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# THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

## III. STATISTICAL THEORY OF THE ENERGY-MOMENTUM LAWS*)

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**Synopsis**

From the atomic conservation laws of energy-momentum the corresponding macroscopic laws are derived with the help of a covariant averaging procedure. The total energy-momentum tensor is found as a statistical expression in terms of atomic quantities. It may be split into a field part $T^{\alpha\beta}_{(f)}$ ($\alpha, \beta = 0, 1, 2, 3$) containing the macroscopic fields and polarizations, which in the rest frame reads:

$$T^{\alpha\beta}_{(f)} = \begin{pmatrix} \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 & (\boldsymbol{E} \wedge \boldsymbol{H})^i \\ (\boldsymbol{E} \wedge \boldsymbol{H})^i & -E^iD^j - H^iB^j + (\frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 - \boldsymbol{M}\cdot\boldsymbol{B})\, g^{ij} \end{pmatrix} \quad (i, j = 1, 2, 3)$$

and a material part $T^{\alpha\beta}_{(m)}$ which forms the relativistic generalization of the usual energy and momentum expressions.

§ 1. *Introduction.* In the papers I and II of this series[1]) the energy-momentum and angular momentum conservation laws for systems consisting of atoms, molecules or ions, carrying electric and magnetic dipoles, have been derived from microscopic theory (cf. § 2.)

In this paper we shall derive the macroscopic conservation laws by an averaging procedure using covariant distribution functions (§ 3). In this way the total macroscopic energy-momentum tensor $T^{\alpha\beta}$ is found as a statistical average of atomic quantities (§ 4). It turns out to contain a "field" part $T^{\alpha\beta}_{(f)}$, which depends only on the macroscopic fields, polarizations and velocities (§ 5) and a "material" part $T^{\alpha\beta}_{(m)}$ (§ 6). From these results the ponderomotive force and torque, which the electromagnetic field exerts on matter, is derived (§ 7–8).

The field energy-momentum tensor $T^{\alpha\beta}_{(f)}$ found here is similar to expressions obtained by Lorentz[2]) and by Einstein and Laub[3]) for the electric dipole case, but is essentially different from the expressions proposed by Minkowski[4]) and Abraham[5]).

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*) Articles I and II appeared in Physica **37** (1967) 284, 297.

§ 2. *The atomic equations.* The energy-momentum conservation laws on the atomic level have the following form

$$\partial_\beta t^{\alpha\beta} = 0, \tag{1}$$

where the total energy-momentum tensor of the system $t^{\alpha\beta}$ is the sum of a material and a field part

$$t^{\alpha\beta} = t^{\alpha\beta}_{(m)} + t^{\alpha\beta}_{(f)}. \tag{2}$$

The material energy-momentum tensor is given in paper II [1]) by

$$t^{\alpha\beta}_{(m)} = \sum_k \rho^{t'}_k u^\alpha_k u^\beta_k - \tfrac{1}{2} \sum_k \Delta^\alpha_{k\varepsilon} \Delta^\beta_{k\zeta} \partial_\gamma (\sigma^{\varepsilon\zeta}_k u^\gamma_k) +$$
$$+ \tfrac{1}{2} c^{-2} \sum_k (u^\alpha_k \sigma^{\beta\gamma}_k D_k u_{k\gamma} + u^\beta_k \sigma^{\alpha\gamma}_k D_k u_{k\gamma}) + \tfrac{1}{2} \sum_k \partial_\gamma (\sigma^{\alpha\gamma}_k u^\beta_k + \sigma^{\beta\gamma}_k u^\alpha_k), \tag{3}$$

where $\rho^{t'}_k$ is the sum of the rest mass density and the internal kinetic and Coulomb energy densities of atom $k$, multiplied by $c^{-2}$, $u^\alpha_k$ the four-velocity of atom $k$, $\Delta^{\alpha\beta}_k = g^{\alpha\beta} + c^{-2} u^\alpha_k u^\beta_k$ ($g^{00} = -1$, $g^{ii} = 1$, $i = 1, 2, 3$), $D_k u^\alpha_k$ its four-acceleration and $\sigma^{\alpha\beta}_k$ the atomic angular momentum density (called $\sigma^{+\alpha\beta}_k$ in II). The field energy-momentum tensor is [1])

$$t^{\alpha\beta}_{(f)} = \sum_{k,l,k\neq l} \{ f^{\alpha\gamma}_l h_k{}^\beta{}_\gamma - \tfrac{1}{4} f_{k\gamma\varepsilon} f^{\gamma\varepsilon}_l g^{\alpha\beta} +$$
$$+ c^{-2} u^\beta_k (f^{\alpha\gamma}_l m_{k\gamma\varepsilon} - m^{\alpha\gamma}_k f_{l\gamma\varepsilon})\, u^\varepsilon_k - c^{-4} u^\alpha_k u^\beta_k u^\gamma_k f_{l\gamma\varepsilon} m^{\varepsilon\zeta}_k u_{k\zeta} \}, \tag{4}$$

where $f^{\alpha\beta}_k$ is the electromagnetic field due to atom $k$, $m^{\alpha\beta}_k$ the polarization tensor due to atom $k$ and $h^{\alpha\beta}_k = f^{\alpha\beta}_k - m^{\alpha\beta}_k$.

The total energy-momentum tensor (2) is symmetric [1])

$$t^{\alpha\beta} = t^{\beta\alpha}. \tag{5}$$

The atomic energy-momentum conservation laws will yield the corresponding macroscopic conservation laws with the help of a covariant averaging procedure.

§ 3. *Retarded distribution functions.* Macroscopic quantities are averages over a number of atoms situated in a macroscopically infinitesimal region, which still contains a sufficient number of particles, such that the principles of statistical mechanics may be applied. As (3) and (4) show, the quantities, which will be averaged are sum functions of one-particle and two-particle dynamical quantities which depend on the retarded values of the atomic positions, the atomic internal coordinates and time derivatives ("fluxions") of these variables. In order to take retardation into account we introduce a one-point "retarded distribution function" [6])

$$f^{\text{ret}}_1(1\,; \boldsymbol{R}, t) \equiv f^{\text{ret}}_1(\boldsymbol{R}_1, \boldsymbol{R}^{(1)}_1, \ldots, \xi_1, \xi^{(1)}_1, \ldots; \boldsymbol{R}, t), \tag{6}$$

which depends on the position $\boldsymbol{R}_1$ of an atom, its velocity $\boldsymbol{R}^{(1)}_1$, its acceleration

