Physica 133A (1985) 370–396
North-Holland, Amsterdam

KINETIC THEORY OF THE COLLECTIVE MODES FOR A DENSE ONE-COMPONENT PLASMA IN A MAGNETIC FIELD

L.G. SUTTORP and J.S. COHEN*
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, the Netherlands

Received 10 June 1985

The collective modes for a classical magnetized Coulomb plasma are determined with the use of kinetic theory. A comparison is made with the results from magnetohydrodynamics. It is shown that only four of the nine independent static transport coefficients are accessible through the damping terms of the purely dissipative modes. One of these is the convective cell mode which decays relatively slowly for strong magnetic fields; it exists solely when the wave vector is purely perpendicular to the field. The oscillating modes contain generalized transport coefficients, evaluated at finite frequencies.

1. Introduction

Collective modes play an important role in the dynamical response of a plasma to external disturbances. In particular, the modes with small wavenumber determine the large-scale behaviour of the dynamical structure factor. Obviously the dynamics of a plasma is influenced by the presence of a magnetic field. In the present paper we shall study the mode spectrum of a magnetized plasma.

For a fluid consisting of neutral particles the collective-mode spectrum is known to consist of a a heat mode, two shear modes and two sound modes. For all these modes the dispersion relation implies that for long wavelengths the frequency of the modes tends to zero. In a plasma the density fluctuations, which are essential for sound propagation, are modified owing to the neutralizing effect of the long-range Coulomb interaction. Consequently, instead of sound waves plasma oscillations occur at a frequency that remains finite for large wavelengths.

To determine the mode spectrum various alternative methods are available. From a macroscopic point of view the collective modes should follow from the

* Present address: Philips Natuurkundig Laboratorium, Eindhoven, the Netherlands.

linearized version of the hydrodynamic equations. For a system of neutral particles the frequencies of the long-wavelength modes are indeed found in this way. The damping is determined by the static transport coefficients. However, for a plasma the usefulness of hydrodynamics is not evident. Indeed, the occurrence of plasma oscillations at a finite frequency, even for vanishing wavenumber, points to the fact that the use of the hydrodynamic limit, which implies a limit of zero frequency, is questionable[1,2]).

An alternative approach to the evaluation of the mode spectrum is furnished by kinetic theory. Here the mode frequencies follow by considering the poles of the matrix elements of the resolvent of the kinetic kernel in the complex frequency plane. In this way it has been shown[2]) that the damping and the dispersion of the collective plasma oscillations is determined by generalized transport coefficients given as kinetic expressions at a finite frequency. The thermal and the viscous modes may still be obtained from macroscopic hydrodynamics.

For a magnetized plasma the spherical isotropy of the plasma is destroyed and one is left with a cylindrical symmetry. As a consequence the mode spectrum will depend on the angle between the wave vector and the magnetic field. Moreover the cyclotron frequency will play a role on a par with the plasma frequency. In view of the findings for an unmagnetized plasma it is obvious that confidence in a (magneto) hydrodynamic evaluation of the mode spectrum is even less justified for a plasma in a magnetic field. Hence a kinetic approach will be adopted from the outset. In a later stage of the treatment the connexion with the magnetohydrodynamic results will be established.

As a model we shall consider a classical one-component plasma, which consists of a system of charged particles in an inert uniform background of opposite charge. The interaction between the particles and with the background is purely electrostatic. The magnetic field is supposed to be stationary and uniform in space.

After a review, in section 2, of the general formalism of kinetic theory for time correlation functions, the frequency matrix, the eigenvalues of which determine the modes, is discussed in section 3. In sections 4–6 the mode frequencies are evaluated. A separate treatment is given of the collective modes in a strongly magnetized plasma and of the modes propagating in a direction transverse to the magnetic field. The final section contains a discussion of the connexion between the kinetic and the magnetohydrodynamic derivation of the mode spectrum.

2. Kinetic equation

The time correlation function C, which describes the motion of the constituent charged particles of a magnetized plasma, is defined by its Fourier–Laplace transform

$$C(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z) = -\mathrm{i} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}zt} \langle \delta f(\boldsymbol{k}, \boldsymbol{p}, t) \delta f(\boldsymbol{k}, \boldsymbol{p}', 0)^* \rangle , \tag{2.1}$$

with $\operatorname{Im} z > 0$. The canonical-ensemble average contains the fluctuating part $\delta f = f - \langle f \rangle$ of the phase-space density

$$f(\boldsymbol{k}, \boldsymbol{p}, t) = \frac{1}{\sqrt{V}} \sum_\alpha \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha(t)} \delta[\boldsymbol{p} - \boldsymbol{p}_\alpha(t)] , \tag{2.2}$$

with $\boldsymbol{r}_\alpha$, $\boldsymbol{p}_\alpha$ the position and momentum of particle α, and V the volume of the plasma. For $\boldsymbol{k} \neq \boldsymbol{0}$ one has $\langle f \rangle = 0$ so that in this case δf in (2.1) may be replaced by f.

With the help of the Mori projection-operator formalism[2,3]) a formal kinetic equation for C may be derived:

$$[z - L_0^B(\boldsymbol{k}, \boldsymbol{p})] C(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z) - \int \mathrm{d}\boldsymbol{p}''\, \varphi(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}'', z) C(\boldsymbol{k}, \boldsymbol{p}'', \boldsymbol{p}', z) = \tilde{C}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') . \tag{2.3}$$

Here L_0^B is the Liouville operator associated with the motion of a single charged particle, of charge e and mass m, in a uniform and stationary magnetic field $\boldsymbol{B}$:

$$L_0^B(\boldsymbol{k}, \boldsymbol{p}) = \frac{\boldsymbol{k}\cdot\boldsymbol{p}}{m} - \mathrm{i}\omega_B (\boldsymbol{p} \wedge \hat{\boldsymbol{B}}) \cdot \boldsymbol{\nabla}_p , \tag{2.4}$$

where ω_B is the Larmor frequency eB/mc and $\hat{\boldsymbol{B}}$ a unit vector in the direction of the field. Furthermore, $\tilde{C}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}')$ is the static correlation function given by $\lim_{z\to\infty} zC(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z)$. As a consequence of the theorem of Bohr and van Leeuwen $\tilde{C}$ is independent of the magnetic field.

The memory kernel φ in (2.3) consists of two parts: $\varphi = \varphi^{\mathrm{s}} + \varphi^{\mathrm{c}}$. The static kernel φ^{s}, which does not depend on z, is given by

$$\varphi^{\mathrm{s}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = -\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{m} n f_0(p) c(k) , \tag{2.5}$$

with n the particle density, $f_0(p)$ the normalized Maxwell–Boltzmann distribution and $c(k)$ the direct correlation function. The collision part φ^{c} of the memory kernel may be written formally as

$$\varphi^{c}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z)nf_0(p') = \left\langle [QL\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^* \frac{1}{z+QLQ} QL\tilde{f}(\boldsymbol{k},\boldsymbol{p})\right\rangle . \tag{2.6}$$

Here L is the Liouville operator in phase space, which for an arbitrary function F on phase space is defined through $\mathrm{d}F/\mathrm{d}t = \mathrm{i}LF$; furthermore $\tilde{f}(\boldsymbol{k},\boldsymbol{p}) = f(\boldsymbol{k},\boldsymbol{p},t=0)$. The projector $Q = 1 - P$ projects an arbitrary function F, which depends parametrically on $\boldsymbol{k}$ and $\boldsymbol{p}$, on the space orthogonal to the one-particle functions $\tilde{f}(\boldsymbol{k},\boldsymbol{p})$,

$$PF(\boldsymbol{k},\boldsymbol{p}) = \int \mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{p}''\langle F(\boldsymbol{k},\boldsymbol{p})\tilde{f}(\boldsymbol{k},\boldsymbol{p}')^*\rangle\langle \tilde{f}(\boldsymbol{k},\boldsymbol{p}')\tilde{f}(\boldsymbol{k},\boldsymbol{p}'')^*\rangle^{-1}\tilde{f}(\boldsymbol{k},\boldsymbol{p}''), \tag{2.7}$$

with the inverse integral kernel

$$\langle \tilde{f}(\boldsymbol{k},\boldsymbol{p})\tilde{f}(\boldsymbol{k},\boldsymbol{p}')^*\rangle^{-1} = \frac{\delta(\boldsymbol{p}-\boldsymbol{p}')}{nf_0(p)} - c(k)\,. \tag{2.8}$$

The collision kernel φ^{c} fulfils a set of symmetry relations and conservation laws. The symmetry of the Liouville operator under parity and under the combined effect of time reversal and of time translation leads to the relations

$$\begin{aligned} &\varphi^{c}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z,\boldsymbol{B}) = \varphi^{c}(-\boldsymbol{k},-\boldsymbol{p},-\boldsymbol{p}',z,\boldsymbol{B})\,,\\ &\varphi^{c}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z,\boldsymbol{B})f_0(p') = \varphi^{c}(-\boldsymbol{k},-\boldsymbol{p}',-\boldsymbol{p},z,-\boldsymbol{B})f_0(p)\,, \end{aligned} \tag{2.9}$$

where the dependence of φ^{c} on the magnetic field has been made explicit. Under complex conjugation the collision kernel transforms according to the rule

$$\varphi^{c}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z,\boldsymbol{B}) = -[\varphi^{c}(-\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',-z^*,\boldsymbol{B})]^*\,. \tag{2.10}$$

The microscopic conservation laws of particle number, momentum and energy give rise to integral expressions for the moments of the collision kernel[2,3]). The derivation of these relations is given in appendix A. Conservation of particle number implies the identity

$$\int \mathrm{d}\boldsymbol{p}\,\varphi^{c}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z) = 0\,. \tag{2.11}$$

From the microscopic conservation of momentum and of energy one derives

$$\int \mathrm{d}\boldsymbol{p}\, \boldsymbol{p}\, \varphi^{\mathrm{c}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z) = \boldsymbol{k} \cdot \boldsymbol{T}(\boldsymbol{k}, \boldsymbol{p}', z)\,, \tag{2.12}$$

$$\int \mathrm{d}\boldsymbol{p}\, \frac{p^2}{2m}\, \varphi^{\mathrm{c}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z) = \boldsymbol{k} \cdot \boldsymbol{J}_\varepsilon(\boldsymbol{k}, \boldsymbol{p}', z) - zE(\boldsymbol{k}, \boldsymbol{p}', z)\,. \tag{2.13}$$

The pressure tensor $\boldsymbol{T}$, the energy density E and the energy flow $\boldsymbol{J}_\varepsilon$ remain finite in the limit $\boldsymbol{k} \to \boldsymbol{0}$, $z \to \mathrm{i}0$.

