Collision avoidance, wave propagation and boundary representations

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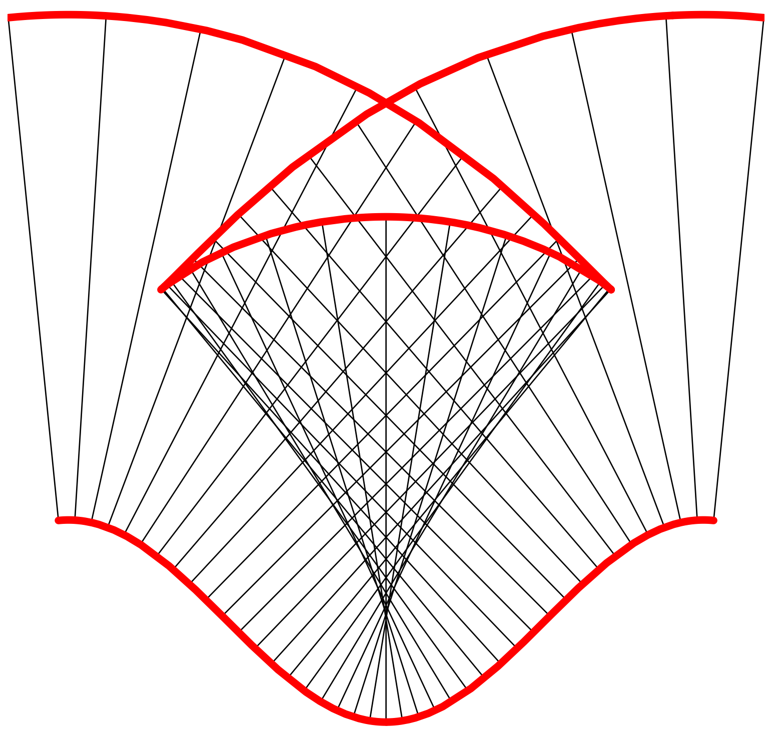
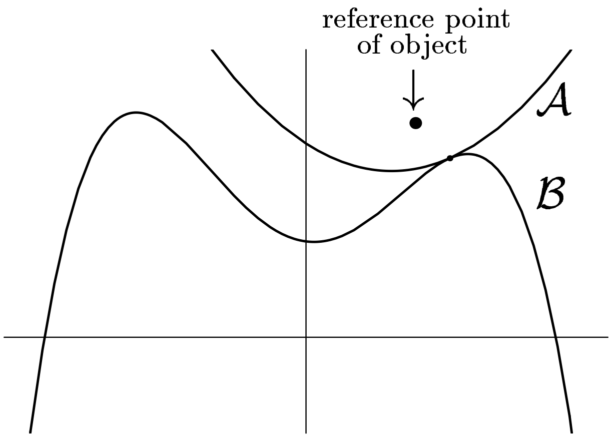
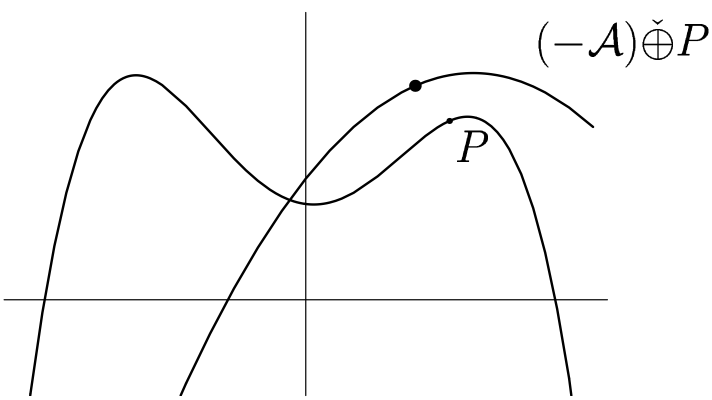


1 Objects in contact

When an object touches another object, this is a limit on its motion. What is the *freespace* boundary, analytically?