$\boldsymbol{R}_1^{(2)}$, higher fluxions $\boldsymbol{R}_1^{(n)}$, the dipole moments indicated by the symbol $\xi_1$, and their fluxions $\xi_1^{(m)}$. Moreover, $\boldsymbol{R}$ and $t$ indicate the reference point. The function (6) is defined in such a way that

$$f_1^{\text{ret}}(1\,;\boldsymbol{R}, t)\, \mathrm{d}\boldsymbol{R}_1\, \mathrm{d}\boldsymbol{R}_1^{(1)} \dots \mathrm{d}\xi_1^{(m)} \tag{7}$$

is the probability (normalized to the number of particles $N$), that a sphere, contracting with the speed of light towards the point $\boldsymbol{R}$ at time $t$, encounters an atom 1 with values of the fluxions $\boldsymbol{R}^{(s)}$ in an interval $\mathrm{d}\boldsymbol{R}_1^{(s)}$ around $\boldsymbol{R}_1^{(s)}$ and fluxions $\xi^{(s)}$ in an interval $\mathrm{d}\xi_1^{(s)}$ around $\xi_1^{(s)}$. In an analogous way a two-point retarded distribution function

$$f_2^{\text{ret}}(1, 2; \boldsymbol{R}, t) \equiv f_2^{\text{ret}}(\boldsymbol{R}_1, \dots, \xi_1^{(m)}, \boldsymbol{R}_2, \dots, \xi_2^{(m)}; \boldsymbol{R}, t) \tag{8}$$

is defined such that

$$f_2^{\text{ret}}(1, 2; \boldsymbol{R}, t)\, \mathrm{d}\boldsymbol{R}_1 \dots \mathrm{d}\xi_1^{(m)}\, \mathrm{d}\boldsymbol{R}_2 \dots \mathrm{d}\xi_2^{(m)} \tag{9}$$

is the probability, normalized to $N(N-1)$, that the contracting sphere encounters an atom 1 with certain values of the fluxions $\boldsymbol{R}_1^{(s)}$ and $\xi_1^{(s)}$ and a different atom 2 with certain values of $\boldsymbol{R}_2^{(s)}$ and $\xi_2^{(s)}$. From this definition it follows that $f_2^{\text{ret}}$ vanishes if $\boldsymbol{R}_1 = \boldsymbol{R}_2$. The two-point function is related to the one-point function by means of

$$\int f_2^{\text{ret}}(1, 2; \boldsymbol{R}, t)\, \mathrm{d}\boldsymbol{R}_2 \dots \mathrm{d}\xi_2^{(m)} = (N-1)\, f_1^{\text{ret}}(1\,; \boldsymbol{R}, t). \tag{10}$$

The retarded distribution function may be related to the ordinary distribution function. A one-point ordinary distribution function

$$f_1(1\,; t) \equiv f_1(\boldsymbol{R}_1, \boldsymbol{R}_1^{(1)}, \dots, \xi_1^{(m)}; t) \tag{11}$$

can be introduced by defining

$$f_1(1\,; t)\, \mathrm{d}\boldsymbol{R}_1\, \mathrm{d}\boldsymbol{R}_1^{(1)} \dots \mathrm{d}\xi_1^{(m)} \tag{12}$$

as the probability (normalized to $N$) of finding an atom in the volume element $\mathrm{d}\boldsymbol{R}_1\, \mathrm{d}\boldsymbol{R}_1^{(1)} \dots \mathrm{d}\xi_1^{(m)}$ around the point $\boldsymbol{R}_1, \boldsymbol{R}_1^{(1)} \dots \xi_1^{(m)}$ at the time $t$. Similarly a two-point ordinary distribution function

$$f_2(1, 2; t_1, t_2) \equiv f_2(\boldsymbol{R}_1, \boldsymbol{R}_1^{(1)}, \dots, \xi_2^{(m)}; t_1, t_2) \tag{13}$$

can be defined by writing the joint probability (normalized to $N(N-1)$) to find an atom in the element $\mathrm{d}\boldsymbol{R}_1\, \mathrm{d}\boldsymbol{R}_1^{(1)} \dots \mathrm{d}\xi_1^{(m)}$ around $\boldsymbol{R}_1, \boldsymbol{R}_1^{(1)}, \dots\dots, \xi_1^{(m)}$ at time $t_1$, and another atom in $\mathrm{d}\boldsymbol{R}_2\, \mathrm{d}\boldsymbol{R}_2^{(1)} \dots \mathrm{d}\xi_2^{(m)}$ around $\boldsymbol{R}_2, \boldsymbol{R}_2^{(1)}, \dots, \xi_2^{(m)}$ at time $t_2$ as:

$$f_2(1, 2; t_1, t_2)\, \mathrm{d}\boldsymbol{R}_1\, \mathrm{d}\boldsymbol{R}_1^{(1)} \dots \mathrm{d}\xi_1^{(m)}\, \mathrm{d}\boldsymbol{R}_2\, \mathrm{d}\boldsymbol{R}_2^{(1)} \dots \mathrm{d}\xi_2^{(m)}. \tag{14}$$

From the definitions one finds that the retarded distribution functions are proportional to the ordinary distribution functions at retarded times:

$$f_1^{\text{ret}}(1; \boldsymbol{R}, t) = \kappa_1 f_1\left(1; t - \frac{|\boldsymbol{R} - \boldsymbol{R}_1|}{c}\right), \tag{15}$$

$$f_2^{\text{ret}}(1, 2; \boldsymbol{R}, t) = \kappa_1\kappa_2 f_2\left(1, 2; t - \frac{|\boldsymbol{R} - \boldsymbol{R}_1|}{c}, t - \frac{|\boldsymbol{R} - \boldsymbol{R}_2|}{c}\right), \tag{16}$$

$$\kappa_i \equiv 1 - \frac{\boldsymbol{R}_i^{(1)} \cdot (\boldsymbol{R} - \boldsymbol{R}_i)}{c|\boldsymbol{R} - \boldsymbol{R}_i|} \qquad (i = 1, 2). \tag{17}$$

