Exploring the conformal model of 3D Euclidean geometry

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1 The conformal model of Euclidean geometry

CGA (conformal geometric algebra) has the Euclidean structure 'baked in' at its deepest level. Its properties are most convincingly introduced by interactive demonstrations.

In this talk, I want to illustrate relationship between algebraic formulation and geometric intuition, and the playful elements of this new geometrical language.

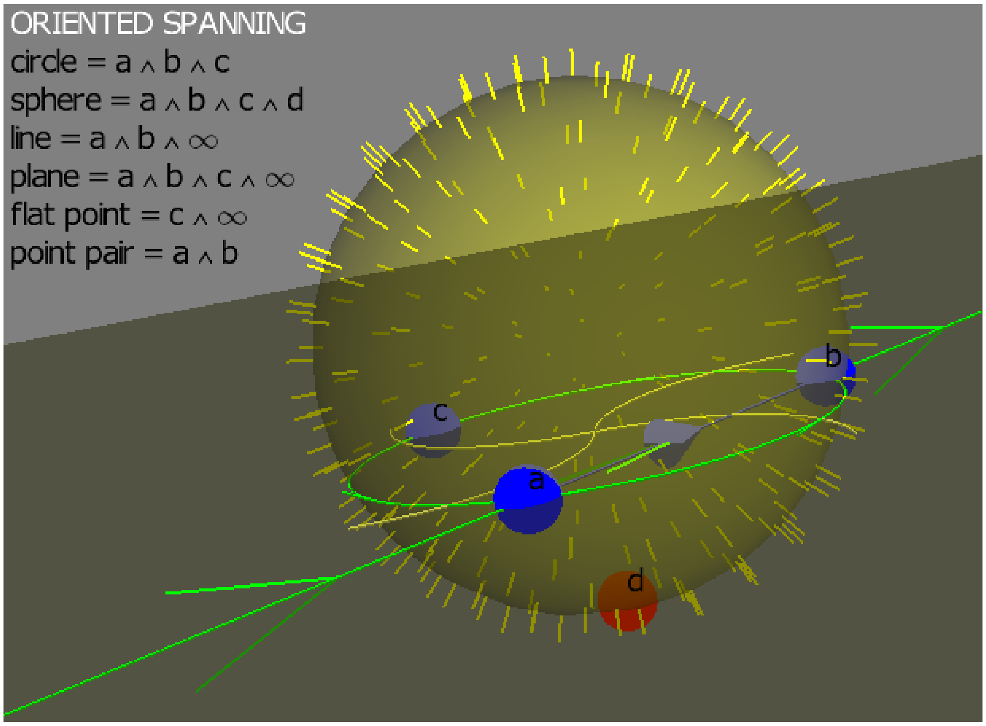
This really requires software. We wrote our own GAviewer in Gaigen, which is C++. But this presentation is about the mathematical model, not about the software.

Three sections to the talk:

1. Introduction of the conformal model
2. A language for Euclidean geometry
3. Structural geometric relationships through versors

2 Spanning oriented 'rounds' and 'flats'

`DEMOspanning()`:



Color coding by 'grade' (1=red, 2=blue, 3=green, 4 = yellow). Note the flat point, a subtly new object – we'll need it.

Geometrically, these objects are *oriented*, algebraically, this is simply done by the outer product being *anti-commutative*.

Containment relationship of these geometrical figures obvious from their definition: suggests an algebra.

3 Dualization of objects

A subspace can be represented in GA in two ways, which we need to distinguish sharply:

- **DIRECT representation of subspace A:**
  With point x as probe,
  $$x \in A \quad \Longleftrightarrow \quad x \wedge A = 0.$$

- **DUAL representation of subspace A:**
  With point x as probe,
  $$x \in A \quad \Longleftrightarrow \quad x \cdot A^* = 0.$$
  (with $A^* = A/I$ and I the pseudoscalar of the space).

The two are of course equivalent because of the duality relationship:

$$(x \wedge A)^* = x \cdot A^*$$

Duals will be depicted by 'complementary' colors. Since the embedding dimension is 5 (we'll see why), the dual of a yellow sphere (4-blade) is red (1-blade).

4 Some basic manipulations in Clifford algebra

1. Outer product $\wedge$ is anti-commutative for vectors: $x \wedge y = -y \wedge x$, associative and linear.

2. Duality laws between inner and outer product:

$$A \cdot B^* = (A \wedge B)^* \quad \text{and} \quad A \wedge B^* = (A \cdot B)^*$$

3. Distributing the inner product, recursive formula:

$$a \cdot (b \wedge C) = (a \cdot b) \wedge C - b \wedge (a \cdot C)$$

4. Geometric product with vector x and blade A:

$$x\,A = x \cdot A + x \wedge A$$

$$x \cdot A = \tfrac{1}{2}(x\,A - A\,x) \quad \text{and} \quad x \wedge A = \tfrac{1}{2}(x\,A + A\,x),$$