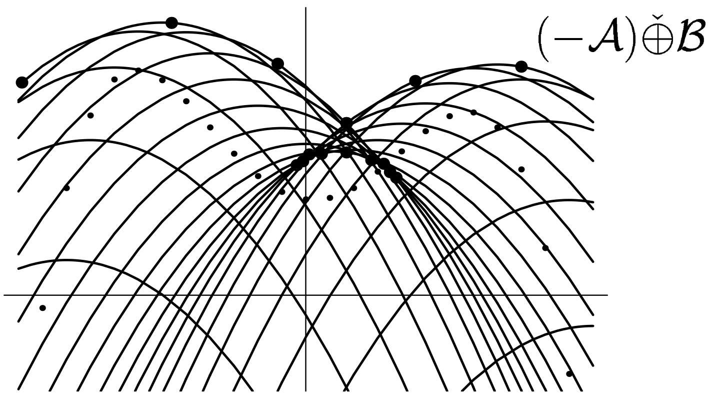
Let us limit ourselves to *translations* only. We get a sketch by determining for each point P of $\mathcal{B}$ which part of space it 'denies' to the reference point of $\mathcal{A}$:



This involves placing the 'stamp' $-\mathcal{A}$ at P, producing a boundary of forbidden area $(-\mathcal{A})\check{\oplus}P$.

2 Collision avoidance and wave propagation

Performing this construction for all points of $\mathcal{B}$ gives:

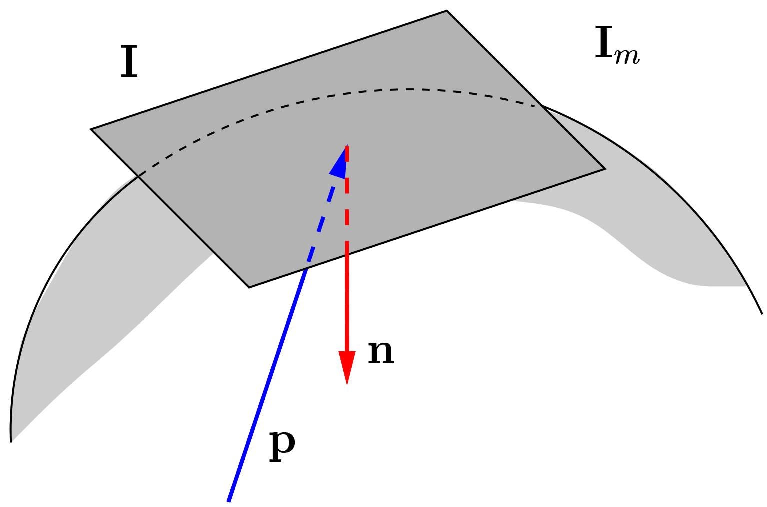


This is reminiscent of *Huygens wave propagation*, with $\mathcal{B}$ as a front of sources, and $-\mathcal{A}$ as propagator; the new wave front is $(-\mathcal{A})\check{\oplus}\mathcal{B}$.

Note that this is a process on *boundaries*, producing a new boundary $(-\mathcal{A})\check{\oplus}\mathcal{B}$ from two boundaries $\mathcal{A}$ and $\mathcal{B}$. Note also that the resulting boundary is not necessarily differentiable (if we take 'outside only') or single-valued (if we take the 'wave front').

We thus have a need for an *analytical description of such boundaries* and their combination.

3 Oriented tangent space

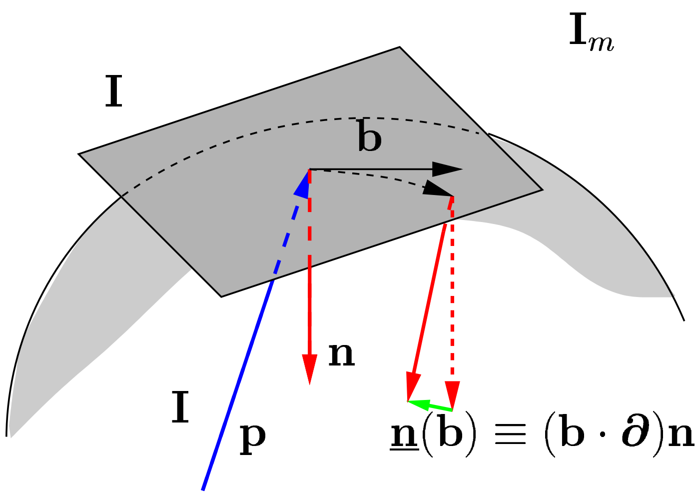


Assume regular boundary in Euclidean m-space with pseudoscalar $\mathbf{I}_m$: at every point $\mathbf{p}$ a tangent space $\mathbf{I}$ of grade $m-1$. Associate a normal vector $\mathbf{n}$ by:

$$\mathbf{n} = -\mathbf{I}\mathbf{I}_m^{-1}.$$

Orient $\mathbf{I}$ such that $\mathbf{n}$ is the *inward pointing normal.* $\mathbf{I}$ characterizes the tangent space at $\mathbf{p}$, which we denote by $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$.

4 Spanning the oriented tangent space



Second order differential structure: differentiate $\mathbf{n}$ to some direction $\mathbf{b}$, defining

$$\underline{\mathbf{n}}(\mathbf{b}) \equiv (\mathbf{b} \cdot \boldsymbol{\partial})\mathbf{n}.$$

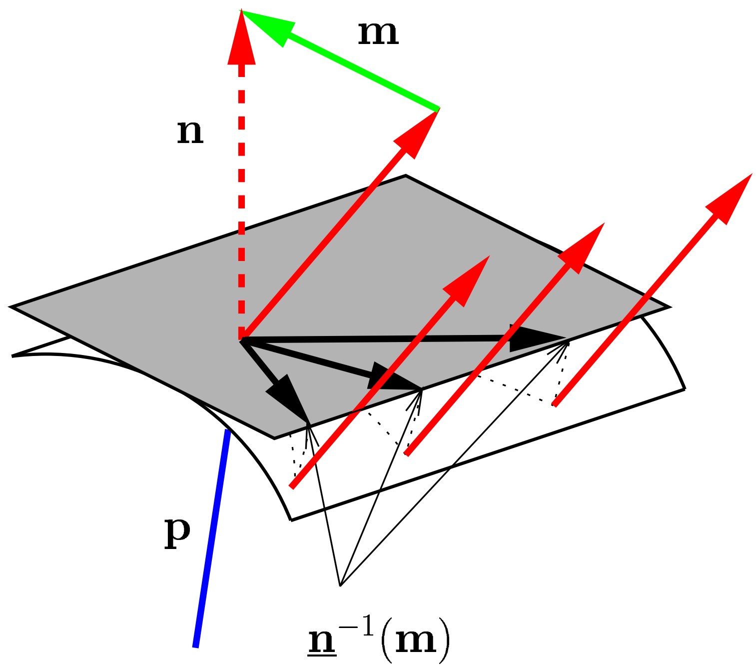
Since $\underline{\mathbf{n}}(\cdot)$ is linear, we can extend it as an outermorphism to all of $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$. Then $\underline{\mathbf{n}}(\mathbf{I})$ denotes the total change of $\mathbf{n}$ on $\mathbf{I}$.

We define the *directed Gaussian curvature* as:

$$\boldsymbol{\kappa} \equiv \underline{\mathbf{n}}(\mathbf{I})\hat{\mathbf{I}}^{-1} \quad \text{at } \mathbf{p}.$$

This relates the spaces with pseudoscalars $\mathbf{I}[\mathbf{p}]$ and $\underline{\mathbf{n}}(\mathbf{I})[\mathbf{p}]$.