With the help of the retarded distribution functions we can define averages of sum functions such that the retardation is taken into account. For physical quantities which are sum functions $a = \sum_k \alpha(k; \boldsymbol{R}, t)$ of one-particle retarded dynamical quantities

$$\alpha(k; \boldsymbol{R}, t) \equiv \alpha(\boldsymbol{R}_k, \boldsymbol{R}_k^{(1)}, \ldots, \boldsymbol{R}_k^{(n)}, \xi_k, \ldots, \xi_k^{(m)}; \boldsymbol{R}, t) \tag{18}$$

we have the avarage:

$$\langle a \rangle = \int \alpha(1; \boldsymbol{R}, t)\, f_1^{\text{ret}}(1; \boldsymbol{R}, t)\, \mathrm{d}V_1, \tag{19}$$

where $\mathrm{d}V_1$ is the element $\mathrm{d}\boldsymbol{R}_1\, \mathrm{d}\boldsymbol{R}_1^{(1)} \ldots \mathrm{d}\boldsymbol{R}_1^{(n)}\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_1^{(m)}$. Similarly for quantities which are sum functions $a = \sum_{k,l,k\neq l} \alpha(k, l; \boldsymbol{R}, t)$ of two-particle retarded quantities

$$\alpha(k, l; \boldsymbol{R}, t) \equiv \alpha(\boldsymbol{R}_k, \ldots, \xi_k^{(m)}, \boldsymbol{R}_l, \ldots, \xi_l^{(m)}; \boldsymbol{R}, t) \tag{20}$$

the average is

$$\langle a \rangle = \int \alpha(1, 2; \boldsymbol{R}, t)\, f_2^{\text{ret}}(1, 2; \boldsymbol{R}, t)\, \mathrm{d}V_1\, \mathrm{d}V_2. \tag{21}$$

If the quantity (20) is independent of $l$, then in (21) the integration over the coordinates 2 can be carried out according to (10). One obtains thus an average of the type (19).

The quantities (7) and (9) are Lorentz invariant, since they are defined as probabilities that a light wave front encounters particles. Therefore the covariant character of the averages (19) and (21) is the same as that of the corresponding microscopic quantities (18) and (20) respectively.

Since the atomic quantities occurring in the equations of § 2 contain space and time derivatives, we are interested in the averages of quantities like $\partial_\mu a$, where $a$ is an atomic quantity of the type just discussed. We want to prove a *lemma* according to which

$$\langle \partial_\mu a \rangle = \partial_\mu \langle a \rangle, \tag{22}$$

i.e., space-time differentiation and averaging commute. If $a$ is a sum function $a = \sum_{k,l,k\neq l} \alpha(k, l; \boldsymbol{R}, t)$ of two-particle retarded quantities, the left-hand side of (22) reads for the case of time differentiation ($\mu = 0$,

$\partial_0 = \partial/\partial ct$):

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int \left\{ \alpha\left( \boldsymbol{R}_1 + \frac{\mathrm{d}\boldsymbol{R}_1}{\mathrm{d}t}\,\Delta t, \ldots, \boldsymbol{R}_1^{(n)} + \frac{\mathrm{d}\boldsymbol{R}_1^{(n)}}{\mathrm{d}t}\,\Delta t, \boldsymbol{R}_2 + \frac{\mathrm{d}\boldsymbol{R}_2}{\mathrm{d}t}\,\Delta t, \ldots \right.\right.$$

$$\left.\left. \ldots, \boldsymbol{R}_2^{(n)} + \frac{\mathrm{d}\boldsymbol{R}_2^{(n)}}{\mathrm{d}t}\,\Delta t; \boldsymbol{R}, t + \Delta t\right) - \alpha(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}; \boldsymbol{R}, t) \right\} \cdot$$

$$\cdot f_2^{\mathrm{ret}}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n+1)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n+1)}; \boldsymbol{R}, t)\, \mathrm{d}V_1^{n+1}\, \mathrm{d}V_2^{n+1}, \tag{23}$$

where for brevity's sake the internal variables $\xi$ have not been written down. Because $\alpha$ is a retarded quantity, the time derivatives appearing here are given by

$$\frac{\mathrm{d}\boldsymbol{R}_i^{(s)}}{\mathrm{d}t} = \frac{\boldsymbol{R}_i^{(s+1)}}{\kappa_i} \qquad (i = 1, 2), \tag{24}$$

which depend on the higher order fluxion $\boldsymbol{R}_i^{(s+1)}$ $(i = 1, 2)$. Since $\boldsymbol{R}_i^{(n+1)}$ has to be considered as an independent variable, we had to introduce a distribution function, which contains the independent variables $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}$, $\boldsymbol{R}_1^{(n+1)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}, \boldsymbol{R}_2^{(n+1)}$. (The index $n + 1$ of the volume elements indicates integration over these variables). From the conservation of particle number and the relation connecting distribution functions of different dimensionality:

$$\int f_2^{\mathrm{ret}}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n+1)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n+1)}; \boldsymbol{R}, t)\, \mathrm{d}\boldsymbol{R}_1^{(n+1)}\, \mathrm{d}\boldsymbol{R}_2^{(n+1)} =$$

$$= f_2^{\mathrm{ret}}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}; \boldsymbol{R}, t) \tag{25}$$

it follows that (23) can be written in the form:

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int \{\alpha(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}; \boldsymbol{R}, t + \Delta t)\,\cdot$$

$$\cdot f_2^{\mathrm{ret}}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}; \boldsymbol{R}, t + \Delta t) - \alpha(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}; \boldsymbol{R}, t)\,\cdot$$

$$\cdot f_2^{\mathrm{ret}}(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_1^{(n)}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_2^{(n)}; \boldsymbol{R}, t)\, \mathrm{d}V_1^n\, \mathrm{d}V_2^n. \tag{26}$$

In this expression one recognizes the right-hand side of the lemma (22), which is thus proved for $\mu = 0$. The case $\mu = 1, 2, 3$ for the space differentiation can be treated in an analogous way. The lemma is valid also for one-particle sum functions, as follows from (10).

§ 4. *Averaged energy-momentum conservation laws.* With the help of the lemma (22) averaging of the atomic conservation laws (1) leads to the macroscopic conservation laws:

$$\partial_\beta T^{\alpha\beta} = 0, \tag{27}$$

where we have written

$$T^{\alpha\beta} = \langle t^{\alpha\beta} \rangle. \tag{28}$$

According to (5) this tensor is symmetric:

$$T^{\alpha\beta} = T^{\beta\alpha}. \tag{29}$$

Just as the atomic energy-momentum tensor $t^{\alpha\beta}$ (eq. (2)) was the sum of a material and a field part, the macroscopic tensor can also be looked upon as the sum of a material and a field contribution:

$$T^{\alpha\beta} = T^{\alpha\beta}_{(m)} + T^{\alpha\beta}_{(f)}. \tag{30}$$

This splitting will not be performed simply such that $T^{\alpha\beta}_{(m)}$ is the average of $t^{\alpha\beta}_{(m)}$, but in a different way for reasons that will appear in the following.