3. Determination of the eigenvalue equation for the collective mode frequencies

The kinetic equation (2.3) is an integral equation for the time correlation function with the formal structure

$$[z - \Sigma(\boldsymbol{k}, z)]C(\boldsymbol{k}, z) = \tilde{C}(\boldsymbol{k})\,, \tag{3.1}$$

where the momentum variables have been suppressed. The kernel Σ has the form

$$\Sigma(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z) = \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{m}\, \delta(\boldsymbol{p} - \boldsymbol{p}') - \mathrm{i}\omega_B(\boldsymbol{p} \wedge \hat{\boldsymbol{B}}) \cdot \nabla_p\, \delta(\boldsymbol{p} - \boldsymbol{p}') + \varphi^{\mathrm{s}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') + \varphi^{\mathrm{c}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z)\,. \tag{3.2}$$

The collective modes of the magnetized one-component plasma are determined by the poles of the matrix elements of the resolvent of Σ,

$$G_{\mu\nu}(\boldsymbol{k}, z) = \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{p}'\, \psi_\mu(\boldsymbol{p}) \left[\frac{1}{z - \Sigma(\boldsymbol{k}, z)}\right](\boldsymbol{p}, \boldsymbol{p}')\, \psi_\nu(\boldsymbol{p}') f_0(p') = \left\langle \mu \left| \frac{1}{z - \Sigma(\boldsymbol{k}, z)} \right| \nu \right\rangle, \tag{3.3}$$

in the complex z-plane. Here the functions $\psi_\mu(\boldsymbol{p})$ with $\mu = 0, \ldots, 4$ are defined as

$$\psi_0(\boldsymbol{p}) = 1\,, \quad \psi_i(\boldsymbol{p}) = \frac{p_i}{(mk_{\mathrm{B}}T)^{1/2}} \quad (i = 1, 2, 3)\,, \qquad \psi_4(\boldsymbol{p}) = \frac{1}{\sqrt{6}}\left(\frac{p^2}{mk_{\mathrm{B}}T} - 3\right). \tag{3.4}$$

The matrix $G_{\mu\nu}$ satisfies the equation[2,3])

$$\sum_{\lambda} [z\delta_{\mu\lambda} - \Omega_{\mu\lambda}(\boldsymbol{k}, z)]G_{\lambda\nu}(\boldsymbol{k}, z) = \delta_{\mu\nu}, \tag{3.5}$$

with the frequency matrix

$$\Omega_{\mu\nu}(\boldsymbol{k}, z) = \langle\mu|\Sigma|\nu\rangle + \left\langle \mu\,|\Sigma\bar{Q}\frac{1}{z - \bar{Q}\Sigma\bar{Q}}\bar{Q}\Sigma|\,\nu \right\rangle. \tag{3.6}$$

Here $\bar{Q}$ is the complement of the projector $\bar{P}$, which projects a momentum-dependent function onto the space spanned by the states $|\mu\rangle$. From (3.5) it is clear that the poles in $G_{\mu\nu}$ are determined by the eigenvalues of the frequency matrix $\Omega_{\mu\nu}$.

The evaluation of the matrix elements $\Omega_{\mu\nu}$ and the subsequent determination of the collective modes is more complicated for a magnetized one-component plasma than for an unmagnetized plasma or a fluid. The reason is that isotropy arguments can not be used to classify the modes as either longitudinal or transverse with respect to the wave vector. In fact, two independent directions, $\boldsymbol{k}$ and $\boldsymbol{B}$, play a role. In the following $\boldsymbol{B}$ will be chosen parallel to the positive z-axis; the wave vector $\boldsymbol{k}$ then points in an arbitrary direction.

The matrix $\Omega_{\mu\nu}$, as given by (3.6), consists of a direct part, which is the first term at the right-hand side, and an indirect part, the remainder of (3.6). The direct part contains contributions due to the free-streaming term, the magnetic-field term and the static memory kernel in (3.2). These contributions are easy to evaluate explicitly[2,3]). For $(\mu, \nu) = (0, 0)$ there is no contribution, while for $(\mu, \nu) = (0, i)$ and $(i, 0)$ one finds $v_0 k_i$ and $v_0(1 - nc)k_i$, respectively, with $i = 1, 2, 3$ and $v_0 = (k_B T/m)^{1/2}$. Furthermore, for both $(\mu, \nu) = (i, 4)$ and $(4, i)$ the result is $\sqrt{2/3}\,v_0 k_i$. The magnetic field only enters in the $(\mu, \nu) = (1, 2)$ and $(2, 1)$ terms, with a contribution $\mathrm{i}\omega_B$ and $-\mathrm{i}\omega_B$, respectively.

The direct part also contains a matrix element of the collision term φ^{c}. As a consequence of particle number conservation (see (2.11)) this matrix element is non-vanishing only for $i \geq 1$, $j \geq 1$. For $(\mu, \nu) = (i, j)$, with $i, j = 1, 2, 3$, it follows from the conservation of momentum in the form (2.12) that the matrix element is proportional to k^2:

$$\langle i|\varphi^{c}|j\rangle = -\mathrm{i}\, v_0^2 M_{ij,mn}(\boldsymbol{k}, z)k_m k_n\,. \tag{3.7}$$

Here $M_{ij,mn}$ is a tensor that is symmetric under the interchanges $i \leftrightarrow m$ and $j \leftrightarrow n$,

$$M_{ij,mn}(\boldsymbol{k}, z) = \frac{\mathrm{i}}{n(k_\mathrm{B}T)^2 V} \left\langle [Q\tau_{jn}(\boldsymbol{k})]^* \frac{1}{z + QLQ} Q\tau_{im}(\boldsymbol{k}) \right\rangle , \tag{3.8}$$

where use has been made of (2.6) and (A.9). For $\boldsymbol{k} \to \boldsymbol{0}$ this tensor only depends on the unit vector $\hat{\boldsymbol{B}}$ and on scalar quantities (z, ω_B, n, T, etc.). One may construct eight covariant tensors of fourth rank possessing the required symmetry properties and depending on a single unit vector $\hat{\boldsymbol{B}}$:

$$\begin{aligned}
T^{(1)}_{ij,mn} &= \delta_{im}\delta_{jn} , \\
T^{(2)}_{ij,mn} &= \delta_{ij}\delta_{mn} + \delta_{in}\delta_{jm} , \\
T^{(3)}_{ij,mn} &= \delta_{im}\hat{B}_j\hat{B}_n , \\
T^{(4)}_{ij,mn} &= \delta_{jn}\hat{B}_i\hat{B}_m , \\
T^{(5)}_{ij,mn} &= \delta_{ij}\hat{B}_m\hat{B}_n + \delta_{mn}\hat{B}_i\hat{B}_j + \delta_{in}\hat{B}_j\hat{B}_m + \delta_{jm}\hat{B}_i\hat{B}_n , \\
T^{(6)}_{ij,mn} &= \hat{B}_i\hat{B}_j\hat{B}_m\hat{B}_n , \\
T^{(7)}_{ij,mn} &= (\delta_{ij}\varepsilon_{mnk} + \delta_{mn}\varepsilon_{ijk} + \delta_{in}\varepsilon_{mjk} + \delta_{jm}\varepsilon_{ink})\hat{B}_k , \\
T^{(8)}_{ij,mn} &= (\hat{B}_i\hat{B}_j\varepsilon_{mnk} + \hat{B}_m\hat{B}_n\varepsilon_{ijk} + \hat{B}_i\hat{B}_n\varepsilon_{mjk} + \hat{B}_j\hat{B}_m\varepsilon_{ink})\hat{B}_k .
\end{aligned} \tag{3.9}$$

Correspondingly, the right-hand side of (3.7) can be represented, for small $\boldsymbol{k}$, by a linear combination of eight terms resulting by contraction of each of the tensors (3.9) with $k_m k_n$. A further simplification of this representation is obtained by employing the symmetry relations (2.9). These imply the following identities for the matrix elements (3.7):

$$\begin{aligned}
\langle i|\varphi^\mathrm{c}|j\rangle(\boldsymbol{k}, z, \boldsymbol{B}) &= \langle i|\varphi^\mathrm{c}|j\rangle(-\boldsymbol{k}, z, \boldsymbol{B}) , \\
\langle i|\varphi^\mathrm{c}|j\rangle(\boldsymbol{k}, z, \boldsymbol{B}) &= \langle j|\varphi^\mathrm{c}|i\rangle(-\boldsymbol{k}, z, -\boldsymbol{B}) ,
\end{aligned} \tag{3.10}$$

where the dependence on $\boldsymbol{k}$, z and $\boldsymbol{B}$ has been indicated. For small $\boldsymbol{k}$ the first relation gives no new information: it is identically satisfied by (3.7). The second relation is fulfilled only if the coefficients of the terms $T^{(3)}_{ij,mn}k_m k_n$ and $T^{(4)}_{ij,mn}k_m k_n$ are equal. In this way only seven independent terms are left in the representation of (3.7). After some rearrangement the matrix element gets the form

$$\langle i|\varphi^\mathrm{c}|j\rangle = -\mathrm{i}v_0^2 k^2 \alpha'_{ij}(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) , \tag{3.11}$$

for small $\boldsymbol{k}$, with a tensor α'_{ij} that depends on $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{B}}$ in the following way:

$$\begin{aligned}\alpha'_{ij}(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) = & a'_1(z)\delta_{ij} + a'_2(z)\hat{k}_i\hat{k}_j \\ & + a'_3(z)(\hat{k}_i\hat{B}_j + \hat{B}_i\hat{k}_j)\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}} + a'_4(z)[\hat{B}_i\hat{B}_j + \delta_{ij}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}})^2] \\ & + a'_5(z)\hat{B}_i\hat{B}_j(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}})^2 + a'_6(z)[(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_i\hat{k}_j - \hat{k}_i(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_j + \varepsilon_{ijk}\hat{B}_k] \\ & + a'_7(z)[(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_i\hat{B}_j - \hat{B}_i(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_j + \varepsilon_{ijk}\hat{B}_k\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}]\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}. \end{aligned} \tag{3.12}$$