5. Versors:
   - Grade preserving: $X \quad \mapsto \quad V\,X\,V^{-1}$
   - Compose through geometric product

5 Incidence

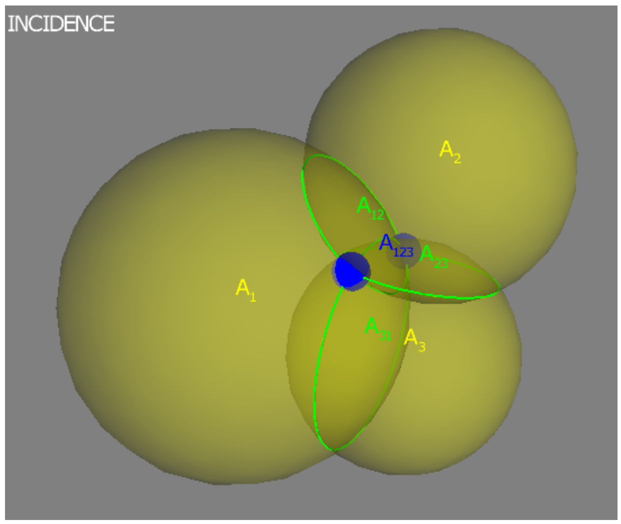
Incidence relationships are dual * to spanning:

$$(A \cap B)^* = B^* \wedge A^*$$

Or more directly:

$$A \cap B = B^* \cdot A$$

This is an oriented incidence. `DEMOincidence()`




```
A1 = -dual(no-ni/2), A2 = -dual(no-2 ni), A3 = -dual(no-ni),
dynamic{ A12  = (A2/I5).A1,};
dynamic{ A23  = (A3/I5).A2,};
dynamic{ A31  = (A1/I5).A3,};
dynamic{ A123 = (A23/A2).A12,};
```

Note the automatic occurrence of *'imaginary' rounds*. (No complex numbers, but simply rounds with negative 'radius squared'.)

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* But you have to be a bit careful: the dual should be relative to the smallest common space, known as the *join*.

6 Closure under incidence

These 'nice' intersections are not all.

What happens when two spheres touch?

```
s1 = pt(-e1)-ni/2, s2 = pt( e1)-ni/2,
dynamic{ both = dual(s1^s2),};
```

These spheres have a common *2-tangent*.

What happens with two parallel planes?

```
p1 = e1, p2 = e2+ni,
dynamic{ both = dual(p1^p2),};
```

These planes have a common *2-attitude*.

So we have the elements of 'Euclidean incidence geometry':

- *rounds* and *dual rounds* (both real and imaginary)
- *flats* and *dual flats*
- *tangents*
- *attitudes*

This is the complete list of objects, we have algebraic closure.

7 Why it works (finally!)

The idea behind the conformal model is to embed the Euclidean metric in the inner product. Introduce the point at infinity ∞.

Now *define* for two points $\mathcal{P}$, $\mathcal{Q}$ represented by vectors p, q:

$$\frac{p}{\infty \cdot p} \cdot \frac{q}{\infty \cdot q} = -\tfrac{1}{2} d_E^2(\mathcal{P}, \mathcal{Q})$$

So $\infty \cdot p \neq 0$. Distance between p and ∞ should be infinite: $\infty \cdot \infty = 0$. Note: $p \cdot p = 0$, so points represented as *null vectors*.

That's all! The rest is GA and interfacing.

In this talk we will use normalized points for convenience:

$$\infty \cdot p = -1, \quad \text{so}: \; p \cdot q = -\tfrac{1}{2} d_E^2(\mathcal{P}, \mathcal{Q}).$$

8 Representation of some common elements

- Midplane between points:

$$x \cdot p = x \cdot q \quad \Longleftrightarrow \quad x \cdot (p - q) = 0$$

  So: $p - q$ is dual representation of midplane.

- Sphere center c radius ρ:

  $$x \cdot c = -\tfrac{1}{2}\rho^2 \quad \Longleftrightarrow \quad x \cdot c - \tfrac{1}{2}\rho^2\, x \cdot \infty = x \cdot (c - \tfrac{1}{2}\rho^2\, \infty) = 0$$

  So: $c - \frac{1}{2}\rho^2\, \infty$ is dual sphere.

- Sphere center c, point a on it:

  $$c - \tfrac{1}{2}\rho^2\, \infty = c + (a \cdot c)\infty = (a \cdot c)\infty - (a \cdot \infty)c = a \cdot (c \wedge \infty)$$

  So: $a \cdot (c \wedge \infty)$ is dual sphere with center c, point a.

- Plane normal $\mathbf{n}$, distance δ:

$$\mathbf{x} \cdot \mathbf{n} = \delta \quad \Longleftrightarrow \quad x \cdot (\mathbf{n} + \delta\, \infty) = 0$$

  So: $\mathbf{n} + \delta\, \infty$ is dual plane.

9 The two sphere representations

Dual representation s of a sphere with center m and point a:

$$s = a \cdot (m \wedge \infty)$$

Direct representation S of the sphere through 4 points a,b,c,d:

$$S = a \wedge b \wedge c \wedge d.$$

Their equivalence is a nice demo of the power of CGA 'reasoning'.