5 Inversion of derivative



The function $\underline{\mathbf{n}}(\cdot)$ is invertible to a *set* of vectors based at $\mathbf{p}$:

$$\underline{\mathbf{n}}^{-1} : \mathcal{G}^1(\mathbf{I}[\mathbf{p}]) \to \mathcal{G}^1(\mathbf{I}[\mathbf{p}]) : \quad \underline{\mathbf{n}}^{-1}(\mathbf{m}) \equiv \{\mathbf{a} \mid \underline{\mathbf{n}}(\mathbf{a}) = \mathbf{m}\}.$$

Such set-valued functions should be added using the *Minkowski sum* $\oplus$:

$$\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

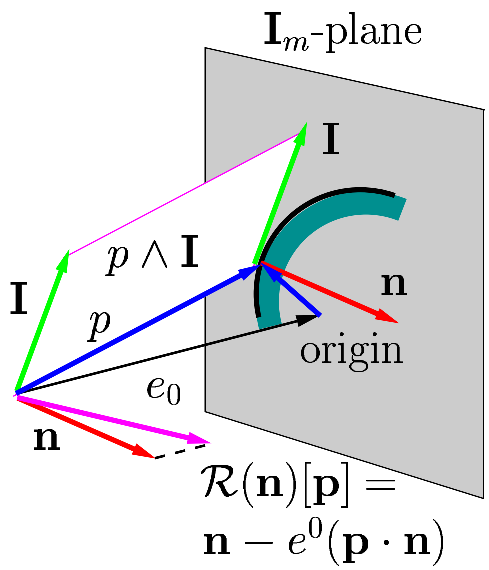
6 Boundary as geometric object

Boundary thus characterized by $\mathbf{p}$ (whose differentials give $\mathbf{I}$ or $\mathbf{n}$), and an 'inside sign' to orient $\mathbf{I}$ or $\mathbf{n}$. Not a *single* geometric object!

Familiar technique: *homogeneous embedding* of E^m into $(m+1)$-dimensional space with pseudoscalar $I_{m+1} \equiv e_0 \mathbf{I}_m$, according to:

$$p = e_0 + \mathbf{p}$$

(**bold** for elements in $\mathcal{G}(\mathbf{I}_m)$, *math* for elements of $\mathcal{G}(I_{m+1})$.)



Then the tangent $\mathbf{I}$ at $\mathbf{p}$ is represented by the homogeneous blade:

$$p \wedge \mathbf{I} = (e_0 + \mathbf{p}) \wedge \mathbf{I}.$$

Its dual is our representation $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ of the normal vector $\mathbf{n}$ at $\mathbf{p}$:

$$\boxed{\mathcal{R}(\mathbf{n})[\mathbf{p}] \equiv (p \wedge \mathbf{I}) I_{m+1}^{-1} = p \cdot (e^0 \mathbf{n}) = \mathbf{n} - e^0(\mathbf{p} \cdot \mathbf{n}).}$$

where e^0 is the reciprocal of e_0, so $e^0 \cdot e_0 = 1$.

7 Invertibility of boundary representation

The boundary representation commutes with differentiation:

$$\underline{\mathcal{R}}(\mathbf{a}) \equiv (\mathbf{a}\cdot\boldsymbol{\partial}_{\mathbf{n}})\mathcal{R}(\mathbf{n}) = p\cdot(e^0\mathbf{a}) = \mathcal{R}(\mathbf{a}) = \mathcal{R}((\mathbf{a}\cdot\boldsymbol{\partial}_{\mathbf{n}})\mathbf{n}).$$

By linearity we can extend it to *any* multivector in the 'differential space at $\mathbf{p}$' $\mathcal{G}(\mathbf{n}\wedge\underline{\mathbf{n}}(\mathbf{I}))[\mathbf{p}]$:

$$\mathcal{R}(\mathbf{A})[\mathbf{p}] = \mathbf{A} - e^0(\mathbf{p}\cdot\mathbf{A}) = p\cdot(e^0\mathbf{A}), \quad \mathbf{A}\in\mathcal{G}(\mathbf{n}\wedge\underline{\mathbf{n}}(\mathbf{I})[\mathbf{p}].$$

It satisfies:

$$p\cdot\mathcal{R}(\mathbf{A}) = p\cdot\big(p\cdot(e^0\mathbf{A})\big) = (p\wedge p)\cdot(e^0\mathbf{A}) = 0,$$

so p is perpendicular to the representation of any vector in the representation of the differential space. If $\boldsymbol{\kappa}\neq 0$, this space is m-dimensional.