The coefficients $a'_j(z)$ satisfy the reality conditions

$$a'_j(z) = [a'_j(-z^*)]^* , \quad j = 1, \dots, 7 , \tag{3.13}$$

as a consequence of (2.10). Each a'_j follows by taking the infinite-wavelength limit of a particular matrix element of φ^c for a special choice of the direction of the wave vector. One has for instance

$$a'_1(z) = \frac{i}{v_0^2} \lim_{\boldsymbol{k}\to\boldsymbol{0}} \frac{1}{k^2} [\langle 1|\varphi^c|1\rangle]_{\boldsymbol{k}=(0,k,0)} . \tag{3.14}$$

The matrix elements $\langle i|\varphi^c|4\rangle$, with $i = 1, 2, 3$, have the general form

$$\langle i|\varphi^c|4\rangle = v_0 M_{im}(\boldsymbol{k}, z)k_m , \tag{3.15}$$

as a consequence of the momentum conservation law (2.12). From (2.6) with (A.9) and (A.18) it follows that the symmetric tensor M_{im} is

$$M_{im}(\boldsymbol{k}, z) = \frac{1}{n(k_B T)^2 V}\sqrt{\frac{2}{3}}\left\langle [QL\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) + Q\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k})]^* \frac{1}{z + QLQ} Q\tau_{im}(\boldsymbol{k})\right\rangle . \tag{3.16}$$

For $\boldsymbol{k}\to\boldsymbol{0}$ the contribution with $\boldsymbol{j}_\varepsilon(\boldsymbol{k})$ can be neglected. Then M_{im} depends on the unit vector $\hat{\boldsymbol{B}}$ and on scalars. Consequently it must be a linear combination of the two tensors

$$T^{(1)}_{im} = \delta_{im} , \quad T^{(2)}_{im} = \hat{B}_i\hat{B}_m . \tag{3.17}$$

The symmetry properties (2.9) of the collision kernel lead to the relations

$$\begin{aligned} \langle i|\varphi^c|4\rangle(\boldsymbol{k}, z, \boldsymbol{B}) &= -\langle i|\varphi^c|4\rangle(-\boldsymbol{k}, z, \boldsymbol{B}) , \\ \langle i|\varphi^c|4\rangle(\boldsymbol{k}, z, \boldsymbol{B}) &= -\langle 4|\varphi^c|i\rangle(-\boldsymbol{k}, z, -\boldsymbol{B}) . \end{aligned} \tag{3.18}$$

Combining (3.15), (3.17) and (3.18) one may write for small $\boldsymbol{k}$

$$\langle i|\varphi^{c}|4\rangle = \langle 4|\varphi^{c}|i\rangle = v_0 k\beta'_i(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z)\,, \tag{3.19}$$

with a vector $\boldsymbol{\beta}'$ depending on $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{B}}$ in the following way:

$$\beta'_i(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) = b'_1(z)\hat{k}_i + b'_2(z)\hat{B}_i\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}\,. \tag{3.20}$$

The coefficients $b'_j(z)$ satisfy the reality conditions

$$b'_j(z) = [b'_j(-z^*)]^*\,,\quad j = 1, 2\,, \tag{3.21}$$

as a result of (2.10).

The static limit of the coefficient $b'_1(z)$ is related to thermodynamic quantities[2,3]). In fact, by taking the double limit $z \to \mathrm{i}0$, $\boldsymbol{k} \to \boldsymbol{0}$ in (3.16) one may prove (see appendix B)

$$\lim_{z\to \mathrm{i}0} b'_1(z) = \sqrt{\frac{2}{3}}\left[\frac{1}{nk_\mathrm{B}}\left(\frac{\partial P}{\partial T}\right)_n - 1\right],\quad \lim_{z\to \mathrm{i}0} b'_2(z) = 0\,. \tag{3.22}$$

Finally, we consider the contribution $\langle 4|\varphi^{c}|4\rangle$ to the direct part of the frequency matrix. From energy conservation in the form (A.18) it follows that this matrix element reads

$$\begin{aligned}\langle 4|\varphi^{c}|4\rangle = \frac{2}{3n(k_B T)^2 V}\Big\langle &[QL\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) + Q\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k})]^* \\ &\times \frac{1}{z + QLQ}[QL\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) + Q\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k})]\Big\rangle\,.\end{aligned} \tag{3.23}$$

For small $\boldsymbol{k}$ the right-hand side is a linear combination of the invariants that can be constructed from $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{B}}$. Up to second order in $\boldsymbol{k}$ one has

$$\langle 4|\varphi^{c}|4\rangle = -\mathrm{i}\gamma'(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) - \mathrm{i}v_0^2k^2\delta'(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z)\,, \tag{3.24}$$

with

$$\gamma'(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) = c'(z)\,,\quad \delta'(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) = d'_1(z) + d'_2(z)(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}})^2\,. \tag{3.25}$$

The symmetry properties

$$\begin{aligned}&\langle 4|\varphi^{c}|4\rangle(\boldsymbol{k}, z, \boldsymbol{B}) = \langle 4|\varphi^{c}|4\rangle(-\boldsymbol{k}, z, \boldsymbol{B})\,,\\ &\langle 4|\varphi^{c}|4\rangle(\boldsymbol{k}, z, \boldsymbol{B}) = \langle 4|\varphi^{c}|4\rangle(-\boldsymbol{k}, z, -\boldsymbol{B})\,,\end{aligned} \tag{3.26}$$

which follow from (2.9), give no new information. From (2.10) one obtains the reality conditions

$$c'(z) = [c'(-z^*)]^* , \quad d_j'(z) = [d_j'(-z^*)]^* , \quad j = 1, 2 . \tag{3.27}$$

For small z the coefficient $c'(z)$ is related to thermodynamic quantities[2,3]). As is shown in appendix B one has in this limit

$$\lim_{z \to i0} \frac{1}{z} c'(z) = i\left(1 - \frac{2c_V}{3k_B}\right) , \tag{3.28}$$

with c_V the isochoric heat capacity.

The indirect part of the frequency matrix $\Omega_{\mu\nu}$ (3.6) may be analyzed in a similar way as has been done above for the contribution $\langle\mu|\varphi^c|\nu\rangle$. Particle number conservation implies the vanishing of indirect contributions to $\Omega_{\mu 0}$ and $\Omega_{0\nu}$. Employing momentum conservation and the cylindrical symmetry of the system one finds that for small $\boldsymbol{k}$ the indirect part of Ω_{ij} has the same form as (3.11)–(3.12), with a tensor α''_{ij} and coefficients a''_j instead of α'_{ij} and a'_j, respectively. Likewise, the indirect part of Ω_{i4} and Ω_{4j} follows from (3.19) with (3.20) by replacing β'_i and b'_j by β''_i and b''_j. For small z both $b''_1(z)$ and $b''_2(z)$ tend to 0. Finally the indirect contribution to Ω_{44} is, for small $\boldsymbol{k}$, of the form (3.24) with (3.25), with the substitutions $\gamma' \to \gamma''$, $\delta' \to \delta''$ and correspondingly $c' \to c''$, $d'_j \to d''_j$. In contrast with (3.28) one has for small z the limit $z^{-1}c''(z) \to 0$.

In conclusion, we have found the following general form for the frequency matrix:

$$\Omega_{\mu\nu}(\boldsymbol{k}, z) = \begin{pmatrix} 0 & v_0 k_x & v_0 k_y & v_0 k_z & 0 \\ v_0(1-nc)k_x & -i v_0^2 k^2 \alpha_{11} & i\omega_B - i v_0^2 k^2 \alpha_{12} & -i v_0^2 k^2 \alpha_{13} & v_0 k \bar{\beta}_1 \\ v_0(1-nc)k_y & -i\omega_B - i v_0^2 k^2 \alpha_{21} & -i v_0^2 k^2 \alpha_{22} & -i v_0^2 k^2 \alpha_{23} & v_0 k \bar{\beta}_2 \\ v_0(1-nc)k_z & -i v_0^2 k^2 \alpha_{31} & -i v_0^2 k^2 \alpha_{32} & -i v_0^2 k^2 \alpha_{33} & v_0 k \bar{\beta}_3 \\ 0 & v_0 k \bar{\beta}_1 & v_0 k \bar{\beta}_2 & v_0 k \bar{\beta}_3 & -i\gamma - i v_0^2 k^2 \delta \end{pmatrix} . \tag{3.29}$$

Here we introduced the abbreviations $\alpha_{ij} = \alpha'_{ij} + \alpha''_{ij}$ and likewise for γ and δ. Furthermore we wrote $\bar{\beta}_i = \beta'_i + \beta''_i + \sqrt{2/3}\hat{k}_i$. Correspondingly one has $a_j = a'_j + a''_j$ and similarly for the coefficients c and d_j. The coefficients $\bar{b}_j$ are defined as $\bar{b}_1 = b'_1 + b''_1 + \sqrt{2/3}$ and $\bar{b}_2 = b'_2 + b''_2$.

