treatments based on microscopic first principles and considerations starting from macroscopic postulates. Only papers of the first category, such as Lorentz's and Einstein-Laub's, can be considered as derivations, whereas treatments of the second category, such as Minkowski's and Abraham's, based on *ad hoc* assumptions, do not give unique results.

§ 1. *Introduction.* In this paper the literature on the energy-momentum tensor in polarized media will be discussed in connexion with the results of the present series of articles. The history of the derivation of the energy-momentum laws from microscopic theory started with Lorentz's electron-theoretical treatment published in 1904[1]). Independently Einstein and Laub[2]) used similar considerations to obtain an expression for (part of) the energy-momentum tensor. The results of both Lorentz and Einstein-Laub came near to the expressions found from the statistical treatment based on microscopic theory, as will be shown in this paper.

In the subsequent literature little attention was paid to the papers of Lorentz and Einstein-Laub. Most of the discussions focused on the proposals put forward by Minkowski[3]) and Abraham[4]), which were suggested by arguments of a macroscopic nature based on *ad hoc* postulates, but not derived from the first principles of microscopic physics.

After a brief survey of the results obtained in the preceding papers of the series (§ 2), a discussion is given of previous work based on microscopic considerations (§§ 3, 4). Work of a macroscopic nature is reviewed in § 5.

§ 2. *Kinetic and macroscopic results derived from microscopic theory.* From the energy-momentum conservation laws valid for a set of charged point

*) Articles I–VI of this series in Physica **37** (1967) and **39** (1968).

particles conservation laws for a system consisting of dipole atoms (or other stable groups of point particles) have been derived[5]. These laws, valid at the "kinetic level" of the theory, have the form

$$\partial_\beta t^{\alpha\beta} = 0, \qquad t^{\alpha\beta} = t^{\alpha\beta}_{(f)} + t^{\alpha\beta}_{(m)}. \tag{1}$$

Here the atomic field energy-momentum tensor is given by (II.26) as

$$t^{\alpha\beta}_{(f)} = \sum_{k,l(k\neq l)} \{f_l^{\alpha\gamma} h_{k\cdot\gamma}^{\ \beta} - \tfrac{1}{4} f_{l\gamma\varepsilon} f_k^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2} u_k^\beta (f_l^{\alpha\gamma} m_{k\gamma\varepsilon} - m_k^{\alpha\gamma} f_{l\gamma\varepsilon})\, u_k^\varepsilon - c^{-4}\, u_k^\alpha u_k^\beta u_k^\gamma f_{l\gamma\varepsilon} m_k^{\varepsilon\zeta} u_{k\zeta}\}, \tag{2}$$

where $f_k^{\alpha\beta}$ is the atomic field tensor due to atom k, $m_k^{\alpha\beta}$ its atomic polarization tensor, $h_k^{\alpha\beta} \equiv f_k^{\alpha\beta} - m_k^{\alpha\beta}$ and u_k^α its four-velocity. Furthermore the atomic material energy-momentum tensor is, according to (II. 36),

$$t^{\alpha\beta}_{(m)} = \sum_k \rho_k^{t'} u_k^\alpha u_k^\beta - \tfrac{1}{2} \sum_k \Delta^\alpha_{k\varepsilon} \Delta^\beta_{k\zeta} \partial_\gamma (\sigma_k^{\varepsilon\zeta} u_k^\gamma) + \tfrac{1}{2} c^{-2} \sum_k (u_k^\alpha \sigma_k^{\beta\gamma} D_k u_{k\gamma} + u_k^\beta \sigma_k^{\alpha\gamma} D_k u_{k\gamma}) + \tfrac{1}{2} \sum_k \partial_\gamma (\sigma_k^{\alpha\gamma} u_k^\beta + \sigma_k^{\beta\gamma} u_k^\alpha), \tag{3}$$

where $\rho_k^{t'}$ is the mass density of atom k, $\Delta_k^{\alpha\beta} = g^{\alpha\beta} + c^{-2} u_k^\alpha u_k^\beta$, $D_k u_k^\alpha$ its four-acceleration and $\sigma_k^{\alpha\beta}$ its internal angular momentum density (called $\sigma_k^{+\alpha\beta}$ in article II). The total atomic energy-momentum tensor is symmetric:

$$t^{\alpha\beta} = t^{\beta\alpha}, \tag{4}$$

as follows from the conservation of angular momentum. In the classical model chosen here self-forces and relativistic contributions to the intra-atomic fields, which keep the stable groups together, are not considered.