Let us consider the average (28) of (2):

$$T^{\alpha\beta} = \langle t^{\alpha\beta}_{(m)} + t^{\alpha\beta}_{(f)} \rangle, \tag{31}$$

where $t^{\alpha\beta}_{(m)}$ and $t^{\alpha\beta}_{(f)}$ are given by (3) and (4). The latter formulae show that $t^{\alpha\beta}_{(m)}$ is a sum function of one-particle atomic quantities, and $t^{\alpha\beta}_{(f)}$ a sum function of two-particle atomic quantities. According to (3), (4), (19) and (21) the tensor (31) can be written as:

$$\begin{aligned} T^{\alpha\beta} = \int \{&\rho_1^{t'} u_1^\alpha u_1^\beta - \tfrac{1}{2}\Delta^\alpha_{1\varepsilon}\Delta^\beta_{1\zeta}\partial_\gamma(\sigma_1^{\varepsilon\zeta}u_1^\gamma) + \tfrac{1}{2}c^{-2}(u_1^\alpha\sigma_1^{\beta\gamma}D_1u_{1\gamma} + u_1^\beta\sigma_1^{\alpha\gamma}D_1u_{1\gamma}) \\ &+ \tfrac{1}{2}\partial_\gamma(\sigma_1^{\alpha\gamma}u_1^\beta + \sigma_1^{\beta\gamma}u_1^\alpha)\}\, f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\,\mathrm{d}V_1 + \int \{f_2^{\alpha\gamma}h_{1\cdot\gamma}^{\beta} - \tfrac{1}{4}f_{1\gamma\varepsilon}f_2^{\gamma\varepsilon}g^{\alpha\beta} \\ &+ c^{-2}\, u_1^\beta(f_2^{\alpha\gamma}\, m_{1\gamma\varepsilon} - m_1^{\alpha\gamma}f_{2\gamma\varepsilon})\, u_1^\varepsilon \\ &- c^{-4}\, u_1^\alpha u_1^\beta u_1^\gamma f_{2\gamma\varepsilon} m_1^{\varepsilon\zeta} u_{1\zeta}\}\, f_2^{\mathrm{ret}}(1,2;\boldsymbol{R},t)\,\mathrm{d}V_1\,\mathrm{d}V_2. \end{aligned} \tag{32}$$

From this total tensor we want to split off a macroscopic field tensor $T^{\alpha\beta}_{(f)}$, which depends on the macroscopic electromagnetic fields and polarizations. These fields $F^{\alpha\beta}$, $H^{\alpha\beta} = F^{\alpha\beta} - M^{\alpha\beta}$ and polarizations $M^{\alpha\beta}$ are defined as averages of the atomic fields $f^{\alpha\beta} = \sum_k f_k^{\alpha\beta}$, $h^{\alpha\beta} = \sum_k h_k^{\alpha\beta}$ and polarizations $m^{\alpha\beta} = \sum_k m_k^{\alpha\beta}$:

$$F^{\alpha\beta} = \int f_1^{\alpha\beta} f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\,\mathrm{d}V_1, \tag{33}$$

$$H^{\alpha\beta} = \int h_1^{\alpha\beta} f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\,\mathrm{d}V_1, \tag{34}$$

$$M^{\alpha\beta} = \int m_1^{\alpha\beta} f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\,\mathrm{d}V_1. \tag{35}$$

These macroscopic fields and polarizations occur in the "noncorrelated" part of the second integral in (32). In fact one can write the two-point distribution function $f_2^{\mathrm{ret}}(1, 2; \boldsymbol{R}, t)$ as the sum of the noncorrelated part $f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\, f_1^{\mathrm{ret}}(2;\boldsymbol{R},t)$ and a correlation function $c_2^{\mathrm{ret}}(1,2;\boldsymbol{R},t)$ defined as:

$$c_2^{\mathrm{ret}}(1,2;\boldsymbol{R},t) = f_2^{\mathrm{ret}}(1,2;\boldsymbol{R},t) - f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\, f_1^{\mathrm{ret}}(2;\boldsymbol{R},t). \tag{36}$$

If one splits $f_2^{\mathrm{ret}}$ in this way, the second integral of (32) becomes:

$$\begin{aligned} &\int f_2^{\alpha\gamma} f_1^{\mathrm{ret}}(2;\boldsymbol{R},t)\,\mathrm{d}V_2 \int h_{1\cdot\gamma}^{\beta} f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\,\mathrm{d}V_1 \\ &\qquad - \tfrac{1}{4}\int f_{1\gamma\varepsilon} f_1^{\mathrm{ret}}(1;\boldsymbol{R},t)\,\mathrm{d}V_1 \int f_2^{\gamma\varepsilon} f_1^{\mathrm{ret}}(2;\boldsymbol{R},t)\,\mathrm{d}V_2\, g^{\alpha\beta} \end{aligned}$$

$$+ c^{-2} \int u_1^\beta m_{1\gamma\varepsilon} u_1^\varepsilon f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1 \int f_2^{\alpha\gamma} f_1^{\text{ret}}(2; \boldsymbol{R}, t) \, \mathrm{d}V_2$$
$$- c^{-2} \int u_1^\beta m_1^{\alpha\gamma} u_1^\varepsilon f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1 \int f_{2\gamma\varepsilon} f_1^{\text{ret}}(2; \boldsymbol{R}, t) \, \mathrm{d}V_2$$
$$- c^{-4} \int u_1^\alpha u_1^\beta u_1^\gamma m_1^{\varepsilon\zeta} u_{1\zeta} f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1 \int f_{2\gamma\varepsilon} f_1^{\text{ret}}(2; \boldsymbol{R}, t) \, \mathrm{d}V_2$$
$$+ \int \{ f_2^{\alpha\gamma} h_1{}^\beta{}_{\cdot\gamma} - \tfrac{1}{4} f_{1\gamma\varepsilon} f_2^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2} u_1^\beta ( f_2^{\alpha\gamma} m_{1\gamma\varepsilon} - m_1^{\alpha\gamma} f_{2\gamma\varepsilon}) \, u_1^\varepsilon$$
$$- c^{-4} u_1^\alpha u_1^\beta u_1^\gamma f_{2\gamma\varepsilon} m_1^{\varepsilon\zeta} u_{1\zeta} \} \, c_2^{\text{ret}} \, (1, 2; \boldsymbol{R}, t) \, \mathrm{d}V_1 \, \mathrm{d}V_2. \qquad (37)$$