The frequency matrix (3.29) is more complicated than the corresponding one for an unmagnetized plasma. Owing to the change from spherical to cylindrical symmetry the matrix no longer has a block structure as in the unmagnetized case. As a consequence the evaluation of the mode frequencies is a more elaborate task, which will be discussed in the next sections.

4. Evaluation of the collective mode frequencies for a strongly magnetized plasma

The mode frequencies are determined by the eigenvalues of the frequency matrix $\Omega_{\mu\nu}$. For $\boldsymbol{k} \to \boldsymbol{0}$ some of these mode frequencies vanish, while others remain finite. It turns out that for general directions of the wave vector there is only one purely dissipative mode, with a damping proportional to k^2. In the special case $\boldsymbol{k} \perp \boldsymbol{B}$ the number of dissipative modes increases to three. This special case will be considered separately in section 6. In this section we will consider the general case of modes with wave vectors in an arbitrary direction.

The dissipative mode is determined by substituting the expansion $z = ak^2 + \cdots$ in the eigenvalue equation $\det(z\delta_{\mu\nu} - \Omega_{\mu\nu}) = 0$ and retaining the leading terms in k^2. One finds in this way

$$z + \mathrm{i}\gamma(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) + \mathrm{i}v_0^2 k^2 \delta(\hat{\boldsymbol{k}}, \hat{\boldsymbol{B}}, z) = 0 \tag{4.1}$$

and hence, by insertion of (3.25) and (3.28),

$$z = -\mathrm{i}\frac{3k_\mathrm{B}}{2c_V} v_0^2[d_1 k_\perp^2 + (d_1 + d_2)k_\parallel^2] . \tag{4.2}$$

The subscripts $\parallel$ and $\perp$ denote components in the direction parallel to and perpendicular to the magnetic field. The coefficients $d_j(z)$ have to be evaluated at the limiting value $z = \mathrm{i}0$ of the mode frequency. From (3.27) it then follows that the mode (4.2) is purely dissipative, with an imaginary frequency, at least up to second order in $\boldsymbol{k}$. The damping in (4.2) is anisotropic, with two independent coefficients d_1 and $d_1 + d_2$.

The remaining modes are, for a general orientation of the wave vector, oscillating modes, with finite frequencies for $\boldsymbol{k} \to \boldsymbol{0}$. These are found from the eigenvalue equation

$$z^4 - [v_0^2 k^2(1 - nc) + \omega_B^2]z^2 + v_0^2 k_\parallel^2(1 - nc)\omega_B^2 = 0 , \tag{4.3}$$

where in the coefficients the limit $\boldsymbol{k} \to \boldsymbol{0}$ is understood. In fact, one has for small $\boldsymbol{k}$ (see appendix B)

$$k^2(1 - nc) = k_\mathrm{D}^2 + \frac{k^2}{nk_\mathrm{B}T\kappa_T} + \mathcal{O}(k^4) , \tag{4.4}$$

with k_D the Debye wave vector, which is related to the plasma frequency ω_p through $v_0 k_\mathrm{D} = \omega_\mathrm{p}$. Furthermore, κ_T is the isothermal compressibility. The solutions of (4.3), with (4.4) inserted, follow as

$$z = \pm\tfrac{1}{2}(w_+ + w_-)\,, \quad z = \pm\tfrac{1}{2}(w_+ - w_-)\,. \tag{4.5}$$

Here the abbreviations

$$w_\pm = (\omega_p^2 + \omega_B^2 \pm 2\omega_p\omega_B\hat{k}_\parallel)^{1/2}\,, \tag{4.6}$$

with $\hat{k}_\parallel = k_\parallel/k$, have been introduced. From (4.5) it is obvious that the case $\boldsymbol{k}\perp\boldsymbol{B}$ is a special one, since $\hat{k}_\parallel = 0$ implies $w_+ = w_-$, so that the second pair of modes then becomes purely dissipative.

The zeroth-order mode frequencies (4.5) simplify in the special case of strong magnetic fields, corresponding to cyclotron frequencies $\omega_B \gg \omega_p$. In that case (4.5) becomes

$$z \simeq \pm\,\omega_B\,, \quad z \simeq \pm\,\omega_p\hat{k}_\parallel\,. \tag{4.7}$$

The damping and the dispersion of the oscillating modes follow by determining the coefficients of order k^2 in the expansion of the mode frequencies with respect to k. In the general case of arbitrary field strengths the results are rather complicated. For $\omega_B \gg \omega_p$, however, one obtains from the eigenvalue equation

$$z = \pm[\omega_B - 2a_6v_0^2k_\perp^2 - (a_6 + a_7)v_0^2k_\parallel^2] - \mathrm{i}(a_1 + \tfrac{1}{2}a_2)v_0^2k_\perp^2 - \mathrm{i}(a_1 + a_4)v_0^2k_\parallel^2\,, \tag{4.8}$$

and

$$z = \pm\,\omega_p\hat{k}_\parallel\left[1 + \frac{v_0^2k^2}{2nk_BT\kappa_T\omega_p^2} + \frac{(\bar{b}_1 + \bar{b}_2)^2v_0^2k^2\hat{k}_\parallel}{2\omega_p(\omega_p\hat{k}_\parallel \pm \mathrm{i}c)}\right] - \tfrac{1}{2}\mathrm{i}(a_1 + a_4)v_0^2k_\perp^2 - \tfrac{1}{2}\mathrm{i}(a_1 + a_2 + 2a_3 + 2a_4 + a_5)v_0^2k_\parallel^2\,. \tag{4.9}$$

In the coefficients a_j, $\bar{b}_j$ and c one should insert the lowest (k^0) order values of z. From the reality conditions (3.13), (3.21) and (3.27) it follows that the mode frequencies (4.8) and (4.9) are located at positions that are symmetric with respect to a reflection in the imaginary z-axis (i.e. with respect to the transformation $z \to -z^*$).

In a recent paper[4]) expressions for the frequencies of the oscillating modes of a plasma in a strong magnetic field have been reported on. Since no explicit formulae for the kinetic coefficients in these expressions have been given a detailed comparison with the results found here is difficult. It can be noted, however, that in some of the contributions to the frequencies of the oscillating modes, as given in ref. 4, the static limit has been taken. As a consequence the mode dispersion contains a term depending on the ratio of the heat capacities c_p and c_V. In (4.8) and (4.9) the kinetic coefficients a_j, $\bar{b}_j$ and c are all to be evaluated at finite frequencies. Such kinetic coefficients also appear in the expressions for the oscillating modes of an unmagnetized plasma[1,2]).

5. Collective mode frequencies for general values of the magnetic field strength

If the plasma is situated in a magnetic field of arbitrary strength the evaluation of its oscillating collective mode frequencies is rather complicated. Complete cylinder symmetry is present only in the particular case that the wave vector is parallel to the magnetic field. It turns out that the oscillating modes are then given, for arbitrary values of ω_B, by expressions (4.8) and (4.9), with the substitutions $k_\parallel \to k$, $k_\perp \to 0$, $\hat{k}_\parallel \to 1$.

For the general case of arbitrary angles between $\boldsymbol{k}$ and $\boldsymbol{B}$ the oscillating mode frequencies are, in zeroth order of $\boldsymbol{k}$, given by (4.5) or by $\pm w_\lambda$, with $w_\lambda = \frac{1}{2}(w_+ + \lambda w_-)$ and $\lambda = \pm 1$. Up to second order in $\boldsymbol{k}$ a straightforward calculation yields

$$z = \pm w_\lambda + \frac{v_0^2 k^2}{2w_\lambda^2(\omega_p^2 + \omega_B^2) - 4\omega_p^2\omega_B^2\hat{k}_\parallel^2}\left(N_1^\pm + \frac{N_2}{\pm w_\lambda + ic}\right), \tag{5.1}$$

with $N_1^\pm$ and N_2 polynomials in w_λ,

$$N_1^\pm = \mp w_\lambda^3\left[\omega_B(\alpha_{12} - \alpha_{21}) - \frac{1}{nk_B T\kappa_T}\right] - i w_\lambda^2[\omega_B^2(\alpha_{11} + \alpha_{22}) + \omega_p^2\alpha_{ij}\hat{k}_i\hat{k}_j]$$
$$\pm w_\lambda\left[\omega_p^2\omega_B\epsilon_{ijm}\alpha_{ij}\hat{k}_m\hat{k}_\parallel - \omega_B^2\hat{k}_\parallel^2\frac{1}{nk_B T\kappa_T}\right] + i\omega_p^2\omega_B^2\hat{k}_\parallel^2\alpha_{ii}, \tag{5.2}$$

$$N_2 = w_\lambda^2[\omega_p^2\bar{\beta}_i\bar{\beta}_j\hat{k}_i\hat{k}_j + \omega_B^2(\bar{\beta}_1^2 + \bar{\beta}_2^2)] - \omega_p^2\omega_B^2\hat{k}_\parallel^2\bar{\beta}_i\bar{\beta}_i, \tag{5.3}$$

where a summation over repeated indices is understood. The coefficients in these polynomials may be expressed in terms of the quantities a_j and $\bar{b}_j$ by inserting (3.12) and (3.20) and their counterparts for α''_{ij} and β''_i. Then one gets