With the help of covariant averaging macroscopic conservation laws have been obtained[6]:

$$\partial_\beta T^{\alpha\beta} = 0, \qquad T^{\alpha\beta} = T^{\alpha\beta}_{(f)} + T^{\alpha\beta}_{(m)}. \tag{5}$$

Here the macroscopic field energy-momentum tensor is according to (III.42):

$$T^{\alpha\beta}_{(f)} = F^{\alpha\gamma} H^{\beta}_{\cdot\gamma} - \tfrac{1}{4} F_{\gamma\varepsilon} F^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2} U^\beta (F^{\alpha\gamma} M_{\gamma\varepsilon} - M^{\alpha\gamma} F_{\gamma\varepsilon})\, U^\varepsilon - c^{-4} U^\alpha U^\beta U^\gamma F_{\gamma\varepsilon} M^{\varepsilon\zeta} U_\zeta, \tag{6}$$

where $F^{\alpha\beta}$ is the macroscopic (Maxwell) field tensor, $M^{\alpha\beta}$ the macroscopic polarization tensor, $H^{\alpha\beta} \equiv F^{\alpha\beta} - M^{\alpha\beta}$ and U^α the macroscopic four-velocity. The field tensor (6) reads in the rest frame (denoted by primes)

$$T^{\alpha\beta}_{(f)} = \begin{pmatrix} \tfrac{1}{2}(\boldsymbol{E}'^2 + \boldsymbol{B}'^2) & \boldsymbol{E}' \wedge \boldsymbol{H}' \\ \boldsymbol{E}' \wedge \boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + (\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{M}'\cdot\boldsymbol{B}')\,\mathsf{U} \end{pmatrix}. \tag{7}$$

The macroscopic material energy-momentum tensor $T^{\alpha\beta}_{(m)}$ has been given as a statistical expression in terms of atomic quantities[7]). It gets a simple form if the relative atomic motion within the correlation domain is non-relativistic and if dynamical effects due to the internal angular momentum of the atoms are disregarded. Then thermodynamical considerations[8]) lead to an equilibrium expression valid for a neutral, polarized fluid:

$$\bar{T}^{\alpha\beta}_{(m)} = c^{-2}U^{\alpha}U^{\beta}(e'_v + \varrho' c^2) + \Delta^{\alpha\beta}p', \tag{8}$$

where e'_v is the energy density, $\varrho' = (v')^{-1}$ the bulk rest mass density and p' the (isotropic) pressure, all measured in the local permanent rest frame. The latter quantities appear in the second law (Gibbs relation) for a neutral, polarized fluid:

$$T'Ds' = De' + p'Dv' - \tfrac{1}{2}F'_{\alpha\beta}D(v'M^{(1)\alpha\beta\prime}) + \tfrac{1}{2}M^{(2)\prime}_{\alpha\beta}v'DF^{\alpha\beta\prime}, \tag{9}$$

where D is the time derivative $\mathrm{d}/\mathrm{d}t$, T' the temperature, s' the specific entropy, $e' \equiv v'e'_v$ the specific energy, $F'_{\alpha\beta}$ the electromagnetic field and $M^{(1)\prime}_{\alpha\beta}$ and $M^{(2)\prime}_{\alpha\beta}$ the polarizations (which in the rest frame represent the electric and magnetic polarization respectively). The relation (9) may be written as:

$$T'Ds' = De' + p'Dv' - \boldsymbol{E}'\cdot D(v'\boldsymbol{P}') + v'\boldsymbol{M}'\cdot D\boldsymbol{B}'. \tag{10}$$

This law was derived (with the help of the canonical ensemble) in the nonrelativistic case and was then generalized to the relativistic case.