Center m is intersection of 3 dual midplanes $(b-a)$, $(c-a)$, $(d-a)$, so dual flat center point is:

$$(m \wedge \infty)^* = (b-a) \wedge (c-a) \wedge (d-a)$$

This helps use relate the two immediately. Since $0 = x \cdot s$,

$$\begin{aligned} 0 &= (x \cdot (a \cdot (m \wedge \infty)))^* \\ &= x \wedge (a \cdot (m \wedge \infty))^* \\ &= x \wedge (a \wedge (m \wedge \infty)^*) \\ &= x \wedge (a \wedge (b-a) \wedge (c-a) \wedge (d-a)) \\ &= x \wedge (a \wedge b \wedge c \wedge d) \end{aligned}$$

This is of the form $0 = x \wedge S$. Done! Run `DEMOspheres()`.

Incidentally, this shows why the representational space is 5-dimensional.

10 Points in coordinates (but don't think that way!)

Our inner product definition can be shown to lead to the embedding of a point with Euclidean position vector $\mathbf{p}$ as:

$$p = o + \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2\,\infty$$

where o is the point at the origin, $\mathbf{p}$ Euclidean position.

So basically like an extra homogeneous coordinate:

$$p = (\mathbf{p}, 1, \tfrac{1}{2}\mathbf{p}^2)^T$$

on the 5D basis $\{o, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \infty\}$ with multiplication table:

$\cdot$	o	$\mathbf{x}$	∞
o	0	0	-1
$\mathbf{x}$	0	$\mathbf{x}^2$	0
∞	-1	0	0

The -1 makes it all work:

$$\begin{aligned} x \cdot y &= (o + \mathbf{x} + \tfrac{1}{2}\mathbf{x}^2\,\infty) \cdot (o + \mathbf{y} + \tfrac{1}{2}\mathbf{y}^2\,\infty) \\ &= (0 + 0 - \tfrac{1}{2}\mathbf{y}^2) + (0 + \mathbf{x} \cdot \mathbf{y} + 0) + (-\tfrac{1}{2}\mathbf{x}^2 + 0 + 0) \\ &= -\tfrac{1}{2}(\mathbf{x} - \mathbf{y})^2 \end{aligned}$$

Nice trick: linearization of a squared distance!

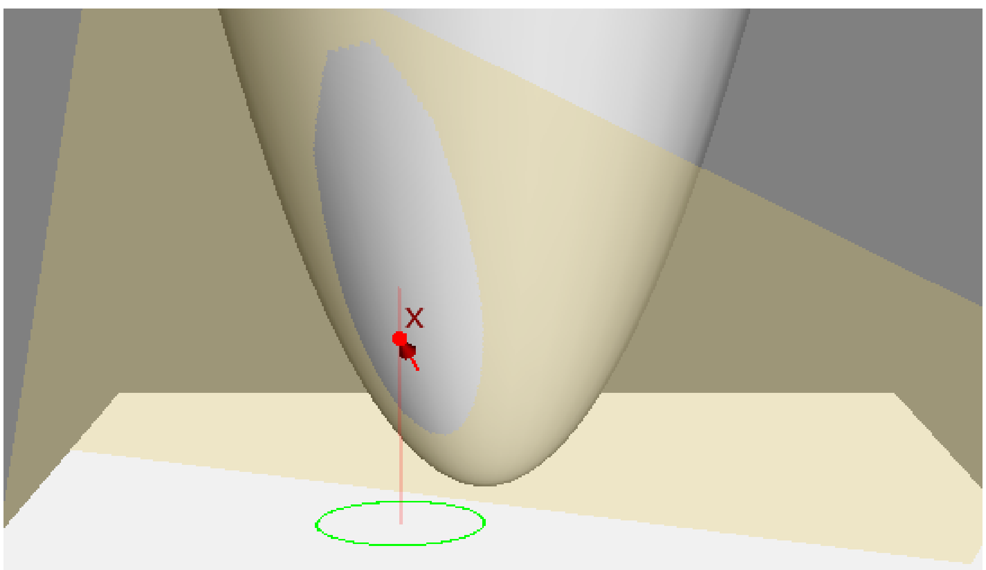
11 Visual demonstration: why blades can be rounds

(Run `DEMOhomogeneous()` to see how homogeneous coordinates are extended in GA.)

Visualize the points

$$p = o + \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2\,\infty$$

for a model of 2-D Euclidean: show Euclidean and ∞ dimension, use homogeneous coordinate dimension to enable translated flats.



Now run `DEMOc2ga_init()`, `DEMOc2ga_x()`, `DEMOc2ga_xy()`.

12 Language: orthogonal specification 1

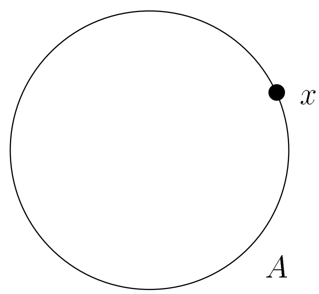
As for 3D vectors, the inner product is about orthogonality:

$$A \cdot B = 0 \quad \Longleftrightarrow \quad A \perp B$$

(We will not prove this.) You can also use this on the duals:

$$A^* \cdot B^* = 0 \quad \Longleftrightarrow \quad A \perp B$$

Play with this by plotting value of inner product for two spheres. `DEMOinnerortho()`.



What about a point x?