$$N_1^{\pm} = \mp w_\lambda^3 \left\{ \omega_B [4a_6 \hat{k}_\perp^2 + 2(a_6 + a_7)\hat{k}_\parallel^2] - \frac{1}{nk_B T\kappa_T} \right\}$$
$$+ \mathrm{i} w_\lambda^2 \{ \omega_p^2 [(2a_1 + a_4)\hat{k}_\perp^4 + (4a_1 + 3a_4 + a_5)\hat{k}_\perp^2 \hat{k}_\parallel^2$$
$$+ 2(a_1 + a_4)\hat{k}_\parallel^4 - (3a_1 + a_2 + a_4)\hat{k}_\perp^2$$
$$- (3a_1 + a_2 + 2a_3 + 4a_4 + a_5)\hat{k}_\parallel^2] - \omega_B^2[(2a_1 + a_2)\hat{k}_\perp^2 + 2(a_1 + a_4)\hat{k}_\parallel^2]\}$$
$$\pm w_\lambda \left\{ \omega_p^2 \omega_B \hat{k}_\parallel^2 [2(a_6 - a_7)\hat{k}_\perp^2 + 2(a_6 + a_7)\hat{k}_\parallel^2] - \omega_B^2 \hat{k}_\parallel^2 \frac{1}{nk_B T\kappa_T} \right\}$$
$$+ \mathrm{i}\omega_p^2 \omega_B^2 \hat{k}_\parallel^2 [(3a_1 + a_2 + a_4)\hat{k}_\perp^2 + (3a_1 + a_2 + 2a_3 + 4a_4 + a_5)\hat{k}_\parallel^2] , \tag{5.4}$$

$$N_2 = w_\lambda^2 [\omega_p^2 (\bar{b}_1 + \bar{b}_2 \hat{k}_\parallel^2)^2 + \omega_B^2 \bar{b}_1^2 \hat{k}_\perp^2] - \omega_p^2 \omega_B^2 [\bar{b}_1^2 \hat{k}_\perp^2 + (\bar{b}_1 + \bar{b}_2)^2 \hat{k}_\parallel^2] \hat{k}_\parallel^2 . \tag{5.5}$$

The coefficients a_j, $\bar{b}_j$ and c are to be evaluated at $z = \pm w_\lambda$.

6. The special case of orthogonal wave vector and magnetic field

The collective mode spectrum for modes with a wave vector perpendicular to the magnetic field is qualitatively different from the spectrum for arbitrary wave vector. Whereas in the general case four of the five modes are oscillating only two of these retain that property in the special case $\hat{k}_\parallel = 0$, as is obvious from the expressions (4.5) for the mode frequency at vanishing wave number. In that case the k^2 contributions to the mode frequencies cannot be obtained from (5.1). In fact, both the numerator and the denominator in the second term of (5.1) vanish for $\hat{k}_\parallel = 0$ and $\lambda = -1$, since in that case $w_\lambda = 0$. No such difficulty arises for $\lambda = +1$, since the zeroth-order frequency then becomes $w_\lambda = (\omega_p^2 + \omega_B^2)^{1/2} \equiv \omega_0$. Putting $\hat{k}_\parallel = 0$ in (5.1) one finds for the propagating modes with $\lambda = +1$

$$z = \pm \left[\omega_0 - 2a_6 \frac{\omega_B}{\omega_0} v_0^2 k^2 + \frac{v_0^2 k^2}{2\omega_0 nk_B T\kappa_T} + \frac{\bar{b}_1^2 v_0^2 k^2}{2(\omega_0 \pm \mathrm{i}c)} \right]$$
$$- \frac{1}{2} \mathrm{i} [\omega_p^2 (a_1 + a_2) + \omega_B^2 (2a_1 + a_2)] \frac{v_0^2 k^2}{\omega_0^2} . \tag{6.1}$$

To obtain the dissipative mode frequencies for the case $\boldsymbol{k} \perp \boldsymbol{B}$ we should return to the expression (3.29) for the frequency matrix $\Omega_{\mu\nu}$. Upon substitution of $k_z = 0$ in (3.29) with (3.12) and (3.20) it follows that $\Omega_{\mu 3} = \Omega_{3\mu} = 0$ for $\mu \neq 3$, so that one of the modes is decoupled. It has the frequency $z = \Omega_{33}$, or

$$z = -\mathrm{i}(a_1 + a_4) v_0^2 k^2 . \tag{6.2}$$

A further analysis of the eigenvalue equation for the remaining modes leads to another dissipative mode:

$$z = -\mathrm{i}\,\frac{\omega_\mathrm{p}^2}{\omega_0^2}\,a_1 v_0^2 k^2\,. \tag{6.3}$$

Of course also the mode (4.2) is recovered. The coefficients a_j in (6.2) and (6.3) must be evaluated at the frequency $z = \mathrm{i}0$. As these are real according to (3.13), the modes (6.2) and (6.3) are purely damped, with an imaginary frequency, up to order k^2. It should be remarked that the expression for the mode (6.3) contains explicitly a factor ω_0^{-2} as compared to that of the modes discussed before. When the strength of the magnetic field increases, the damping of this mode thus vanishes relatively fast as compared to that of the other modes. So for intense magnetic fields the mode (6.3) becomes long-lived.

The singular property of long-livedness of the mode (6.3) shows that it is not connected in a continuous way to the modes with $k_\parallel \neq 0$. In fact, the mode frequencies given in (6.2) and (6.3) cannot be obtained from the general formula by putting $\lambda = -1$ and taking the limit $\hat{k}_\parallel = 0$. Indeed, since for $\lambda = -1$ and $\hat{k}_\parallel \to 0$ one has

$$w_\lambda \simeq \frac{\omega_\mathrm{p}\omega_\mathrm{B}}{\omega_0}\,\hat{k}_\parallel\,, \tag{6.4}$$

the polynomial (5.2) becomes for small $\hat{k}_\parallel$

$$N_1^- \simeq \frac{\mathrm{i}\omega_\mathrm{p}^2\omega_B^2}{\omega_0^2}\,[\omega_\mathrm{p}^2(2a_1 + a_4) + \omega_B^2(a_1 + a_4)]\hat{k}_\parallel^2\,, \tag{6.5}$$

while $N_2 = \mathcal{O}(\hat{k}_\parallel^4)$. Then one finds from (5.1) for $\hat{k}_\parallel \to 0$:

$$z \simeq \pm\,\frac{\omega_\mathrm{p}\omega_B}{\omega_0}\,\hat{k}_\parallel - \frac{\mathrm{i}}{2\omega_0^2}\,[\omega_\mathrm{p}^2(2a_1 + a_4) + \omega_B^2(a_1 + a_4)]v_0^2 k^2\,. \tag{6.6}$$

Hence for small but nonvanishing $k_\parallel$ two of the oscillating modes become nearly degenerate, with almost vanishing frequencies and with the same damping coefficients. This damping coefficient differs from those of (6.2) and (6.3). In fact, it is the arithmetic mean of these. For strong fields it does not tend to zero, so that none of the modes (6.6) becomes longlived. The mode (6.3) indeed has a unique property.

The discontinuity in the frequency spectrum of the collective modes is a nice example of a general phenomenon in spectral theory. Indeed, discontinuities in an eigenvalue spectrum may arise if a degeneracy occurs for a particular value of a parameter, as is a well-known fact in perturbation theory. The prescription to obtain the perturbed eigenvalues depends critically on the degree of degeneracy of the unperturbed problem. The present calculation of the mode frequencies can be considered as an example of a perturbation analysis. The unperturbed problem corresponds to the case of vanishing wave vector.

7. Magnetohydrodynamics

To establish the connexion between magnetohydrodynamics and the kinetic theory of a magnetized plasma we calculate in this section the modes that follow from the linearized magnetohydrodynamic equations. Subsequently, we investigate the relation with the results from the preceding sections.

The linearized magnetohydrodynamic equations read

$$\frac{\partial}{\partial t}\delta n(\boldsymbol{r},t)+n\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{r},t)=0\,,$$

$$nm\frac{\partial}{\partial t}\boldsymbol{v}(\boldsymbol{r},t)+\boldsymbol{\nabla}\delta P(\boldsymbol{r},t)-\boldsymbol{\nabla}\cdot\boldsymbol{\eta}:\boldsymbol{\nabla}\boldsymbol{v}(\boldsymbol{r},t)=ne\boldsymbol{E}(\boldsymbol{r},t)+nm\omega_B\boldsymbol{v}(\boldsymbol{r},t)\wedge\hat{\boldsymbol{B}}\,,$$

$$\frac{\partial}{\partial t}\delta T(\boldsymbol{r},t)+\frac{T}{nc_V}\left(\frac{\partial P}{\partial T}\right)_n\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{r},t)-\frac{1}{nc_V}\boldsymbol{\nabla}\cdot\boldsymbol{\lambda}\cdot\boldsymbol{\nabla}\delta T(\boldsymbol{r},t)=0\,. \tag{7.1}$$

The hydrodynamic velocity is denoted by $\boldsymbol{v}(\boldsymbol{r},t)$. The local fluctuations of the particle density n, the temperature T and the hydrostatic pressure P are written as $\delta n(\boldsymbol{r},t)$, $\delta T(\boldsymbol{r},t)$ and $\delta P(\boldsymbol{r},t)$, respectively. The local electric field satisfies the Maxwell equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t)=e\delta n(\boldsymbol{r},t)\,. \tag{7.2}$$

The thermal conductivity tensor $\boldsymbol{\lambda}$ depends on the magnetic field and satisfies the Onsager symmetry

$$\lambda_{ij}(\boldsymbol{B})=\lambda_{ji}(-\boldsymbol{B})\,. \tag{7.3}$$

Upon expansion in covariant tensors we have[5,6])

$$\lambda_{ij}=\lambda_\perp\delta_{ij}+(\lambda_\parallel-\lambda_\perp)\hat{B}_i\hat{B}_j+\lambda_t\varepsilon_{ijk}\hat{B}_k\,. \tag{7.4}$$