The total energy-momentum tensor may be split in a different way[9])

$$T^{\alpha\beta} = T^{\alpha\beta}_{[f]} + T^{\alpha\beta}_{[m]}, \tag{11}$$

where a material energy-momentum tensor has been introduced of the form

$$T^{\alpha\beta}_{[m]} = c^{-2}\,U^{\alpha}U^{\beta}(e'_{v0} + \varrho' c^2) + \Delta^{\alpha\beta}p'_0; \tag{12}$$

in contrast with (8) it contains an energy density e'_{v0} and a pressure p'_0 defined for the system *without* fields, but at the same temperature and density. The corresponding field tensor $T^{\alpha\beta}_{[f]}$ reads for the case of a substance which obeys linear constitutive relations (cf. (VI. 23)):

$$\begin{aligned} T^{\alpha\beta}_{[f]} = {} & F^{\alpha\gamma}H^{\beta}_{\cdot\gamma} - \tfrac{1}{4}F_{\gamma\varepsilon}H^{\gamma\varepsilon}g^{\alpha\beta} + c^{-2}U^{\beta}(F^{\alpha\gamma}M_{\gamma\varepsilon} - M^{\alpha\gamma}F_{\gamma\varepsilon})\,U^{\varepsilon} \\ & + \tfrac{1}{2}\Delta^{\alpha\beta}\left(c^{-2}\,v'\frac{\partial\kappa}{\partial v'}\,F_{\gamma\varepsilon}U^{\varepsilon}U_{\zeta}F^{\gamma\zeta} + \tfrac{1}{2}v'\frac{\partial\chi}{\partial v'}\,F_{\gamma\varepsilon}\Delta^{\gamma}_{\zeta}\Delta^{\varepsilon}_{\eta}F^{\zeta\eta}\right) \\ & + \tfrac{1}{2}c^{-2}U^{\alpha}U^{\beta}\left(c^{-2}T'\frac{\partial\kappa}{\partial T'}\,F_{\gamma\varepsilon}U^{\varepsilon}U_{\zeta}F^{\gamma\zeta} + \tfrac{1}{2}T'\frac{\partial\chi}{\partial T'}\,F_{\gamma\varepsilon}\Delta^{\gamma}_{\zeta}\Delta^{\varepsilon}_{\eta}F^{\zeta\eta}\right), \end{aligned} \tag{13}$$

where κ and χ are the electric and magnetic susceptibilities, v' the specific volume and T' the temperature in the rest frame. In the rest frame this tensor reads

$$T^{\alpha\beta}_{[f]} = \begin{vmatrix} \frac{1}{2}\left(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}' + \boldsymbol{E}'^2 T' \frac{\partial\kappa}{\partial T'} + \boldsymbol{B}'^2 T' \frac{\partial\chi}{\partial T'}\right) & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + \frac{1}{2}\left(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}' + \boldsymbol{E}'^2 v' \frac{\partial\kappa}{\partial v'} + \boldsymbol{B}'^2 v' \frac{\partial\chi}{\partial v'}\right)\mathrm{U} \end{vmatrix} \tag{14}$$

The ponderomotive force density is defined as the negative divergence of the field energy-momentum tensor. With the field tensor $T^{\alpha\beta}_{(f)}$ from splitting (5) one has the ponderomotive force density $F^\alpha = -\partial_\beta T^{\alpha\beta}_{(f)}$, corresponding thus to a material energy-momentum tensor $T^{\alpha\beta}_{(m)}$ which contains an energy density and a pressure defined for the system in the presence of fields. The space components ($\alpha = 1, 2, 3$) of this four-force read for uniform and constant velocity in the rest frame

$$\boldsymbol{F}' = (\boldsymbol{\nabla}'\boldsymbol{E}')\cdot\boldsymbol{P}' + (\boldsymbol{\nabla}'\boldsymbol{B}')\cdot\boldsymbol{M}' + \partial_0'(\boldsymbol{P}'\wedge\boldsymbol{B}' - \boldsymbol{M}'\wedge\boldsymbol{E}'). \tag{15}$$

This constitutes a generalization of the Kelvin force density.