Now the macroscopic rest mass transport $\varrho' U^\alpha$ is defined as the average:

$$\varrho' U^\alpha = \int \rho_1' u_1^\alpha f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1, \qquad (38)$$

where $\rho_1'$ is the rest mass density of an atom (cf. I (23)[1]) in the rest frame (indicated by a dash) and $u_1^\alpha$ its four-velocity. Imposing the usual condition for a velocity:

$$U^\alpha U_\alpha = -c^{-2} \qquad (39)$$

we have determined the bulk rest mass density $\varrho'$ by (38). Furthermore one can write the atomic velocity as

$$u_1^\alpha = U^\alpha + \hat{u}_1^\alpha \qquad (40)$$

where $\hat{u}_1^\alpha$ is called the velocity fluctuation. With this splitting and formulae (33)–(35) the expression (37) becomes:

$$F^{\alpha\gamma} H^\beta{}_{\cdot\gamma} - \tfrac{1}{4} F_{\gamma\varepsilon} F^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2} U^\beta (F^{\alpha\gamma} M_{\gamma\varepsilon} - M^{\alpha\gamma} F_{\gamma\varepsilon}) \, U^\varepsilon$$
$$- c^{-4} U^\alpha U^\beta U^\gamma F_{\gamma\varepsilon} M^{\varepsilon\zeta} U_\zeta$$
$$+ c^{-2} \int (u_1^\beta u_1^\varepsilon - U^\beta U^\varepsilon) \, m_{1\gamma\varepsilon} f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1 \int f_2^{\alpha\gamma} f_1^{\text{ret}}(2; \boldsymbol{R}, t) \, \mathrm{d}V_2$$
$$- c^{-2} \int (u_1^\beta u_1^\varepsilon - U^\beta U^\varepsilon) \, m_1^{\alpha\gamma} f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1 \int f_{2\gamma\varepsilon} f_1^{\text{ret}}(2; \boldsymbol{R}, t) \, \mathrm{d}V_2$$
$$- c^{-4} \int (u_1^\alpha u_1^\beta u_1^\gamma u_{1\zeta} - U^\alpha U^\beta U^\gamma U_\zeta) \, m_1^{\varepsilon\zeta} f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1 \int f_{2\gamma\varepsilon} f_1^{\text{ret}}(2; \boldsymbol{R}, t) \, \mathrm{d}V_2$$
$$+ \int \{ f_2^{\alpha\gamma} h_1{}^\beta{}_{\cdot\gamma} - \tfrac{1}{4} f_{1\gamma\varepsilon} f_2^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2} u_1^\beta ( f_2^{\alpha\gamma} m_{1\gamma\varepsilon} - m_1^{\alpha\gamma} f_{2\gamma\varepsilon}) \, u_1^\varepsilon$$
$$- c^{-4} u_1^\alpha u_1^\beta u_1^\gamma f_{2\gamma\varepsilon} m_1^{\varepsilon\zeta} u_{1\zeta} \} \, c_2^{\text{ret}}(1, 2; \boldsymbol{R}, t) \, \mathrm{d}V_1 \, \mathrm{d}V_2. \qquad (41)$$

This expression shows that in the total tensor $T^{\alpha\beta}$ a part $T_{(f)}^{\alpha\beta}$ occurs, which depends on the macroscopic fields and polarizations

$$T_{(f)}^{\alpha\beta} = F^{\alpha\gamma} H^\beta{}_{\cdot\gamma} - \tfrac{1}{4} F_{\gamma\varepsilon} F^{\gamma\varepsilon} g^{\alpha\beta}$$
$$+ c^{-2} U^\beta (F^{\alpha\gamma} M_{\gamma\varepsilon} - M^{\alpha\gamma} F_{\gamma\varepsilon}) \, U^\varepsilon - c^{-4} U^\alpha U^\beta U^\gamma F_{\gamma\varepsilon} M^{\varepsilon\zeta} U_\zeta. \qquad (42)$$

Thus in a straightforward manner the macroscopic field energy-momentum tensor has been found. Then from (30) and (32) with (41) the statistical expression for the material energy-momentum tensor also follows:

$$T_{(m)}^{\alpha\beta} = \int \{ \rho_1' u_1^\alpha u_1^\beta - \tfrac{1}{2} \Delta_{1\varepsilon}^\alpha \Delta_{1\zeta}^\beta \partial_\gamma (\sigma_1^{\varepsilon\zeta} u_1^\gamma) + \tfrac{1}{2} c^{-2} (u_1^\alpha \sigma_1^{\beta\gamma} D_1 u_{1\gamma} + u_1^\beta \sigma_1^{\alpha\gamma} D_1 u_{1\gamma})$$
$$+ \tfrac{1}{2} \partial_\gamma (\sigma_1^{\alpha\gamma} u_1^\beta + \sigma_1^{\beta\gamma} u_1^\alpha) \} \, f_1^{\text{ret}}(1; \boldsymbol{R}, t) \, \mathrm{d}V_1$$
$$+ \text{(the velocity fluctuations and correlations of (41))}. \qquad (43)$$

Before discussing this expression we shall rewrite it. In the first integral the velocities $u_1^\alpha$ will be split according to (40). Futhermore we also write

$$D_1 u_1^\alpha = DU^\alpha + D_1 {}'u_1^\alpha, \tag{44}$$

where $DU^\alpha$ is the bulk acceleration

$$DU^\alpha \equiv U^\beta \partial_\beta U^\alpha, \tag{45}$$

and where the last term of (44) is called the acceleration fluctuation. Finally the average total mass density $\varrho^{t'}$ and the average internal angular momentum density $\Sigma^{\alpha\beta}$ are defined by:

$$\varrho^{t'} = \int \rho_1^{t'} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1, \tag{46}$$

$$\Sigma^{\alpha\beta} = \int \sigma_1^{\alpha\beta} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1. \tag{47}$$

The material energy-momentum tensor (43) may now be written with the help of (33), (40)–(42) and (44)–(47) as:

$$\begin{aligned}
T_{(m)}^{\alpha\beta} = {} & \varrho^{t'} U^\alpha U^\beta - \tfrac{1}{2}\Delta_\varepsilon^\alpha \Delta_\zeta^\beta \partial_\gamma(\Sigma^{\varepsilon\zeta} U^\gamma) + \tfrac{1}{2}c^{-2}(U^\alpha \Sigma^{\beta\gamma} DU_\gamma + U^\beta \Sigma^{\alpha\gamma} DU_\gamma) \\
& + \tfrac{1}{2}\partial_\gamma\,(\Sigma^{\alpha\gamma} U^\beta + \Sigma^{\beta\gamma} U^\alpha) + \int [\rho_1^{t'}(u_1^\alpha u_1^\beta - U^\alpha U^\beta) - \tfrac{1}{2}\Delta_{1\varepsilon}^\alpha \Delta_{1\zeta}^\beta \partial_\gamma(\sigma_1^{\varepsilon\zeta} u_1^\gamma) \\
& + \tfrac{1}{2}\Delta_\varepsilon^\alpha \Delta_\zeta^\beta \partial_\gamma(\sigma_1^{\varepsilon\zeta} U^\gamma) \\
& + \tfrac{1}{2}c^{-2}\{(u_1^\alpha D_1 u_{1\gamma} - U^\alpha DU_\gamma)\,\sigma_1^{\beta\gamma} + (u_1^\beta D_1 u_{1\gamma} - U^\beta DU_\gamma)\,\sigma_1^{\alpha\gamma}\} \\
& + \tfrac{1}{2}\partial_\gamma(\sigma_1^{\alpha\gamma}\hat{u}_1^\beta + \sigma_1^{\beta\gamma}\hat{u}_1^\alpha)]\, f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\
& + c^{-2}F^{\alpha\gamma} \int (u_1^\beta u_1^\varepsilon - U^\beta U^\varepsilon)\, m_{1\gamma\varepsilon} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\
& - c^{-2}F_{\gamma\varepsilon} \int (u_1^\beta u_1^\varepsilon - U^\beta U^\varepsilon)\, m_1^{\alpha\gamma} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\
& - c^{-4}F_{\gamma\varepsilon} \int (u_1^\alpha u_1^\beta u_1^\gamma u_{1\zeta} - U^\alpha U^\beta U^\gamma U_\zeta)\, m_1^{\varepsilon\zeta} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\
& + \int \{f_2^{\alpha\gamma} h_1{}^\beta{}_{\cdot\gamma} - \tfrac{1}{4} f_{1\gamma\varepsilon} f_2^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2}u_1^\beta (f_2^{\alpha\gamma} m_{1\gamma\varepsilon} - m_1^{\alpha\gamma} f_{2\gamma\varepsilon})\, u_1^\varepsilon \\
& - c^{-4} u_1^\alpha u_1^\beta u_1^\gamma f_{2\gamma\varepsilon} m_1^{\varepsilon\zeta} u_{1\zeta}\}\, c_2^{\text{ret}}(1, 2;\, \boldsymbol{R}, t)\, \mathrm{d}V_1\, \mathrm{d}V_2,
\end{aligned} \tag{48}$$

where the tensor $\Delta_\beta^\alpha$ is defined with the bulk velocity $U^\alpha$:

$$\Delta_\beta^\alpha = \delta_\beta^\alpha + c^{-2}U^\alpha U_\beta. \tag{49}$$

In this way the expressions (42) and (48) are obtained for both the field and material parts $T_{(f)}^{\alpha\beta}$ and $T_{(m)}^{\alpha\beta}$ of the symmetric total energy momentum tensor $T^{\alpha\beta}$. They were not simply the averages of the atomic tensors $t_{(f)}^{\alpha\beta}$ and $t_{(m)}^{\alpha\beta}$, which together form the total atomic energy-momentum tensor $t^{\alpha\beta}$. In the following sections the physical content of the macroscopic field and material energy-momentum tensors will be discussed briefly.

§ 5. *The macroscopic field energy-momentum tensor.* In the local momentary rest frame the field energy-momentum tensor (42) reads – in three-dimentional notation –

$$T^{\alpha\beta}_{(f)} = \begin{pmatrix} T^{00}_{(f)} & T^{0i}_{(f)} \\ T^{i0}_{(f)} & T^{ij}_{(f)} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}E^2 + \frac{1}{2}B^2 & (\boldsymbol{E} \wedge \boldsymbol{H})^i \\ (\boldsymbol{E} \wedge \boldsymbol{H})^i & -E^iD^j - H^iB^j + (\frac{1}{2}E^2 + \frac{1}{2}B^2 - \boldsymbol{M}\cdot\boldsymbol{B})\, g^{ij} \end{pmatrix}, \tag{50}$$

where $T^{00}_{(f)}$ is the field energy density, $cT^{0i}_{(f)}$ the Poynting vector, $c^{-1}T^{i0}_{(f)}$ the field momentum density and $T^{ij}_{(f)}$ the field ("Maxwell") pressure.

For substances that are isotropic as far as the polarizations are concerned the vector $\boldsymbol{D}$ is parallel to the field $\boldsymbol{E}$, and $\boldsymbol{H}$ parallel to $\boldsymbol{B}$ (in the rest frame). Then the tensor (50) is symmetric, and therefore the field energy-momentum tensor is symmetric in *all* Lorentz frames. For an anisotropic system the energy-momentum tensor is asymmetric in general.

For electric dipole substances ($\boldsymbol{M} = 0$) the results for $T^{i0}_{(f)}$ and $T^{ij}_{(f)}$ were given already by Lorentz[2]) and by Einstein and Laub[3]) on the basis of electron-theoretical arguments. Minkowski's[4]) and Abraham's[5]) tensors differ essentially from (50): both have for $T^{00}_{(f)}$ and in the bracket of $T^{ij}_{(f)}$ the expression $\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$, Minkowski writes for $T^{i0}_{(f)}$ the vector $\boldsymbol{D} \wedge \boldsymbol{B}$ and Abraham symmetrizes Minkowski's $T^{00}_{(f)}$. (For a discussion and for later literature see article VII of this series).

The simple expression (50) is valid only in the rest frame. The general expression (42) contains the local velocity $\boldsymbol{v} = \boldsymbol{\beta}c$ at the reference point $(\boldsymbol{R}, t)$. Its components read in three-dimensional notation:

$$T^{00}_{(f)} = \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 + \boldsymbol{P}\cdot\boldsymbol{E} - \gamma^2\boldsymbol{\beta}\cdot(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E}) - \gamma^4(\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\Omega^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B}), \tag{51}$$

$$T^{0i}_{(f)} = \{\boldsymbol{E}\wedge\boldsymbol{H} - \gamma^2\boldsymbol{\beta}\cdot(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E})\,\boldsymbol{\beta} - \gamma^4(\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\Omega^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B})\,\boldsymbol{\beta}\}^i \quad (i = 1, 2, 3), \tag{52}$$

$$T^{i0}_{(f)} = \{\boldsymbol{E}\wedge\boldsymbol{H} - \gamma^2\beta^2(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E}) + \gamma^2\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{E} + \boldsymbol{M}\wedge\boldsymbol{B}) - \gamma^4(\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\Omega^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B})\,\boldsymbol{\beta}\}^i \quad (i = 1, 2, 3), \tag{53}$$

$$T^{ij}_{(f)} = -E^iD^j - H^iB^j + (\tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 - \boldsymbol{M}\cdot\boldsymbol{B})\, g^{ij} + \gamma^2\{\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{E} + \boldsymbol{M}\wedge\boldsymbol{B}) - \boldsymbol{P}\wedge\boldsymbol{B} + \boldsymbol{M}\wedge\boldsymbol{E}\}^i\,\beta^j - \gamma^4(\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\Omega^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B})\,\beta^i\beta^j \quad (i, j = 1, 2, 3), \tag{54}$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ and $\Omega^2 = \mathsf{U} - \boldsymbol{\beta}\boldsymbol{\beta}$. These expressions show the influence of the velocity of the medium.