Similarly, the viscosity tensor $\boldsymbol{\eta}$ satisfies the Onsager symmetry

$$\eta_{ijmn}(\boldsymbol{B}) = \eta_{mnij}(-\boldsymbol{B}) \tag{7.5}$$

and is symmetric in the indices ij and mn. Consequently, we may expand it in seven independent linear combinations of the tensors $T^{(i)}$ as defined in (3.9). We choose the independent combinations such that the contraction has a form similar to (3.12)[5,6]),

$$\begin{aligned}\eta_{ijmn} = f_1(-T^{(1)}_{imjn} + T^{(2)}_{imjn}) + f_2 T^{(1)}_{imjn} + f_3(T^{(3)}_{imjn} + T^{(4)}_{imjn}) \\ + f_4(T^{(5)}_{imjn} - T^{(3)}_{imjn} - T^{(4)}_{imjn}) + f_5 T^{(6)}_{imjn} + f_6 T^{(7)}_{imjn} + f_7 T^{(8)}_{imjn} \,.\end{aligned} \tag{7.6}$$

The coefficients f_i are related to the seven viscosity coefficients as introduced in ref. 5, viz.

$$\begin{aligned}&f_1 = -\eta_1 + 2\eta_2 \,, \quad f_2 = \tfrac{1}{3}\eta_1 + \eta_V - 2\zeta \,, \quad f_3 = -\eta_1 + \eta_3 + 3\zeta \,, \\ &f_4 = \eta_1 - 2\eta_2 + \eta_3 \,, \quad f_5 = 2\eta_1 + 2\eta_2 - 4\eta_3 \,, \quad f_6 = \tfrac{1}{2}\eta_4 \,, \quad f_7 = -\tfrac{1}{2}\eta_4 - \eta_5 \,.\end{aligned} \tag{7.7}$$

The coefficients $\eta_{1,2,\ldots,5}$ are the shear viscosities; the coefficient η_V is the volume viscosity and ζ describes a cross effect between shear and volume viscosity. From the requirement of positive entropy production it follows that

$$\begin{aligned}&\lambda_\perp \geqslant 0 \,, \quad \lambda_\parallel \geqslant 0 \,, \quad \eta_i \geqslant 0 \quad (i = 1, 2, 3, V) \,, \\ &2\eta_2 \geqslant \eta_1 \,, \quad \eta_1 \eta_V \geqslant 3\zeta^2 \,.\end{aligned} \tag{7.8}$$

To find the collective modes we apply a Fourier–Laplace transform to the equations (7.1). Then we obtain the determinant of the resulting system of equations for the quantities $\delta n(\boldsymbol{k}, z)$, $\boldsymbol{v}(\boldsymbol{k}, z)$ and $\delta T(\boldsymbol{k}, z)$. Subsequently we calculate the zeros of this determinant for long wavelengths.

In general we find only one purely damped mode, viz. the heat mode

$$z = \frac{-\mathrm{i}}{n c_V} (\lambda_\perp k_\perp^2 + \lambda_\parallel k_\parallel^2) \,, \tag{7.9}$$

where terms of higher order in k have been omitted. As compared to the corresponding expression for a neutral gas the heat mode contains the specific heat c_V instead of c_p. This feature occurs already in the unmagnetized one-component plasma[1,2]). It is due to the suppression of density fluctuations by the local self-consistent electric field.

The remaining modes have an oscillating nature, at least when the wave vector has an arbitrary orientation with respect to the magnetic field.

Specifically, we have for very strong magnetic fields, i.e. $\omega_B \gg \omega_p$,

$$z = \pm\left[\omega_B + \frac{1}{nm}(-\eta_4 k_\perp^2 + \eta_5 k_\parallel^2)\right] - \frac{i}{nm}\left[\left(-\frac{5}{6}\eta_1 + 2\eta_2 + \frac{1}{2}\eta_V - \zeta\right)k_\perp^2 + \eta_3 k_\parallel^2\right], \tag{7.10}$$

and

$$z = \pm\omega_p \hat{k}_\parallel\left(1 + \frac{c_s^2 k^2}{2\omega_p^2}\right) - \frac{i}{nm}\left[\frac{1}{2}\eta_3 k_\perp^2 + \left(\frac{2}{3}\eta_1 + \frac{1}{2}\eta_V + 2\zeta\right)k_\parallel^2\right]. \tag{7.11}$$

The sound velocity c_s depends in the usual way on the thermodynamic quantities,

$$c_s^2 = \frac{c_p}{c_V}\frac{1}{nm\kappa_T} = \frac{1}{nm\kappa_T} + \frac{T}{n^2 m c_V}\left(\frac{\partial P}{\partial T}\right)_n^2. \tag{7.12}$$

The imaginary parts of z in (7.10) and (7.11) are negative definite, so that indeed the modes are damped. In fact, one easily verifies with the use of (7.8)

$$\begin{aligned} -\tfrac{5}{6}\eta_1 + 2\eta_2 + \tfrac{1}{2}\eta_V - \zeta &\geq 2\eta_2 - \eta_1 + \frac{1}{2}\left(\sqrt{\tfrac{1}{3}\eta_1} - \sqrt{\eta_V}\right)^2 \geq 0\,, \\ \tfrac{2}{3}\eta_1 + \tfrac{1}{2}\eta_V + 2\zeta &\geq \frac{1}{2}(2\sqrt{\tfrac{1}{3}\eta_1} - \sqrt{\eta_V})^2 \geq 0\,. \end{aligned} \tag{7.13}$$

The oscillation frequency of the mode (7.10) is shifted from the Larmor frequency by an amount depending on the viscosity coefficients η_4 and η_5. Hence in the presence of a magnetic field the transport coefficients not only determine the damping of the modes but in some cases also influence the dispersion.

For arbitrary values of the magnitude of the magnetic field the calculation of the modes becomes relatively simple if the wave vector is parallel to the magnetic field, i.e. $k_\perp = 0$. Then one finds that the expressions (7.10) and (7.11) for $k_\perp = 0$ are valid for arbitrary values of ω_B.

Another particularly interesting case occurs when the wave vector is purely perpendicular to the magnetic field, i.e. $k_\parallel = 0$. In this case there are, apart from the heat mode, two purely dissipative modes: a viscous mode

$$z = -\frac{i\eta_3}{nm}k^2 \tag{7.14}$$

and a vortex mode

$$z = -\frac{\mathrm{i}(-\eta_1 + 2\eta_2)}{nm} \frac{\omega_\mathrm{p}^2}{\omega_0^2} k^2 , \tag{7.15}$$

which is known as the convective cell[7]); here $\omega_0 = (\omega_\mathrm{p}^2 + \omega_B^2)^{1/2}$, as before. The dissipative modes (7.14) and (7.15) are not obtained by taking the limit $k_\| \to 0$ in the expressions of the mode frequencies for general wave vector. The situation is quite similar to that considered in the previous section.

In addition, for $k_\| = 0$, we find two oscillating modes, viz.

$$z = \pm\left[\omega_0 + \left(\frac{1}{2}c_\mathrm{s}^2 - \frac{\eta_4\omega_B}{nm}\right)\frac{k^2}{\omega_0}\right] - \mathrm{i}\left[\omega_\mathrm{p}^2\left(-\frac{1}{3}\eta_1 + \eta_2 + \frac{1}{2}\eta_V - \zeta\right)\right.$$
$$\left. + \omega_B^2\left(-\frac{5}{6}\eta_1 + 2\eta_2 + \frac{1}{2}\eta_V - \zeta\right)\right]\frac{k^2}{nm\omega_0^2} . \tag{7.16}$$

Now that we have calculated the modes from the magnetohydrodynamic equations we can make a comparison with the results that we have obtained from kinetic theory. For the heat mode (7.9) the relation with (4.2) is obvious,

$$\lambda_\perp = \frac{3n}{2m} k_\mathrm{B}^2 T d_1 , \quad \lambda_\| = \frac{3n}{2m} k_\mathrm{B}^2 T (d_1 + d_2) , \tag{7.17}$$

where $d_j = d_j(z = \mathrm{i}0)$. The heat mode is a purely hydrodynamic mode; in the general framework of kinetic theory it occurs in the simultaneous limit $\boldsymbol{k} \to \boldsymbol{0}$, $z \to \mathrm{i}0$. Similarly, the connexion is clear for the purely dissipative modes at $k_\| = 0$. A comparison of (7.14) and (7.15) with (6.2) and (6.3) yields

$$\eta_3 = nk_\mathrm{B}T(a_1 + a_4) , \quad -\eta_1 + 2\eta_2 = nk_\mathrm{B}Ta_1 , \tag{7.18}$$

where again the a_j are to be evaluated at $z = \mathrm{i}0$.