If the field tensor is defined as in the splitting (11) one gets a ponderomotive force density $\mathscr{F}^\alpha = -\partial_\beta T^{\alpha\beta}_{[f]}$, corresponding to a material tensor $T^{\alpha\beta}_{[m]}$, which contains an energy density and a pressure defined for the system in the absence of fields. Its space components read for uniform and constant velocity in the rest frame

$$\mathscr{F}' = -\tfrac{1}{2}\boldsymbol{E}'^2\boldsymbol{\nabla}'\kappa - \boldsymbol{\nabla}'\left(\tfrac{1}{2}v'\frac{\partial\kappa}{\partial v'}\boldsymbol{E}'^2\right) - \tfrac{1}{2}\boldsymbol{B}'^2\boldsymbol{\nabla}'\chi - \boldsymbol{\nabla}'\left(\tfrac{1}{2}v'\frac{\partial\chi}{\partial v'}\boldsymbol{B}'^2\right) + \partial_0'\{(\kappa+\chi)\,\boldsymbol{E}'\wedge\boldsymbol{B}'\}. \tag{16}$$

This is a generalization of the Helmholtz force density.

§ 3. *Discussion of treatments based on microscopic theory.* In this section theories using microscopic concepts will be considered. Lorentz[1]) dealt with the problem of the electromagnetic forces in a medium using the methods of his electron theory. Although the treatment of matter is rather sketchy, since both a clear atomic picture and adequate statistical methods were lacking at the time, he nevertheless arrived at results for the electric dipole case which as far as the field terms are concerned are in agreement with (7). In fact Lorentz found for the field momentum density (times c) and flow

$$\boldsymbol{E}'\wedge\boldsymbol{B}', \quad -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{B}'\boldsymbol{B}' + \tfrac{1}{2}(\boldsymbol{E}'^2 + \boldsymbol{B}'^2)\,\mathrm{U}. \tag{17}$$

As for the material terms no explicit expressions for the pressure are given, except for a contribution $-\frac{1}{5}\boldsymbol{P}'\boldsymbol{P}' - \frac{1}{10}\boldsymbol{P}'^2\mathrm{U}$. The latter term may indeed be found if the principal value part is split off from the potential pressure[7]).

Independently Einstein and Laub[2]) derived the field momentum density and flow for an electric and magnetic dipole substance. The magnetic terms were obtained by an argument of analogy only. They found the expressions

$$\boldsymbol{E}' \wedge \boldsymbol{H}', \quad -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + (\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' + \tfrac{1}{2}\boldsymbol{M}'^2)\,\mathrm{U}. \tag{18}$$

The electric terms are the same as those in (7), but an extra magnetic term $\frac{1}{2}\boldsymbol{M}'^2$ is present in the scalar part of the field pressure. The material terms are not considered at all, so that the validity of the field expressions (18) cannot be assessed. Gans[10]) disputed the correctness of Einstein and Laub's arguments, as Pauli mentions in his review[11]). However Gans's critique concerned only the splitting of the total force density into a Lorentz force density (acting on the charges) and a ponderomotive force (acting on the polarizations); this splitting is arbitrary since only total forces can be measured (v. also article III, § 7).