§ 6. *The macroscopic material energy-momentum tensor.* The material energy-momentum tensor (48) contains as a first term a quantity of which the time-time component is $\gamma^2 \varrho^{t'} c^2$ (and thus $\varrho^{t'} c^2$ in the rest frame); its time-space component (the bulk energy transport times $c^{-1}$) and its space-time component (the bulk momentum density times $c$) are equal to $\gamma^2 \varrho^{t'} c^2 \boldsymbol{\beta}$; its space-space components give the bulk momentum transport tensor $\gamma^2 \varrho^{t'} c^2 \boldsymbol{\beta}\boldsymbol{\beta}$.

Furthermore (48) shows a number of terms, which contain the macroscopic internal angular momentum density $\Sigma^{\alpha\beta}$, defined by (47). It may be remarked that, in contrast with its atomic counterpart $\sigma_k^{\alpha\beta}$, this antisymmetric tensor is not purely space-space-like in the momentary rest frame.

Finally the material energy-momentum tensor contains fluctuation and correlation parts, which in the general case are rather lengthy expressions. They constitute the relativistic generalization of the usual expressions, which are valid if the relative particle velocities within a domain of the dimension of the correlation length around the point of observation have nonrelativistic magnitudes. This will be shown explicitly in article IV.

If macroscopic electromagnetic fields are absent, the total energy-momentum tensor reduces to an expression, which has the same form as (48). However, its numerical value is different, since the distribution functions for vanishing fields have then to be employed.

§ 7. *The ponderomotive force.* Using (30) we can write the conservation law (27) of energy-momentum as

$$\partial_\beta T^{\alpha\beta}_{(m)} = F^\alpha, \tag{55}$$

where the force density $F^\alpha$ is given in terms of the macroscopic fields and polarizations by

$$F^\alpha = -\partial_\beta T^{\alpha\beta}_{(f)}. \tag{56}$$

With the help of the expression (42) for $T^{\alpha\beta}_{(f)}$ and the (macroscopic) Maxwell equations:

$$\partial_\beta H^{\alpha\beta} = c^{-1} J^\beta, \tag{57}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{58}$$

where $J^\alpha$ is the macroscopic four-current density, the force (56) becomes a sum of two contributions:

$$F^\alpha = F^\alpha_{(L)} + F^\alpha_{(P)}, \tag{59}$$

where the first term is the macroscopic Lorentz force density:

$$F^\alpha_{(L)} = c^{-1} F^{\alpha\beta} J_\beta, \tag{60}$$

and the second the *ponderomotive force density*:

$$F^\alpha_{(P)} = \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\, M_{\beta\gamma} - c^{-2}\varrho' D\{v'(F^{\alpha\beta} M_{\beta\gamma} - M^{\alpha\beta} F_{\beta\gamma})\, U^\gamma\} + c^{-4}\varrho' D(v' U^\alpha U^\beta F_{\beta\gamma} M^{\gamma\varepsilon} U_\varepsilon). \tag{61}$$

Here use has been made of the macroscopic proper mass conservation:

$$\partial_\alpha(\varrho' U^\alpha) = 0, \tag{62}$$

which follows from the definition (38). Furthermore the specific volume $v' = (\varrho')^{-1}$ and the operator $D = U^\alpha \partial_\alpha$ have been introduced.

In the momentary rest frame, denoted by (0), the expression (61) for $\alpha = 0$ becomes in three-dimensional notation

$$F^{0(0)}_{(P)} = \varrho' \boldsymbol{E}^{(0)} \cdot \frac{\partial(v' \boldsymbol{P}^{(0)})}{c\,\partial t^{(0)}} - \boldsymbol{M}^{(0)} \cdot \frac{\partial \boldsymbol{B}^{(0)}}{c\,\partial t^{(0)}} + 2(\boldsymbol{E}^{(0)} \wedge \boldsymbol{M}^{(0)}) \cdot \partial_0^{(0)} \boldsymbol{\beta}^{(0)}, \tag{63}$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$. The combination of terms occurring here gets a simple form if one introduces quantities defined in a frame in which the medium is – locally – at rest all the time. This frame (denoted by a prime) is a *succession* of Lorentz frames, not a Lorentz frame itself. In fact (cf. I (46), (47)) one has:

$$\frac{\partial \boldsymbol{P}'}{\partial t'} = \frac{\partial \boldsymbol{P}^{(0)}}{\partial t^{(0)}} - \frac{1}{c} \frac{\partial \boldsymbol{v}^{(0)}}{\partial t^{(0)}} \wedge \boldsymbol{M}^{(0)}, \tag{64}$$

$$\frac{\partial \boldsymbol{B}'}{\partial t'} = \frac{\partial \boldsymbol{B}^{(0)}}{\partial t^{(0)}} - \frac{1}{c} \frac{\partial \boldsymbol{v}^{(0)}}{\partial t^{(0)}} \wedge \boldsymbol{E}^{(0)}. \tag{65}$$

With these formulae one obtains from (63) the relation

$$F^{0(0)}_{(P)} = \varrho' \boldsymbol{E}' \cdot \frac{\partial(v' \boldsymbol{P}')}{c\,\partial t'} - \boldsymbol{M}' \cdot \frac{\partial \boldsymbol{B}'}{c\,\partial t'}. \tag{66}$$

The space part ($\alpha = 1, 2, 3$) of (61) is the ordinary three-dimensional ponderomotive force. In the momentary rest frame it reads (omitting the suffixes (0)):

$$\begin{aligned} \boldsymbol{F}_{(P)} = {} & (\boldsymbol{\nabla} \boldsymbol{E}) \cdot \boldsymbol{P} + (\boldsymbol{\nabla} \boldsymbol{B}) \cdot \boldsymbol{M} + \varrho' \partial_0 (v' \boldsymbol{P} \wedge \boldsymbol{B}) \\ & - \varrho' \partial_0 (v' \boldsymbol{M} \wedge \boldsymbol{E}) - \partial_0 \boldsymbol{\beta} \wedge (\boldsymbol{P} \wedge \boldsymbol{E} + \boldsymbol{M} \wedge \boldsymbol{B}) + \partial_0 \boldsymbol{\beta} (\boldsymbol{P} \cdot \boldsymbol{E}). \end{aligned} \tag{67}$$

The last two terms represent relativistic effects containing the acceleration.