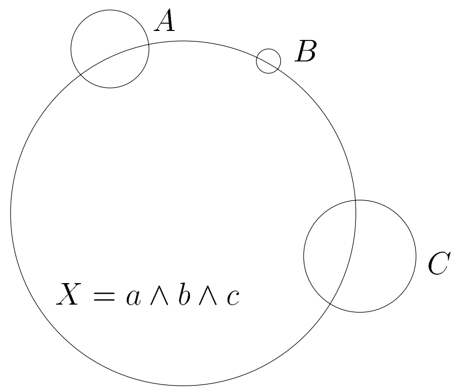
$$x \cdot A^* = 0 \quad \Leftrightarrow \quad x^* \perp A \quad \Leftrightarrow \quad x \in A \quad \Leftrightarrow \quad x \wedge A = 0$$

Since x^* is like a sphere of radius zero, it is consistent. This is why *Euclidean points are small dual spheres* (and *not* small spheres).

Weird, but consistent.

13 Language: orthogonal specification 2

Make the object X orthogonally cutting three spheres A, B, C.



$X \cdot A = 0$, so $X \wedge a = 0$ with $a = A^*$ a dual sphere. Similar for others. So:

$$X = a \wedge b \wedge c$$

In particular, if A etc. are spheres of radius zero, we get a circle containing the points a, b, c.

```
a = no-ni/2,
b = no-ni/4,
c = no-ni/8,
dynamic{ X = a^b^c,}
```

The wedge of points (i.e. small *dual* spheres) gives *direct* representations of rounds.

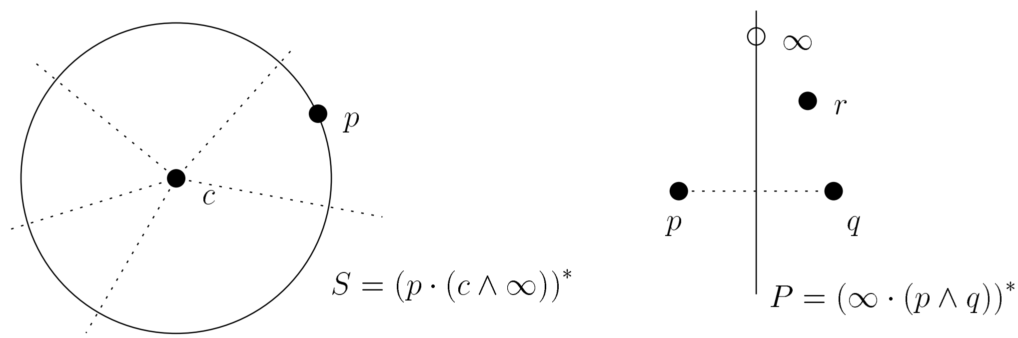
14 Dual sphere and dual plane revisited

The object $p \cdot (c \wedge \infty)$ is the dual sphere with center c, through p. We have seen the algebra; now get the 'language' aspect:

$$(p \cdot (c \wedge \infty))^* = p \wedge (c \wedge \infty)^*$$

So this contains point p and is orthogonal to $c \wedge \infty$.

Apparently 'being orthogonal to $c \wedge \infty$' is 'staying equidistant to c'. We can imagine a flat point as having 'rays' to infinity.



Similarly, we know that $p - q$ is the dual midplane of p and q. Write it multiplicatively

$$p - q = \infty \cdot (p \wedge q)$$

and note that we can give it the immediate interpretation: it is a dual object which contains ∞ and is orthogonal to $p \wedge q$.

So we learn. What is $r \cdot (p \wedge q)$?

15 Language: orthogonal specification 3

Q: What is $o \wedge \mathbf{e}_1 \wedge \infty$?

A: Immediately, we see that the points o and ∞ should be on it, and that it should cut $\mathbf{e}_1{}^*$ orthogonally. But that is the origin plane with normal vector $\mathbf{e}_1$. So this is a line in $\mathbf{e}_1$ direction.

Alternatively, see this as $o \wedge (\mathbf{e}_1 \wedge \infty)$, i.e. the point at the origin with the attitude $\mathbf{e}_1 \wedge \infty$ attached.

Q: What is $p \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$?

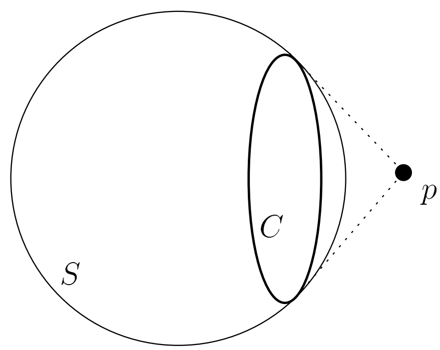
A: See `DEMOortho()`.

16 Language: orthogonal specification 4

All this is consistent with the incidence operation:

$$(A \cap B)^* = B^* \wedge A^*$$

By our semantics, the right hand side is an object orthogonal to A and B; its dual is in both A and B. (Taking the dual relative to the join then makes it the largest set common to A and B.)



Q: Specify the contour circle seen on a sphere S from a point p.

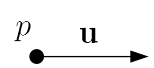
A: $(s \wedge (s \cdot (p \wedge \infty)))^* = s \cdot (s \wedge (p \wedge \infty)^*)$.