For the oscillating modes the situation is more subtle. For $\omega_B \gg \omega_\mathrm{p}$ these modes have dispersion relations, given by (4.8) and (4.9), that contain the coefficients a_j, $\bar{b}_j$ and c. Since $z \not\to \mathrm{i}0$ if $\boldsymbol{k} \to \boldsymbol{0}$ these coefficients are to be evaluated at finite values of their argument. So in principle they are not the phenomenological transport coefficients which appear in the magnetohydrodynamic considerations. However, for a strongly coupled plasma we expect collisions to dominate the collective behaviour[1]), so that $\omega_0/\omega_\mathrm{c} \ll 1$, where ω_c is the collision frequency. Then the coefficients $a_j(z)$, $\bar{b}_j(z)$ and $c(z)$ are well approximated by their values near $z = \mathrm{i}0$. These follow with the use of (3.22) and (3.28), so that we can make the replacements

$$\bar{b}_1 \to \sqrt{\frac{2}{3}} \frac{1}{nk_{\mathrm{B}}} \left(\frac{\partial P}{\partial T}\right)_n, \quad \bar{b}_2 \to 0, \quad c \to \mathrm{i}z\left(1 - \frac{2c_V}{3k_{\mathrm{B}}}\right). \tag{7.19}$$

Now (4.8) and (4.9) get the same form as (7.10) and (7.11) and we can make the following identifications:

$$\begin{aligned} &-\tfrac{5}{6}\eta_1 + 2\eta_2 + \tfrac{1}{2}\eta_V - \zeta = nk_{\mathrm{B}}T(a_1 + \tfrac{1}{2}a_2), \\ &\tfrac{2}{3}\eta_1 + \tfrac{1}{2}\eta_V + 2\zeta = nk_{\mathrm{B}}T(\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2 + a_3 + a_4 + \tfrac{1}{2}a_5), \\ &\eta_4 = 2nk_{\mathrm{B}}Ta_6, \quad \eta_5 = -nk_{\mathrm{B}}T(a_6 + a_7), \end{aligned} \tag{7.20}$$

where $a_j = a_j(z = \mathrm{i}0)$. One more relation is found from the comparison of the mode dispersion relations for arbitrary values of the strength of the magnetic field and an arbitrary direction of the wave vector. The relations are conveniently written in terms of the coefficients f_j, which were defined in (7.7), viz.

$$f_j = nk_{\mathrm{B}}Ta_j(z = \mathrm{i}0). \tag{7.21}$$

Thus we have related the phenomenological viscosity coefficients η_j, ζ and the thermal conductivity coefficients $\lambda_\perp$, $\lambda_\parallel$ to the kinetic coefficients a_j and d_j. For these kinetic coefficients we have derived expressions that contain the collision kernel, e.g. (3.14).

We have now established the connexion between kinetic theory and magnetohydrodynamics as far as the collective modes are concerned. The magnetohydrodynamic equations contain static transport coefficients, whereas kinetic theory shows that several collective modes are determined by generalized transport coefficients at finite frequencies. Only the purely dissipative modes follow correctly from the magnetohydrodynamic equations. The thermal conductivity coefficients $\lambda_\perp$, $\lambda_\parallel$, the viscosity coefficient η_3 and the combination $\eta_1 - 2\eta_2$ are accessible through the damping coefficients of these modes. The remaining viscosity coefficients, however, appear only in the oscillating modes in the form of generalized transport coefficients at finite frequencies. Their static counterparts do not play a role in the mode spectrum.

Acknowledgement

This investigation is part of the research programme of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)", which is financially supported by the "Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (Z.W.O.)".

Appendix A

Integral relations for the collision kernel

The microscopic conservation of the number of particles is expressed by the continuity equation

$$\mathrm{i}Ln(\boldsymbol{k}) = \mathrm{i}L \sum_{\alpha} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} = -\mathrm{i}\boldsymbol{k} \cdot \sum_{\alpha} \frac{\boldsymbol{p}_\alpha}{m} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} . \tag{A.1}$$

Upon projection onto the space orthogonal to the one-particle phase functions one gets

$$QL \int \mathrm{d}\boldsymbol{p}\, \tilde{f}(\boldsymbol{k}, \boldsymbol{p}) = 0 , \tag{A.2}$$

and hence, with the help of (2.6),

$$\int \mathrm{d}\boldsymbol{p}\, \varphi^{\mathrm{c}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = 0 . \tag{A.3}$$

The microscopic momentum balance equation follows by writing

$$\mathrm{i}L\boldsymbol{g}(\boldsymbol{k}) = \mathrm{i}L \sum_{\alpha} \boldsymbol{p}_\alpha\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} = -\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\tau}^{\mathrm{kin}}(\boldsymbol{k}) + \sum_{\alpha} \dot{\boldsymbol{p}}_\alpha\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} , \tag{A.4}$$

with the kinetic pressure tensor

$$\boldsymbol{\tau}^{\mathrm{kin}}(\boldsymbol{k}) = \sum_{\alpha} \frac{\boldsymbol{p}_\alpha \boldsymbol{p}_\alpha}{m} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} , \tag{A.5}$$

and using the equation of motion (cf. refs. 8, 9)

$$\dot{\boldsymbol{p}}_\alpha = -\frac{\mathrm{i}}{V} \sum_{\boldsymbol{q}(\neq \boldsymbol{0})} \frac{e^2 \boldsymbol{q}}{q^2} \sum_{\beta(\neq \alpha)} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}} + \omega_B \boldsymbol{p}_\alpha \wedge \hat{\boldsymbol{B}} . \tag{A.6}$$

The subsidiary condition $\boldsymbol{q} \neq \boldsymbol{0}$ accounts for the effects of the neutralizing background. The microscopic momentum balance equation becomes

$$\mathrm{i}L\boldsymbol{g}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k}) - \mathrm{i}ne^2\frac{\boldsymbol{k}}{k^2}n(\boldsymbol{k}) + \omega_B\boldsymbol{g}(\boldsymbol{k})\wedge\hat{\boldsymbol{B}}. \tag{A.7}$$

Here $\boldsymbol{\tau}$ is the sum of the kinetic pressure tensor (A.5) and the potential pressure tensor, which for small $\boldsymbol{k}$ is given by[8,9])

$$\boldsymbol{\tau}^{\mathrm{pot}}(\boldsymbol{k}) = \frac{1}{2V}\sum_{\boldsymbol{q}(\neq \boldsymbol{0},\neq\boldsymbol{k})}\frac{e^2}{q^2}\left(\boldsymbol{U}-\frac{2\boldsymbol{q}\boldsymbol{q}}{q^2}\right)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}, \tag{A.8}$$

with $\boldsymbol{U}$ the unit tensor.

The momentum balance equation (A.7) gives rise to an expression for the first moment of the collision kernel. Projection with the operator Q yields

$$QL\int \mathrm{d}\boldsymbol{p}\,\boldsymbol{p}\,\tilde{f}(\boldsymbol{k},\boldsymbol{p}) = -\frac{1}{\sqrt{V}}\,Q\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k}) \tag{A.9}$$

and hence, with (2.6),

$$\int \mathrm{d}\boldsymbol{p}\,\boldsymbol{p}\,\varphi^{\mathrm{c}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z) = \boldsymbol{k}\cdot\boldsymbol{T}(\boldsymbol{k},\boldsymbol{p}',z), \tag{A.10}$$

with the tensor $\boldsymbol{T}$ defined by

$$\boldsymbol{T}(\boldsymbol{k},\boldsymbol{p},z)nf_0(p) = -\frac{1}{\sqrt{V}}\left\langle[QL\tilde{f}(\boldsymbol{k},\boldsymbol{p})]^*\frac{1}{z+QLQ}\,Q\boldsymbol{\tau}(\boldsymbol{k})\right\rangle. \tag{A.11}$$

For small $\boldsymbol{k}$ the right-hand side of (A.10) is linear in $\boldsymbol{k}$, since $\boldsymbol{T}$ has a finite limit for $\boldsymbol{k}\to\boldsymbol{0}$. It should be remarked that the magnetic force term in (A.7) has been annihilated by the projector Q, so that no such term appears in (A.9). However, the tensor $\boldsymbol{T}$ still depends on the magnetic field through the Liouville operator in the denominator at the right-hand side of (A.11).

The microscopic energy balance follows by starting from the kinetic energy:

$$\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) = \sum_\alpha \frac{p_\alpha^2}{2m}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} \tag{A.12}$$

and deriving its time derivative as

$$\mathrm{i}L\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{j}_{\varepsilon}^{\mathrm{kin}} - \frac{\mathrm{i}}{V}\sum_{\boldsymbol{q}(\neq \boldsymbol{0})}\frac{e^2\boldsymbol{q}}{q^2}\cdot\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\boldsymbol{p}_\alpha}{m}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,, \tag{A.13}$$

where (A.6) has been used. The kinetic energy flow $\boldsymbol{j}_{\varepsilon}^{\mathrm{kin}}$ is defined as

$$\boldsymbol{j}_{\varepsilon}^{\mathrm{kin}} = \sum_{\alpha}\frac{\boldsymbol{p}_\alpha}{m}\frac{p_\alpha^2}{2m}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,. \tag{A.14}$$

It should be noted that the magnetic field term in (A.6) has dropped out from (A.13). The microscopic potential energy is given by[8,9])

$$\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) = \frac{1}{2V}\sum_{\boldsymbol{q}(\neq 0,\neq \boldsymbol{k})}\frac{e^2}{q^2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,, \tag{A.15}$$

for small values of $\boldsymbol{k}$. The total energy satisfies the conservation law

$$\mathrm{i}L[\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) + \varepsilon^{\mathrm{pot}}(\boldsymbol{k})] = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{j}_{\varepsilon}(\boldsymbol{k})\,, \tag{A.16}$$

with the total microscopic energy flow $\boldsymbol{j}_\varepsilon = \boldsymbol{j}_\varepsilon^{\mathrm{kin}} + \boldsymbol{j}_\varepsilon^{\mathrm{pot}}$, given by (A.14) and by[8,9])

$$\boldsymbol{j}_{\varepsilon}^{\mathrm{pot}}(\boldsymbol{k}) = \frac{1}{V}\sum_{\boldsymbol{q}(\neq 0,\neq \boldsymbol{k})}\frac{e^2}{q^2}\left(\boldsymbol{U} - \frac{\boldsymbol{q}\boldsymbol{q}}{q^2}\right)\cdot\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\boldsymbol{p}_\alpha}{m}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,, \tag{A.17}$$

for small $\boldsymbol{k}$.