In the meantime purely macroscopic proposals*) had been put forward by Minkowski[3]) and soon afterwards, by Abraham[4]). Their field tensors became the subject of many discussions, amongst which treatments based on microscopic theory. For the discussion of the latter it will be convenient to have at our disposal the explicit expressions for these tensors. Minkowski's field energy-momentum tensor reads in the rest frame

$$T^{\alpha\beta'}_{(f)M} = \begin{pmatrix} \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}') & \boldsymbol{E}' \wedge \boldsymbol{H}' \\ \boldsymbol{D}' \wedge \boldsymbol{B}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}')\mathrm{U} \end{pmatrix}, \tag{19}$$

while Abraham's tensor is

$$T^{\alpha\beta'}_{(f)A} = \begin{pmatrix} \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}') & \boldsymbol{E}' \wedge \boldsymbol{H}' \\ \boldsymbol{E}' \wedge \boldsymbol{H}' & \begin{matrix} -\frac{1}{2}(\boldsymbol{E}'\boldsymbol{D}' + \boldsymbol{D}'\boldsymbol{E}' + \boldsymbol{H}'\boldsymbol{B}' + \boldsymbol{B}'\boldsymbol{H}') \\ + \frac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{B}'\cdot\boldsymbol{H}')\,\mathrm{U} \end{matrix} \end{pmatrix}. \tag{20}$$

In a later paper Abraham[12]) remarked that the derivation of the expression for the energy-momentum tensor should be based on microscopic theory, but he limited himself to a discussion of possible approaches. Dällenbach[13]) tried to follow an electron-theoretical line, but he did not give a proper derivation, since the material part of the energy-momentum tensor is not considered. He ends up with Minkowski's field tensor by generalizing electrostatic arguments. Frenkel[14]) starts from microscopic considerations but since he believed that covariance includes form invariance, he was not able to arrive at a conclusion, as he states explicitly. Ott[15]) would prefer a

*) For their discussion see section 5.

microscopic starting point, but arrives at his result (the Minkowski tensor) with *ad hoc* arguments of a macroscopic nature. Marx and Györgyi[16]) follow the same line as Einstein and Laub and advocate Abraham's proposal. Since the material tensor is not considered, a unique result cannot be found from such a method. Rancoita[17]) uses some microscopic concepts, but gives no derivation on this basis since in his treatment several arguments of a macroscopic origin are used. His final result however is the correct expression (7).

The averaging is performed in a rather loose way by all authors quoted in this section: sometimes averaging is performed over small (time or space) regions, sometimes the averaging is not even specified.

The ponderomotive force in nonrelativistic approximation has been derived from a microscopic basis by Mazur and de Groot[18]) for an electrostatic dipole system and extended to higher multipoles by Kaufman[19]).

§ 4. *Relativistic dynamics of a composite particle.* The problem of the derivation of the relativistic energy-momentum tensor from microscopic theory contains in its first stage the determination of the relativistic equations of motion of a stable group (called "atom" here) of charged (spinless) point particles. From the energy-momentum conservation laws for an atom carrying a charge, an electric dipole moment and a magnetic dipole moment[5])[20]) one may deduce equations of motion of the form:

$$\frac{\mathrm{d}G^\alpha}{\mathrm{d}s} = c^{-1}eF^{\alpha\beta}U_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\,M_{\beta\gamma} - c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}\,(F^{\alpha\beta}M_{\beta\gamma}U^\gamma), \tag{21}$$

$$\frac{\mathrm{d}S^{\alpha\beta}}{\mathrm{d}s} = G^\alpha U^\beta - G^\beta U^\alpha + \Delta^\alpha_\varepsilon M^{\varepsilon\gamma}F^\beta_{\cdot\gamma} - \Delta^\beta_\varepsilon F^{\alpha\gamma}M^\varepsilon_{\cdot\gamma}, \tag{22}$$

where the abbreviation

$$G^\alpha = mU^\alpha + c^{-2}S^{\alpha\beta}\dot{U}_\beta - c^{-2}\Delta^\alpha_\beta M^{\beta\gamma}F_{\gamma\varepsilon}U^\varepsilon \tag{23}$$