17 A natural language for Euclidean geometry

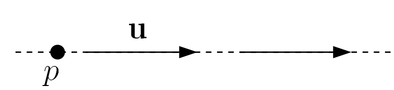
We get an algebraically founded *language for Euclidean elements*, containing precise descriptions of useful (but sloppy) classical elements.

For instance, the riches of vector-related concepts for 1-directions. At the origin o:

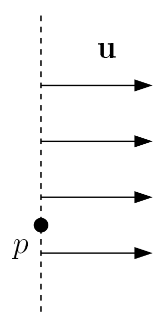
- 'normal vector $\mathbf{u}$' is dual plane $\mathbf{u}$
- 'direction vector $\mathbf{u}$' is the attitude $\mathbf{u} \wedge \infty$
- 'tangent vector $\mathbf{u}$' is $o \wedge \mathbf{u}$
- 'position vector $\mathbf{u}$' is the line element $o \wedge \mathbf{u} \wedge \infty$



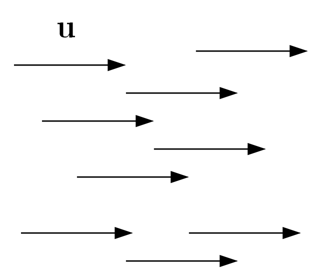
tangent vector $p \wedge (p \cdot (\mathbf{u}\,\infty))$



line vector $p \wedge (\mathbf{u}\,\infty)$



normal vector $p \cdot (\mathbf{u}\infty)$



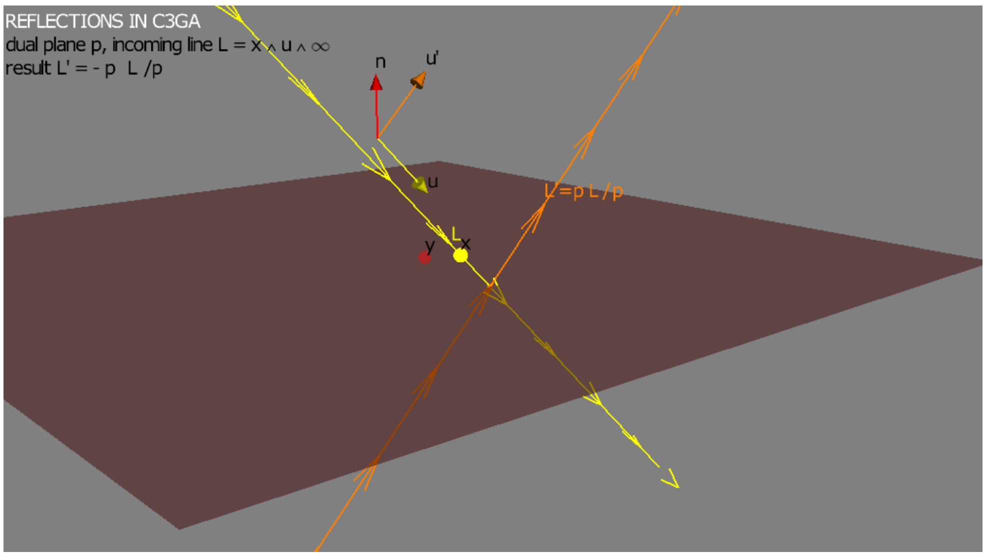
free vector $\mathbf{u}\,\infty$

18 Towards versors: objects as operators

An object can be used as a mirror to act on other objects.

```
dynamic{mirror: MX = M X/M,}
```

The object X can be any blade, and so can the mirror M.



Footnote: Structurally OK, but actually, for correct orientation:

$$\mathbf{X} \quad \mapsto \quad (-1)^{x(m+1)}\mathbf{M}\,\mathbf{X}\mathbf{M}^{-1}$$

19 Generalization of classical formulas

In any standard graphics book you find for the reflection of a vector $\mathbf{u}$ in a plane with unit normal $\mathbf{n}$ (in the origin!):

$$\mathbf{u} \quad \mapsto \quad \mathbf{u} - 2\,(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}$$

Using 3D Euclidean GA we convert this to the versor expression:

$$\mathbf{u} \quad \mapsto \quad -\,\mathbf{n}\,\mathbf{u}\,\mathbf{n}^{-1}$$

Embed in CGA: the full ray is $\ell_0 = o \wedge \mathbf{u} \wedge \infty$, so

$$\begin{aligned}
\ell_0 = o \wedge \mathbf{u} \wedge \infty \ \mapsto \ & o \wedge (-\mathbf{n}\,\mathbf{u}\,\mathbf{n}^{-1}) \wedge \infty \\
= \ & (-\mathbf{n}\,o\,\mathbf{n}^{-1}) \wedge (-\mathbf{n}\,\mathbf{u}\mathbf{n}^{-1}) \wedge (-\mathbf{n}\,\infty\,\mathbf{n}^{-1}) \\
= \ & -\mathbf{n}\,(o \wedge \mathbf{u} \wedge \infty)\,\mathbf{n}^{-1} \\
= \ & -\mathbf{n}\,\ell_0\,\mathbf{n}^{-1}
\end{aligned}$$

since versors are GP preserving and linear, therefore $\wedge$-preserving. Same formula it does the whole line!