From these m conditions we can retrieve p, and hence $\mathbf{p}$. These representative vectors can be constructed by differentiation of $\mathcal{R}(\mathbf{n})$, so *the boundary representation is invertible* (if $\boldsymbol{\kappa}\neq 0$):

$$\boxed{p = \mathcal{R}(\mathbf{I}_m)\mathbf{I}_m^{-1}e_0,}$$

which is simply the dual in $\mathcal{G}(\mathbf{I}_{m+1})$.

8 Example: spherical boundaries

Sphere (radius ρ, center $\mathbf{c}$) defined by implicit scalar function:

$$\phi_{\mathbf{p}} = (\mathbf{p} - \mathbf{c})^2 - \rho^2 = 0.$$

Then $\mathbf{n}$ is computed as:

$$\mathbf{n} = \pm \frac{\boldsymbol{\partial}_{\mathbf{p}} \phi(\mathbf{p})}{|\boldsymbol{\partial}_{\mathbf{p}} \phi(\mathbf{p})|} = \pm \frac{\mathbf{p} - \mathbf{c}}{|\rho|}.$$

We orient the inward normal for a hole or a blob; this can be achieved by a sign for ρ: *positive is a blob, negative is a hole*:

$$\mathbf{n} = \frac{\mathbf{c} - \mathbf{p}}{\rho}$$

So $\mathbf{p}[\mathbf{n}] = \mathbf{c} - \rho\, \mathbf{n}$. The representation is now:

$$\boxed{\mathcal{R}(\mathbf{n}) = \mathbf{n} - e^0(\mathbf{c} \cdot \mathbf{n} - \rho)}$$

This satisfies $(e_0 + \mathbf{c}) \cdot \mathcal{R}(\mathbf{n}) = \rho$, so resides in plane perpendicular to $(e_0 + \mathbf{c})$, at distance ρ (see figure on next slide).

Inversion: $\underline{\mathcal{R}}(\mathbf{m}) = \mathcal{R}(\mathbf{m}) = \mathbf{m} - e^0(\mathbf{c} \cdot \mathbf{m})$, so that $\mathcal{R}(\mathbf{I}_m) = \mathcal{R}(\mathbf{n}) \wedge \mathcal{R}(-\mathbf{I}) = \mathbf{I}_m - e^0(\mathbf{c} \cdot \mathbf{I}_m + \rho \mathbf{I})$. Its dual is:

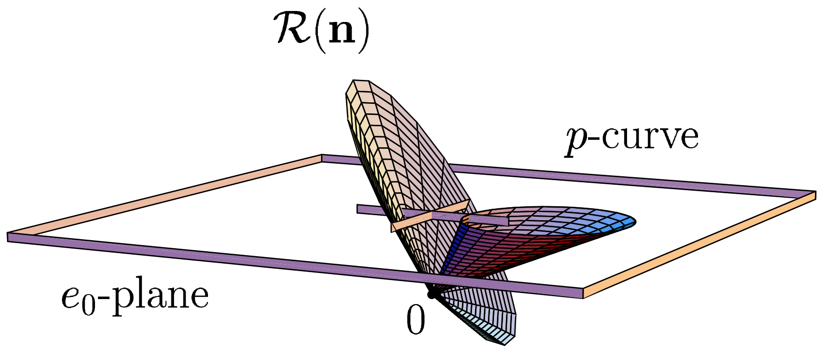
$$\mathcal{R}(\mathbf{I}_m)\mathbf{I}_m^{-1} e_0 = e_0 - e^0(\mathbf{c} \wedge 1)e_0 - \rho\, e^0 \mathbf{I}\mathbf{I}_m^{-1} e_0 = e_0 + \mathbf{c} - \rho\, \mathbf{n},$$

so this retrieves $\mathbf{p}[\mathbf{n}] = \mathbf{c} - \rho\, \mathbf{n}$.