is used. Here s is the proper time of the atom, U^α its four-velocity, $\dot{U}^\alpha$ its four-acceleration, e its charge, m its mass (rest mass and internal kinetic and Coulomb energies of the constituent particles), $S^{\alpha\beta}$ its internal angular momentum, $M^{\alpha\beta}$ its polarization tensor and $F^{\alpha\beta}$ the external field. For a magnetic dipole atom (21), (22) and (23) reduce to

$$\frac{\mathrm{d}G^\alpha}{\mathrm{d}s} = c^{-1}eF^{\alpha\beta}U_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\,M_{\beta\gamma}, \tag{24}$$

$$\frac{\mathrm{d}S^{\alpha\beta}}{\mathrm{d}s} = G^\alpha U^\beta - G^\beta U^\alpha + M^{\alpha\gamma}F^\beta_{\cdot\gamma} - F^{\alpha\gamma}M^\beta_{\cdot\gamma}, \tag{25}$$

where now

$$G^\alpha = mU^\alpha + c^{-2}S^{\alpha\beta}\dot{U}_\beta - c^{-2}M^{\alpha\beta}F_{\beta\gamma}U^\gamma. \tag{26}$$

These equations are exactly the same as those written down by Frenkel[21]) for a single particle with intrinsic spin in classical dynamics.

An important point in the formulation of the relativistic dynamics of a composite particle is the definition of the centre of gravity. In the frame work of our theory it was sufficient to introduce a definition of an approximate centre of gravity which is correct up to the second order in the atomic parameters r_{ki}. In this way an explicit construction of the centre of gravity could be indicated in a unique way. It is known[22]) that an exact definition can only be given at the expense of the constructive character: in fact the exact definition does *not* determine the centre of gravity in a unique way. If such a definition is nevertheless used atomic equations of motion may be derived, which have the same form as those of the present theory, as has been shown by Vlieger[23]). His final results contain an unspecified tensor $T^{\alpha\beta}$, which is assumed to be symmetric. However its form can be deduced from the microscopic energy-momentum laws to be:

$$T^{\alpha\beta}(R) = \sum_i c \int m_i \frac{\mathrm{d}R_i^\alpha}{\mathrm{d}\tau_i} \frac{\mathrm{d}R_i^\beta}{\mathrm{d}\tau_i} \delta(R_i - R)\, \mathrm{d}\tau_i + \sum_{i \neq j} \{f^{\alpha\gamma}_{(in)i}(R)\, f_{(in)j}{}^{\beta}{}_{\cdot\gamma}(R) - \tfrac{1}{4} f^{\gamma\varepsilon}_{(in)i}(R)\, f_{(in)j\gamma\varepsilon}(R)\, g^{\alpha\beta}\}, \tag{27}$$

where i, j number the constituents of the composite particle ("atom"), m_i the mass of particle i, $\mathrm{d}R_i^\alpha/\mathrm{d}\tau_i$ its four-velocity, R_i^α its position and $f^{\alpha\beta}_{(in)i}$ the intra-atomic field due to particle i. This tensor is indeed symmetric. If (27) is inserted in Vlieger's final expressions one obtains the results of papers I and II of this series, if the Darwin approximation is used.

In the literature the dynamics of a magnetic dipole particle has often been discussed. Relativistic dynamics yielded the equation (21) with (23), which in the rest frame, in three-dimensional notation and including terms of order c^{-1} reads

$$m \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = (\boldsymbol{\nabla} \boldsymbol{B}) \cdot \boldsymbol{M} - \frac{1}{c} \frac{\partial}{\partial t} (\boldsymbol{M} \wedge \boldsymbol{E}) = \boldsymbol{M} \cdot \boldsymbol{\nabla} \boldsymbol{B} - \frac{1}{c} \frac{\partial \boldsymbol{M}}{\partial t} \wedge \boldsymbol{E}. \tag{28}$$

Tellegen[24]) finds a different result, because he uses nonrelativistic dynamics. Remarkably enough however he states that formula (28) describes the force on a "magnetic-charge dipole" of moment $\boldsymbol{M}$.