Now apply a Euclidean rigid body versor A from CGA to move the situation to general position and orientation. Versors, so:

$$\begin{aligned}
& A\,(-\mathbf{n}\,\ell_0\,\mathbf{n}^{-1})\,A^{-1} \\
& \qquad = \ -(A\,\mathbf{n}\,A^{-1})\,(A\,\ell_0\,A^{-1})\,(A\,\mathbf{n}^{-1}\,A^{-1}) \\
& \qquad = \ -n\,\ell\,n^{-1}
\end{aligned}$$

where ℓ and n are A-transformed line and dual plane. Same formula does *any* situation of lines and planes. Also, orientation and position done in one go!

`DEMOreflect()`.

20 Universal operations: exponentials of bivectors

The algebra takes care of the versor properties. You know from 3D the rotors:

$$\begin{aligned}\exp(\mathbf{e}_1 \wedge \mathbf{e}_2\, \phi) &= 1 + \phi\, (\mathbf{e}_1 \wedge \mathbf{e}_2) + \tfrac{1}{2!}\phi^2 (\mathbf{e}_1 \wedge \mathbf{e}_2)^2 + \cdots \\ &= (1 - \tfrac{1}{2!}\phi^2 + \cdots) + (\phi - \tfrac{1}{3!}\phi^3 + \cdots)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\ &= \cos\phi + \sin\phi\, \mathbf{e}_1 \wedge \mathbf{e}_2\end{aligned}$$

Because $(\mathbf{e}_1 \wedge \mathbf{e}_2)^2 = -1$, we get trigonometry.

In CGA we also have translators. Null vector ∞ is used:

$$\exp(\mathbf{t} \wedge \infty) = 1 + (\mathbf{t} \wedge \infty) + \tfrac{1}{2!}(\mathbf{t} \wedge \infty)^2 + \cdots = 1 + (\mathbf{t} \wedge \infty)$$

Because $(\mathbf{t} \wedge \infty)^2 = 0$, we get additive properties. Composition:

$$\begin{aligned}\exp(\mathbf{t}_2 \wedge \infty)\ \exp(\mathbf{t}_1 \wedge \infty) &= (1 + \mathbf{t}_2 \wedge \infty)\,(1 + \mathbf{t}_1 \wedge \infty) \\ &= 1 + \mathbf{t}_2 \wedge \infty + \mathbf{t}_1 \wedge \infty + 0 \\ &= 1 + (\mathbf{t}_2 + \mathbf{t}_1) \wedge \infty \\ &= \exp((\mathbf{t}_2 + \mathbf{t}_1) \wedge \infty)\end{aligned}$$

And of course we can translate o to any point:

$$\exp(-\mathbf{p} \wedge \infty/2)\, o\ \exp(\mathbf{p} \wedge \infty/2) = o + \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2\, \infty$$

And there are more such versors, for scaling, transversion, loxodromes and other conformal transformations. E.g. scaling:

$$\exp(\gamma o \wedge \infty) = \cosh\gamma + \sinh\gamma\, o \wedge \infty$$

21 Visualizing versor action

We choose to visualize a versor by its 'trail' after repeated action on an object.

```
dynamic{ vtrail(V,A,30),}
V = exp(- e1^ni/2);
A = no,
```

Rotation is exponent of dual line:

```
x = no, y = no,
dynamic{L: line = x^y^ni,};
dynamic{V: V    = exp(dual(line)/10);};
```

To make a screw, compose it with a translation:

```
dynamic{V: V = exp(-e1^ni/10) exp(dual(line)/10);};
```

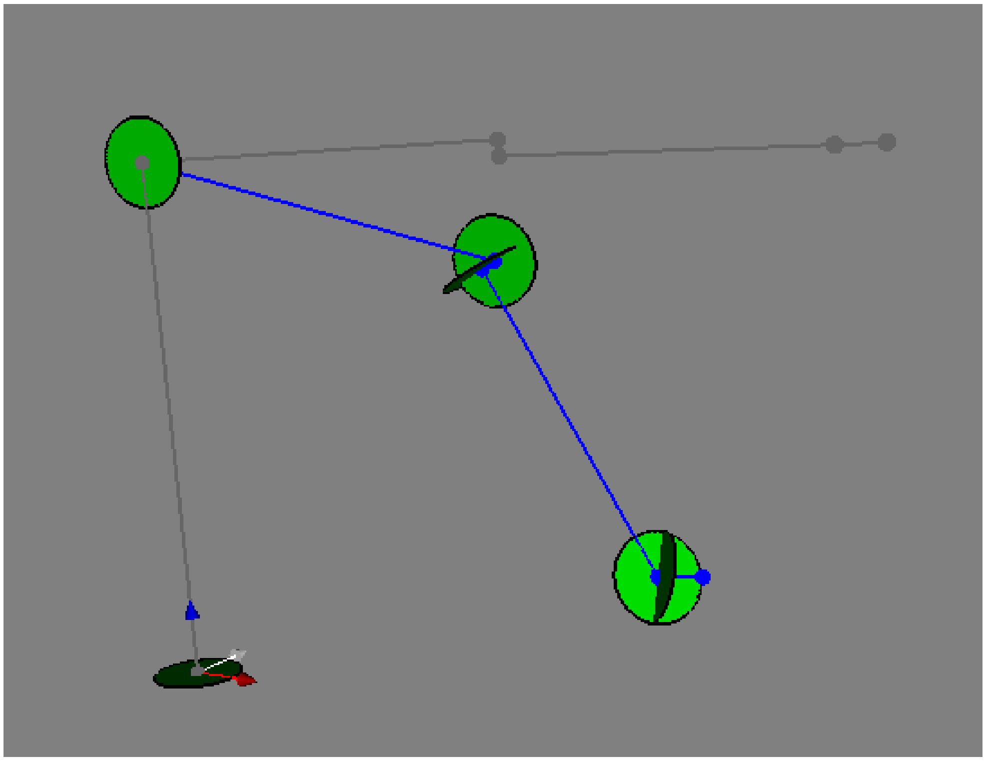
Or couple a rotation and a translation:

```
dynamic{V: V = exp(-(y-x)^ni/10) exp(dual(line)/10);};
```