From the microscopic energy balance equation (A.16) one may derive an expression for the second moment of the collision kernel. In fact, applying the operator Q to (A.16) one gets

$$QL\int \mathrm{d}\boldsymbol{p}\,\frac{p^2}{2m}\tilde{f}(\boldsymbol{k},\boldsymbol{p}) = -\frac{1}{\sqrt{V}}\,Q[L\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) + \boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k})]\,. \tag{A.18}$$

The collision kernel (2.6) thus fulfils the relation

$$\begin{aligned}&\int \mathrm{d}\boldsymbol{p}\,\frac{p^2}{2m}\,\varphi^{\mathrm{c}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',z)nf_0(p')\\ &\qquad = -\frac{1}{\sqrt{V}}\left\langle[QL\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^*\,\frac{1}{z+QLQ}\,[L\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) + \boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k})]\right\rangle.\end{aligned} \tag{A.19}$$

Since one has $QLP\,\varepsilon^{\text{pot}}(\boldsymbol{k}) = 0$, the right-hand side may be written as

$$-\frac{\boldsymbol{k}}{\sqrt{V}}\cdot\left\langle[QL\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^*\frac{1}{z+QLQ}\boldsymbol{j}_\varepsilon(\boldsymbol{k})\right\rangle$$

$$+\frac{z}{\sqrt{V}}\left\langle[QL\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^*\frac{1}{z+QLQ}\varepsilon^{\text{pot}}(\boldsymbol{k})\right\rangle$$

$$-\frac{1}{\sqrt{V}}\langle[QL\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^*\varepsilon^{\text{pot}}(\boldsymbol{k})\rangle\,. \tag{A.20}$$

The third term can be shown to be proportional to $\boldsymbol{k}$ for small $\boldsymbol{k}$. In fact, by writting $Q = 1 - P$, this term falls apart into two contributions. The first, without the projector, becomes upon using (A.16) and then (A.13)

$$\frac{1}{\sqrt{V}}\langle[\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^*\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\text{pot}}(\boldsymbol{k})\rangle-\frac{1}{V^{3/2}}\left\langle[\tilde{f}(\boldsymbol{k},\boldsymbol{p}')]^*\sum_{\boldsymbol{q}(\neq\boldsymbol{0})}\frac{e^2\boldsymbol{q}}{q^2}\cdot\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\boldsymbol{p}_\alpha}{m}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\right\rangle. \tag{A.21}$$

Inserting the expression for $\tilde{f}$ and performing the momentum average one finds that the second term vanishes as a consequence of the antisymmetry in $\boldsymbol{q}$. The first term has the form

$$n^2 f_0(p')\frac{1}{V}\sum_{\boldsymbol{q}(\neq\boldsymbol{0},\neq\boldsymbol{k})}\frac{e^2}{q^2}h(q)\boldsymbol{k}\cdot\left(\boldsymbol{U}-\frac{\boldsymbol{q}\boldsymbol{q}}{q^2}\right)\cdot\frac{\boldsymbol{p}'}{m}=\frac{2}{3}n^2 f_0(p')\frac{\boldsymbol{k}\cdot\boldsymbol{p}'}{m}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\frac{e^2}{q^2}h(q)\,, \tag{A.22}$$

where the isotropy of the summation over $\boldsymbol{q}$ has been used.

The second contribution, with the projector P, to the third term of (A.20) is

$$-f_0(p')\frac{\boldsymbol{k}\cdot\boldsymbol{p}'}{m}[1-nc(k)]\frac{1}{V}\langle[n(\boldsymbol{k})]^*\varepsilon^{\text{pot}}(\boldsymbol{k})\rangle\,, \tag{A.23}$$

as follows by employing the definition (2.7). For small $\boldsymbol{k}$ the inverse structure factor $1 - nc(k)$ is proportional to k^{-2} (see (B.4)). However, the average in (A.23) is proportional to k^2 (see (B.5)). Hence, for small $\boldsymbol{k}$ the expression (A.23) becomes

$$-3nk_{\mathrm{B}}Tf_0(p')\frac{\boldsymbol{k}\cdot\boldsymbol{p}'}{m}\left(\frac{1}{nk_{\mathrm{B}}T\kappa_T}-1\right). \tag{A.24}$$

From (A.20), (A.22) and (A.24) it follows that the left-hand side of (A.19) may indeed be written in the form (2.13), with finite $\boldsymbol{J}_\varepsilon$ and E for small $\boldsymbol{k}$ and z.

Appendix B

The low-frequency limit of matrix elements of the collision kernel

In this appendix the relations (3.22) and (3.28), which determine the low-frequency limits of the matrix elements (3.19) and (3.24), will be derived.

For small $\boldsymbol{k}$ and $z \to \mathrm{i}0$ the expression (3.15) with (3.16) becomes

$$\langle i|\varphi^{\mathrm{c}}|4\rangle = \frac{v_0}{n(k_{\mathrm{B}}T)^2 V}\sqrt{\frac{2}{3}}\left\langle [QL\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* \frac{1}{QLQ} Q[\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k})]_i\right\rangle$$
$$= \frac{v_0}{n(k_{\mathrm{B}}T)^2 V}\sqrt{\frac{2}{3}}\langle [\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* Q[\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k})]_i\rangle\,, \tag{B.1}$$

where we used $QLP\varepsilon^{\mathrm{pot}} = 0$ in the last equality. With the definition (2.7) of the projector we get

$$\frac{v_0}{n(k_{\mathrm{B}}T)^2}\sqrt{\frac{2}{3}}\left\{\frac{1}{V}\langle [\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^*[\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k})]_i\rangle - \frac{1}{V^2}\langle [\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* n(\boldsymbol{k})\rangle \frac{1-nc(k)}{n}\langle [n(\boldsymbol{k})]^*[\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k})]_i\rangle\right\}. \tag{B.2}$$

The canonical average in the first term within the braces is given by[9])

$$\frac{1}{V}\langle [\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^*\boldsymbol{\tau}(\boldsymbol{k})\rangle = \left[k_{\mathrm{B}}T^2\left(\frac{\partial P}{\partial T}\right)_n - n(k_{\mathrm{B}}T)^2\right]\boldsymbol{U}\,, \tag{B.3}$$

for small $\boldsymbol{k}$. It should be remarked here that the fluctuation formulae of ref. 9 are valid for a magnetized plasma as well. The second term in (B.2) is proportional to k^2 for small $\boldsymbol{k}$, so that it may be ignored in the limit $\boldsymbol{k}\to\boldsymbol{0}$. In fact, the inverse structure factor is, for small $\boldsymbol{k}$, proportional to k^{-2};

$$1 - nc(k) = \frac{k_{\mathrm{D}}^2}{k^2} + \frac{1}{nk_{\mathrm{B}}T\kappa_T} + \mathcal{O}(k^2)\,, \tag{B.4}$$

with κ_T the isothermal compressibility. On the other hand, the two canonical averages are each proportional to k^2 for small wave vectors[9]),

$$\frac{1}{V}\langle[\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* n(\boldsymbol{k})\rangle = 3nk_{\mathrm{B}}T\frac{k^2}{k_{\mathrm{D}}^2}\left(\frac{1}{nk_{\mathrm{B}}T\kappa_T} - 1\right), \tag{B.5}$$

$$\frac{1}{V}\langle[n(\boldsymbol{k})]^* \boldsymbol{\tau}(\boldsymbol{k})\rangle = nk_{\mathrm{B}}T\frac{k^2}{k_{\mathrm{D}}^2}\frac{1}{nk_{\mathrm{B}}T\kappa_T}\,\boldsymbol{U}. \tag{B.6}$$

Substitution of (B.3) into (B.2) and comparison with (3.19)–(3.20) completes the proof of (3.22).

To prove (3.28) we start from (3.23), which for small $\boldsymbol{k}$ and z reads

$$\langle 4|\varphi^{\mathrm{c}}|4\rangle = \frac{2}{3n(k_{\mathrm{B}}T)^2 V}\left\langle [QL\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* \frac{1}{z + QLQ} QL\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\right\rangle . \tag{B.7}$$

Again using the identity $QLP\varepsilon^{\mathrm{pot}} = 0$ we may write the canonical average as

$$\langle[\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^*(QLQ - zQ)\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\rangle , \tag{B.8}$$

where terms of higher order in z have been neglected.

The first part of this expression, which contains QLQ, vanishes on account of its antisymmetry with respect to time reversal. Substituting the projector in the second part we get

$$\langle 4|\varphi^{\mathrm{c}}|4\rangle = -\frac{2z}{3n(k_{\mathrm{B}}T)^2}\left\{\frac{1}{V}\langle[\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^*\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\rangle - \frac{1}{V^2}\langle[\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* n(\boldsymbol{k})\rangle \frac{1 - nc(k)}{n}\langle[n(\boldsymbol{k})]^*\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\rangle\right\} . \tag{B.9}$$

Again, the second term is of order k^2, as follows from (B.4) and (B.5). Since one has[9])

$$\frac{1}{V}\langle[\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^*\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\rangle = nk_{\mathrm{B}}T^2\left(c_V - \frac{3}{2}k_{\mathrm{B}}\right), \tag{B.10}$$

for small $\boldsymbol{k}$, the proof of (3.28) is established.

References

1) M. Baus and J.-P. Hansen, Phys. Rep. **59** (1980) 1.
2) M. Baus, Physica **79A** (1975) 377.
3) D. Forster, Phys. Rev. A **9** (1974) 943.
4) M.C. Marchetti, T.R. Kirkpatrick and J.R. Dorfman, Phys. Rev. A **29** (1984) 2960.
5) S.R. de Groot and P. Mazur, Non-equilibrium Thermodynamics (North-Holland, Amsterdam, 1962) p. 314; G.J. Hooyman, P. Mazur and S.R. de Groot, Physica **21** (1955) 355.
6) S.I. Braginskii, in: Reviews of Plasma Physics, vol. I, M.A. Leontovich, ed. (Consultants Bureau, New York, 1965) p. 252 ff.
7) H. Okuda and J.M. Dawson, Phys. Rev. Lett. **28** (1972) 1625; Phys. Fluids **16** (1973) 408.
8) B. Bernu and P. Vieillefosse, Phys. Rev. A **18** (1978) 2345.
9) L.G. Suttorp and J.S. Cohen, Physica **133A** (1985) 357, previous article.