§ 5. *Discussion of treatments based on macroscopic arguments.* Many authors try to tackle the problem of finding the energy-momentum tensor, especially its field part, by means of macroscopic *ad hoc* arguments, instead of deriving it from microscopic theory[20]). No explicit expressions for the material part of the energy-momentum tensor are given, and often the latter tensor is not even considered. The problem then remains to a large extent undetermined.

Much of the discussion centred on the relative merits of Minkowski's and Abraham's field tensors (19) and (20). Minkowski[3]) adopted as a guiding principle form invariance in all Lorentz frames; this is however a requirement which may not be imposed on the theory. With the help of this principle and the expressions for the field energy density, the field energy flow and the field pressure due to Maxwell, Poynting and Heaviside he arrived at (19), which contains as momentum density $c^{-1}\boldsymbol{D}' \wedge \boldsymbol{B}'$. Indeed Minkowski's field energy-momentum tensor has the same form (19), but without primes, in an arbitrary Lorentz frame.

Abraham[4]) adopts Hertz's symmetrized field pressure expression and also symmetrizes the time-space and space-time field tensor components by writing $c^{-1}\boldsymbol{E}' \wedge \boldsymbol{H}'$ for the momentum density. In this way he obtains a completely symmetrical field energy-momentum tensor, even for anisotropic media. In a later paper[12]) he mentions as an argument in favour of this symmetry the fact that the microscopic tensor is symmetric. However this argument ensures only the symmetry of the total energy-momentum tensor, *not* the symmetry of the field tensor. As a different argument in favour of the equality of space-time and time-space components (in the rest frame) of the field tensor Planck's remark that energy transport and momentum density are equal (apart from a factor c^2) is often quoted[25]). However again such an argument can only be used for the *total* energy-momentum tensor.

A much discussed argument in favour of the asymmetric Minkowski tensor was put forward by Von Laue[26]). According to this argument the energy transport velocity, which is the quotient of the energy flow and the energy density, should transform in such a way that the addition theorem for velocities is satisfied. The Minkowski field tensor does satisfy this criterion, but Abraham's field tensor does not. However Tang and Meixner[27]) invalidated Von Laue's argument by showing that it may only be applied to the total energy transport and has no physical content for the field energy transport alone. (Moreover the criterion had to be amended somewhat in order to be valid for the total energy transport).

Often reasonings which start from macroscopic variational principles are considered as derivations of the form of the field energy-momentum tensor[28]). In this way Minkowski's, Abraham's and other expressions have been found. Such arguments are not convincing since they start from postulated macroscopic Lagrangians which themselves are not derived from first principles. Various other *ad hoc* macroscopic postulates[29]) have been put forward in order to justify the choice of a particular form of the field energy-momentum tensor[20]).

A somewhat special class of theories is based on thermodynamical considerations. In the framework of a theory by Kluitenberg and de Groot[30]) a relativistic Gibbs relation and the symmetric character of the material energy-momentum tensor were postulated. As a result a field energy-momentum tensor is obtained which is symmetric, and which comes very near to the expression (7). De Sa[31]) and Meixner[32]) discuss various possibilities of the splitting of the total energy-momentum tensor into a material and a field part. They rightly conclude that thermodynamical view points do not allow to specify the material part sufficiently well; hence the field part remains then undetermined. Chu, Haus and Penfield[33]) postulate a form for the first law of thermodynamics and the symmetrical character of the material energy-momentum tensor. Since this starting point is equivalent to Kluitenberg and de Groot's their resulting field energy-momentum tensor is also the same, apart from some diagonal terms. Chu, Haus and Penfield follow the same thermodynamical reasoning as Prigogine and Mazur[34]) in order to compare the material pressure in systems with and without fields.

In general it may be stated that the solution of the problem to derive the field energy-momentum tensor in polarized media remains undetermined as long as macroscopic arguments are used as guiding principles. The problem only becomes well-defined if the total (field plus matter) energy-momentum laws are considered. This can be achieved if one starts from the microscopic conservation laws (*v.* § 2).

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