In the special case that $\boldsymbol{\beta}$ is constant in time and space, the expression (67) in the rest frame simplifies to:

$$\boldsymbol{F}_{(P)} = (\boldsymbol{\nabla} \boldsymbol{E}) \cdot \boldsymbol{P} + (\boldsymbol{\nabla} \boldsymbol{B}) \cdot \boldsymbol{M} + \partial_0 (\boldsymbol{P} \wedge \boldsymbol{B}) - \partial_0 (\boldsymbol{M} \wedge \boldsymbol{E}), \tag{68}$$

a result which also follows immediately from (50) and (56) for $\alpha = 1, 2, 3$. Alternatively, with the help of the Maxwell equations, this form may be written as:

$$\begin{aligned} \boldsymbol{F}_{(P)} & = \\ & = -c^{-1} \boldsymbol{J} \wedge \boldsymbol{M} + \boldsymbol{P} \cdot \boldsymbol{\nabla} \boldsymbol{E} + \boldsymbol{M} \cdot \boldsymbol{\nabla} \boldsymbol{H} + \tfrac{1}{2} \boldsymbol{\nabla} M^2 + (\partial_0 \boldsymbol{P}) \wedge \boldsymbol{H} - (\partial_0 \boldsymbol{M}) \wedge \boldsymbol{E}. \end{aligned} \tag{69}$$

The first term is not absorbed into the Lorentz force in order to maintain form invariance of the latter under Lorentz transformations. In the electrostatic case (68) or (69) reduces to the well-known Kelvin force

$$\boldsymbol{F}_{(P)} = (\boldsymbol{\nabla E})\cdot\boldsymbol{P} = \boldsymbol{P}\cdot\boldsymbol{\nabla E}. \tag{70}$$

In the general case of arbitrary motion the ponderomotive force (61) for $\alpha = 0$ and $\alpha = 1, 2, 3$ respectively has the form:

$$F^0_{(P)} = -(\partial_0 \boldsymbol{E})\cdot\boldsymbol{P} - (\partial_0 \boldsymbol{B})\cdot\boldsymbol{M} + c^{-1}\varrho'\gamma \frac{\mathrm{d}}{\mathrm{d}t}\{\gamma v'\boldsymbol{\beta}\cdot(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E})\}$$
$$+ c^{-1}\varrho'\gamma \frac{\mathrm{d}}{\mathrm{d}t}\{\gamma^3 v'(\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\Omega^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B})\}, \tag{71}$$

$$\boldsymbol{F}_{(P)} = (\boldsymbol{\nabla E})\cdot\boldsymbol{P} + (\boldsymbol{\nabla B})\cdot\boldsymbol{M} + c^{-1}\varrho'\gamma \frac{\mathrm{d}}{\mathrm{d}t}\{\gamma v'(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E})\}$$
$$-c^{-1}\varrho'\gamma \frac{\mathrm{d}}{\mathrm{d}t}\{\gamma v'\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{E} + \boldsymbol{M}\wedge\boldsymbol{B})\}$$
$$+ c^{-1}\varrho'\gamma \frac{\mathrm{d}}{\mathrm{d}t}\{\gamma^3 v'(\boldsymbol{P} - \boldsymbol{\beta}\wedge\boldsymbol{M})\cdot\Omega^2\cdot(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B})\,\boldsymbol{\beta}\}, \tag{72}$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, $\Omega^2 = \mathsf{U} - \boldsymbol{\beta\beta}$ and the substantial time derivative $c^{-1}\,\mathrm{d}/\mathrm{d}t \equiv \partial_0 + \boldsymbol{\beta}\cdot\boldsymbol{\nabla}$. These formulae show the influence of arbitrary (relativistic) motion on the ponderomotive force $\boldsymbol{F}_{(P)}$.

§ 8. *The ponderomotive torque.* From the conservation of energy-momentum (27) and the symmetry (29) of the energy-momentum tensor one proves the conservation law of total angular momentum:

$$\partial_\gamma(x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}) = 0. \tag{73}$$

With (30) and (56) this can be written as

$$\partial_\gamma(x^\alpha T^{\beta\gamma}_{(m)} - x^\beta T^{\alpha\gamma}_{(m)}) = x^\alpha F^\beta - x^\beta F^\alpha + T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)}. \tag{74}$$

The space-space part ($\alpha = i$, $\beta = j$; $i, j = 1, 2, 3$) of this equation becomes, with (54) and (59):

$$\partial_\gamma(x^i T^{j\gamma}_{(m)} - x^j T^{i\gamma}_{(m)}) = \{\boldsymbol{R}\wedge\boldsymbol{F}_{(L)} + \boldsymbol{R}\wedge\boldsymbol{F}_{(P)} + \gamma^2\Omega^2\cdot(\boldsymbol{P}\wedge\boldsymbol{E} + \boldsymbol{M}\wedge\boldsymbol{B})$$
$$+ \gamma^2\boldsymbol{\beta}\wedge(\boldsymbol{P}\wedge\boldsymbol{B} - \boldsymbol{M}\wedge\boldsymbol{E})\}^k \qquad (i, j, k = 1, 2, 3, \text{cycl.}). \tag{75}$$

Here one recognizes in the first term a torque due to the Lorentz force $\boldsymbol{F}_{(L)}$. The other terms represent the "ponderomotive torque" exerted on the medium: it contains the torque due to the ponderomotive force $\boldsymbol{F}_{(P)}$ and besides two other terms which in the rest frame reduce to $\boldsymbol{P}\wedge\boldsymbol{E} + \boldsymbol{M}\wedge\boldsymbol{B}$. For a system isotropic with respect to polarization the last two terms of (75) disappear.

§ 9. *Concluding remarks.* The purpose of this paper was to derive an expression for the *total* macroscopic energy-momentum tensor $T^{\alpha\beta}$ in terms of atomic parameters. This tensor was seen to contain a field part $T^{\alpha\beta}_{(f)}$, depending on the macroscopic fields and polarizations, and a material part $T^{\alpha\beta}_{(m)}$. In the subsequent paper we shall discuss the content of $T^{\alpha\beta}_{(m)}$. On the other hand it will then also be shown that different splittings of $T^{\alpha\beta}$ into "field" and "material" parts may be used.

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