At all times, you can change A, for all objects transform the same: planes, spheres, dual planes, circles, etc.

22 A robot

Use these elements to represent and animate a PUMA robot.



Run `DEMOpuma_init()`, `dynamic{puma(atime);}`, animate.

23 Rigid body motions as ratios of flats

A translation versor is a ratio of two flat points:

```
p1 = no, label(p1);
p2 = no, label(p2);
dynamic{o1: o1 = p1^ni,};
dynamic{o2: o2 = p2^ni,};
dynamic{V: V = o2/o1;};
dynamic{vtrail(V,A),}
```

A rotation versor is a ratio of two (dual) planes:

```
n1 = (e3ga) e1, label(n1);
n2 = (e3ga) e2, label(n2);
dynamic{o1: o1 = p1.(n1^ni),};
dynamic{o2: o2 = p2.(n2^ni),};
```

General rigid body motions (screws) are ratios of lines:

```
dynamic{o1: o1 = p1^n1^ni,}
dynamic{o2: o2 = p2^n2^ni,}
```

Can find more general transformations in this way.

24 Why versors are essential

A proper geometrical system has a nice transformation property: any object made by any construction should transform *covariantly* under the symmetries A of the system:

$$X \circ Y \quad \mapsto \quad A(X \circ Y) = A(X) \circ A(Y)$$

Recipe: If you have a geometry characterized by symmetries, find a GA model in which those symmetry operations are represented as *versors*. Base all operations on (sums of) geometric products. This gives *automatically* a structure-preserving geometrical system.

$$X \circ Y = X\,Y \quad \mapsto \quad (AXA^{-1})\,(AYA^{-1}) = A(X\,Y)A^{-1}$$

(and linear). So all permissible constructions are covariant. You can't go wrong. This is extremely convenient in programming.

Using versors, objects and their relationships transform identically – so the geometric relationship is preserved.

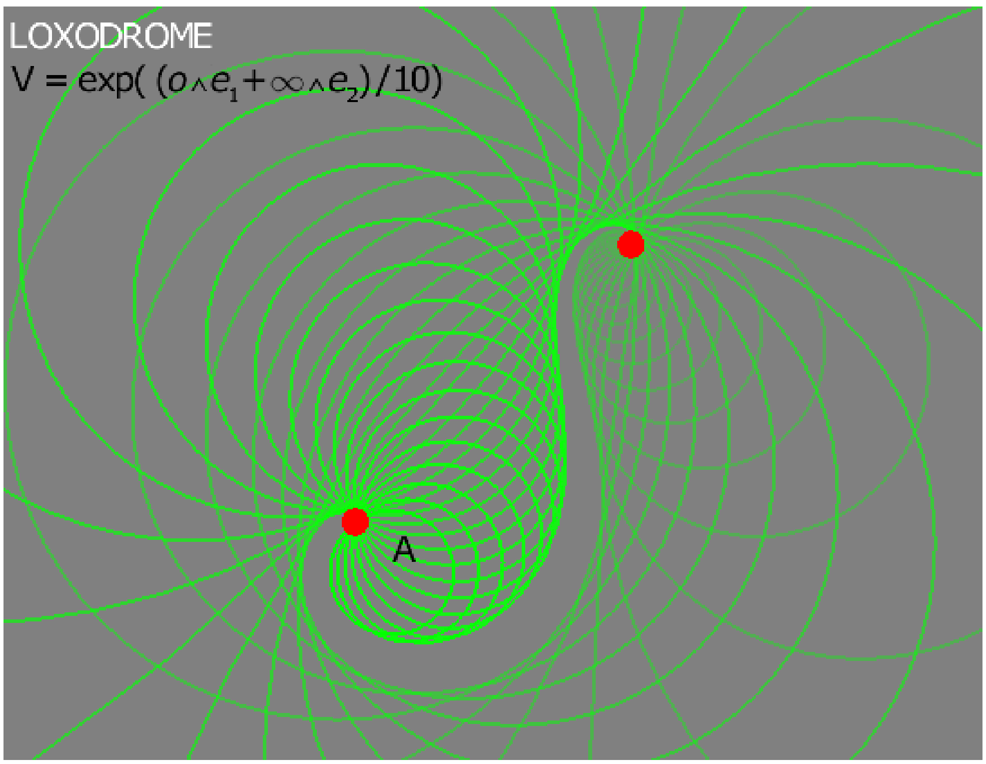
25 Non-Euclidean versors

The conformal model is overkill for Euclidean operations, but it is the smallest algebra in which their symmetries become versors.