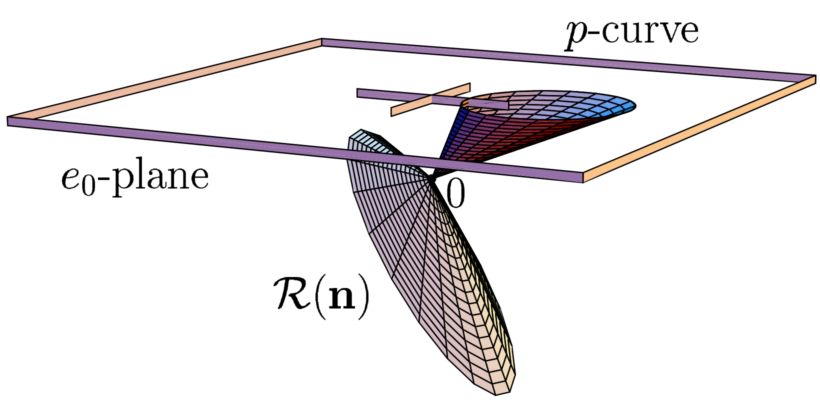
Curvature: note that $\underline{\mathbf{n}}(\mathbf{a}) = -\mathbf{a}/\rho$, so

$$\boldsymbol{\kappa} = \underline{\mathbf{n}}(\hat{\mathbf{I}})\mathbf{I}^{-1} = \frac{1}{\rho^{m-1}}.$$

9 The representation graphically



(a) *a circular hole*



(b) *a circular blob*

10 Towards an isometric representation

The representation $\mathcal{R}(\cdot)$ is not isometric:

$$\mathcal{R}(\mathbf{a}) \cdot \mathcal{R}(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + (e^0 \cdot e^0)(\mathbf{p} \cdot \mathbf{a})(\mathbf{p} \cdot \mathbf{b}) \neq \mathbf{a} \cdot \mathbf{b} = \mathcal{R}(\mathbf{a} \cdot \mathbf{b}),$$

unless e_0 is a null vector!

So, make e_0 and e^0 the null vectors on a null cone in the *Minkowski space* $\mathcal{C}\ell_{m+1,1}$. Define the pseudoscalar for this space:

$$I_{m+1,1} \equiv (e_0 \wedge e^0)\mathbf{I}_m \equiv E\mathbf{I}_m.$$

A vector $\mathbf{p}$ is represented as:

$$p' \equiv e_0 + \mathbf{p} - e^0\mathbf{p}^2/2,$$

and e_0 represents the point at the origin, and $-e^0$ the point at infinity. This is the *generalized homogeneous model of Eulidean space* recently proposed by [Li *et al.*].

Blades now represent *spheres* of E^m (due to definition of p'). A *flat* in E^m is represented as a *blade containing* e^0 (i.e. a sphere through infinity).

11 Isometric representation

The tangent $\mathbf{I}$ at $\mathbf{p}$ is represented as:

$$e^0 \wedge p' \wedge \mathbf{I}.$$

Its dual is our *new boundary representation* $\mathcal{R}'(\mathbf{n})[\mathbf{p}]$:

$$\boxed{\mathcal{R}'(\mathbf{n})[\mathbf{p}] \equiv (e^0 \wedge p' \wedge \mathbf{I})\mathbf{I}_m^{-1}E = p \cdot (e^0\mathbf{n}) = \mathcal{R}(\mathbf{n})[\mathbf{p}],}$$

so it is numerically the same as the previous, but algebraically much nicer, for it preserves the geometric product:

$$\mathcal{R}'(\mathbf{a})\mathcal{R}'(\mathbf{b}) = \mathcal{R}'(\mathbf{ab}).$$

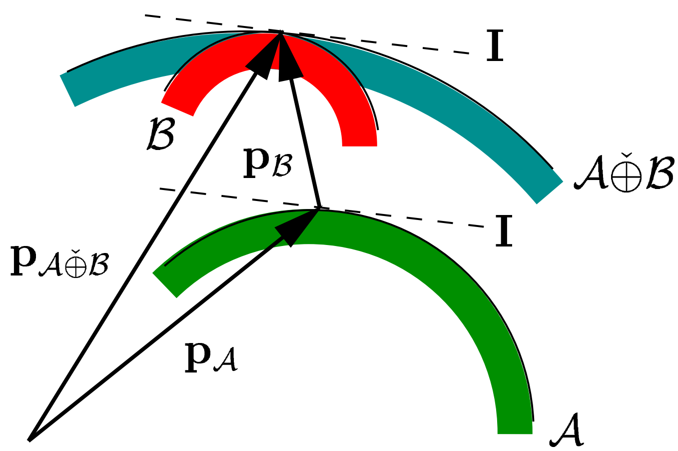
It is of course as invertible as $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ was, now through the dual in $\mathcal{G}(E\mathbf{I}_m)$:

$$\boxed{e^0 \wedge p = -\mathcal{R}'(\mathbf{I}_m)\mathbf{I}_m^{-1}E,}$$

which yields $\mathbf{p}$.

12 Boundary propagation

Propagation combines two boundaries $\mathcal{A}$ and $\mathcal{B}$ to produce a boundary $\mathcal{A}\check{\oplus}\mathcal{B}$ according to the following rules (which can be taken as the definition of propagation, or derived from basic principles).



- The resulting position vector after combining a point $\mathbf{p}_\mathcal{A}$ on $\mathcal{A}$ and a point $\mathbf{p}_\mathcal{B}$ on $\mathcal{B}$ is the position $\mathbf{p}_\mathcal{A} + \mathbf{p}_\mathcal{B}$:

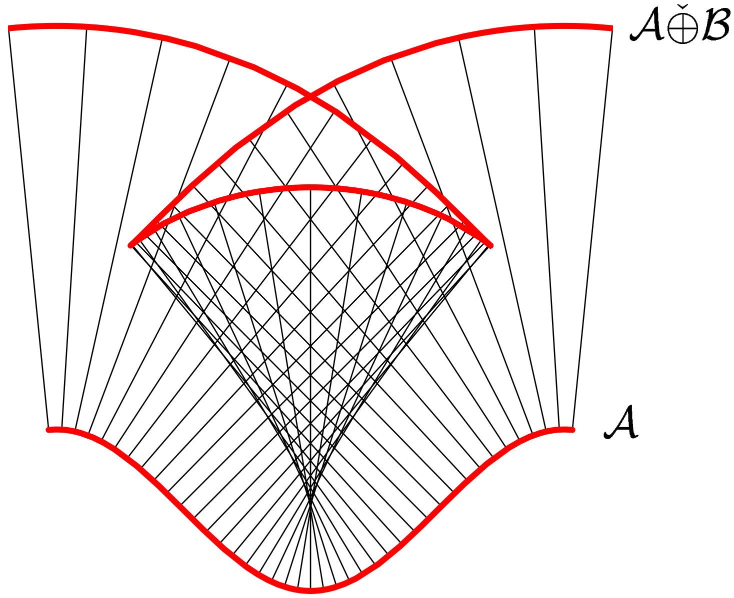
$$\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}} = \mathbf{p}_\mathcal{A} + \mathbf{p}_\mathcal{B}$$

- The points $\mathbf{p}_\mathcal{A}$ and $\mathbf{p}_\mathcal{B}$ *must* have the same inward pointing normal (to $\mathcal{A}$ and $\mathcal{B}$, respectively), and this is also the inward pointing normal at the resulting position in the resulting boundary. Symbolically:

$$\mathbf{n}_{\mathcal{A}\check{\oplus}\mathcal{B}}[\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}] = \mathbf{n}_\mathcal{A}[\mathbf{p}_\mathcal{A}] = \mathbf{n}_\mathcal{B}[\mathbf{p}_\mathcal{B}].$$

13 Swallowtails

Propagation of a concavity $\mathcal{A}$ by a large enough sphere $\mathcal{B}$ leads to swallowtails:



At the 'cusps', what is the nature of the irregularity? First order? Second order? Zero 'velocity' along boundary?