We get other versors too, in fact, all conformal transformations. Here's a sample, the *loxodromic* transformation generated by:

$$V = \exp(o \wedge \mathbf{e}_1 + \infty \wedge \mathbf{e}_2)$$

It is nice that all these are now available in n-D.



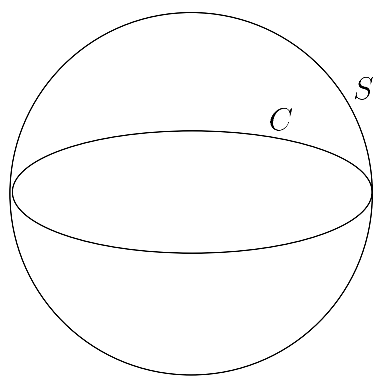
26 Euclidean Projection: unusual

Set up a projector and play with it. Note how non-flat points project.

```
dynamic{PX = (X.P)/P,};
P = dual(no-ni/2),
X = (no+ni/10)^e2^e3,
dynamic{AX = alpha( dual(P)^X, 0.2)}; // has X, hits P straight
```

Plane on a sphere gives a sticking out dual sphere, which indeed hits P straight and therefore is 'in' P.

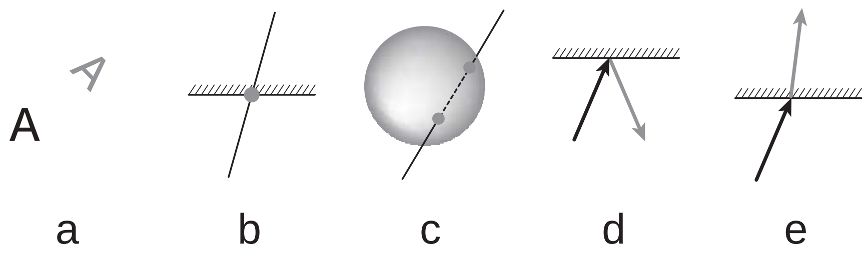
Q: Specify the sphere S containing a circle C as equator.

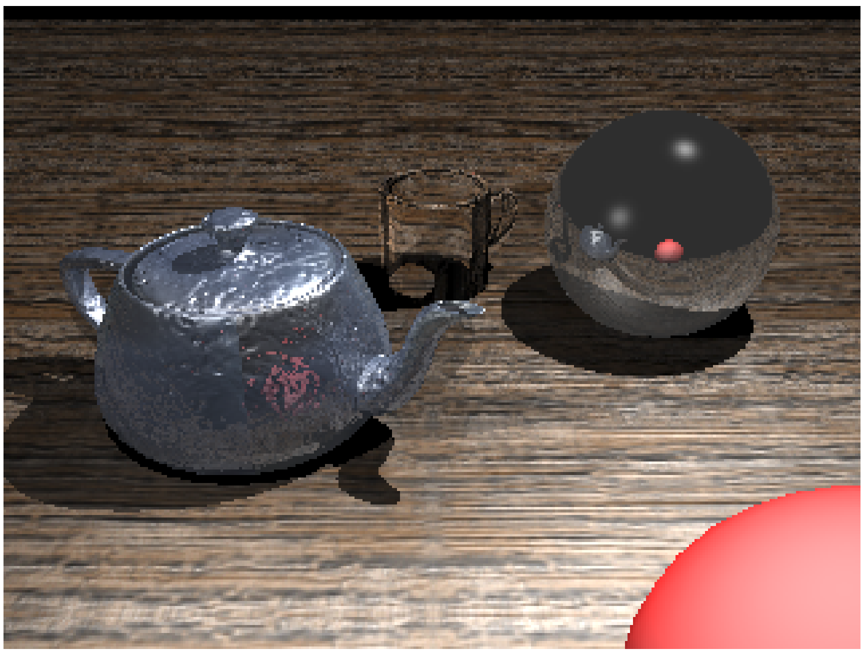


A: It is $S^* = (\,(\infty \wedge C)/C$, rejection of ∞ from C.

27 But, is it worth it? A 2001 ray tracer application

Basic operations:





model	features	rendering
3D LA	vectors and matrices only	1.0 ×
3D GA	quaternions, subspaces direct	2.5 ×
4D LA	homogeneous pts, lines ad hoc	1.0 ×
4D GA	affine subspaces, univ.lin.map	3.0 ×
5D GA	spheres and circles, RBMs as versors	5.7 ×

28 Conclusion

- In CGA, we have a consistent and compact framework for Euclidean geometry.
- The equations strongly suggest a specification/construction language – its nouns, its verbs.
- It should help to adapt our thinking to these new primitives and relations.
- Playing with software helps build this intuition.
- For uncorrupted students, it is easy and natural.
- But even we can still learn this...

Tutorials and GAviewersoftware for the DEMOs will appear on:

`http://www.science.uva.nl/ga/`