14 Propagation in the embedded representation

Our boundary representation $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ provides direct implementation of the definition of propagation:

Let $\mathbf{p}_\mathcal{A}[\mathbf{n}]$ *be defined as:*

$$\mathbf{p}_\mathcal{A}[\mathbf{n}] = \{\mathbf{x} \in \mathcal{A} \mid \mathbf{n}_\mathcal{A}[\mathbf{x}] = \mathbf{n}\},$$

(set-valued!) and similarly for $\mathbf{p}_\mathcal{B}[\cdot]$.
Then the propagation result of two boundaries

$$\mathcal{R}(\mathbf{n})[\mathbf{p}_\mathcal{A}] = \mathbf{n} - e^0(\mathbf{p}_\mathcal{A}[\mathbf{n}] \cdot \mathbf{n})$$

and

$$\mathcal{R}(\mathbf{n})[\mathbf{p}_\mathcal{B}] = \mathbf{n} - e^0(\mathbf{p}_\mathcal{B}[\mathbf{n}] \cdot \mathbf{n})$$

is

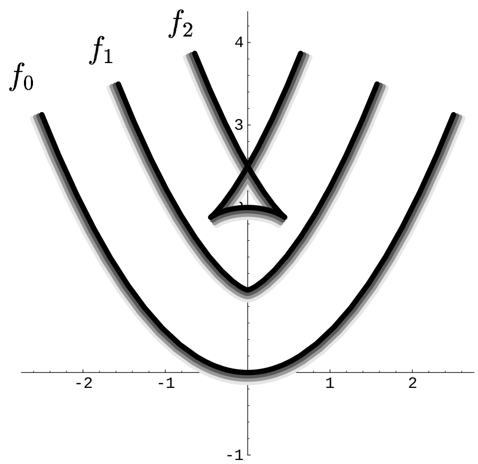
$$\begin{aligned}
\mathcal{R}(\mathbf{n})[\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}] &= \mathbf{n} - e^0\left(\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}} \cdot \mathbf{n}\right) \\
&= \mathbf{n} - e^0\left((\mathbf{p}_\mathcal{A}[\mathbf{n}] \oplus \mathbf{p}_\mathcal{B}[\mathbf{n}]) \cdot \mathbf{n}\right) \\
&= \mathbf{n} - e^0\left((\mathbf{p}_\mathcal{A}[\mathbf{n}] \cdot \mathbf{n}) \oplus (\mathbf{p}_\mathcal{B}[\mathbf{n}] \cdot \mathbf{n})\right) \\
&= \mathcal{R}(\mathbf{n})[\mathbf{p}_\mathcal{A}] + \mathcal{R}(\mathbf{n})[\mathbf{p}_\mathcal{B}] - \mathbf{n}.
\end{aligned}$$

So basically, *the* e^0 *components add up* (we use the $\oplus$ since there may be several values of $\mathbf{p}[\mathbf{n}]$ for a given $\mathbf{n}$ in each boundary, if the boundaries are non-convex).

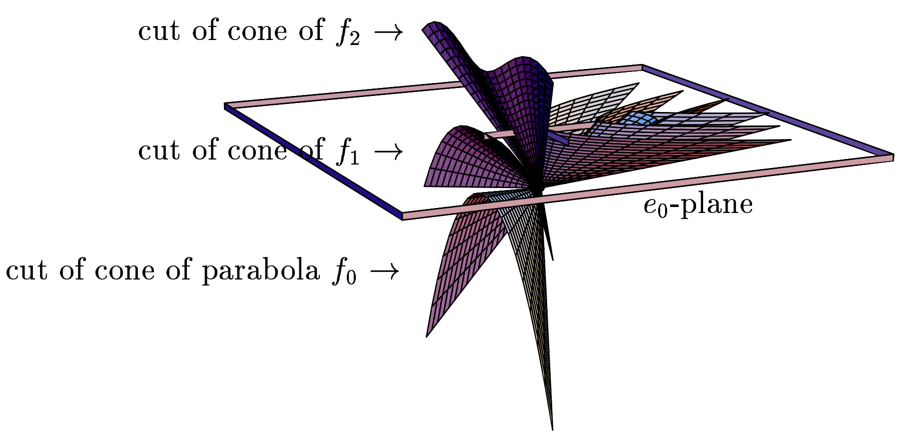
Note: *requires reparametrization to same* $\mathbf{n}$*!*

15 Example: circular propagation of parabola

In the spatial domain, a 'swallowtail' develops in the wave front:

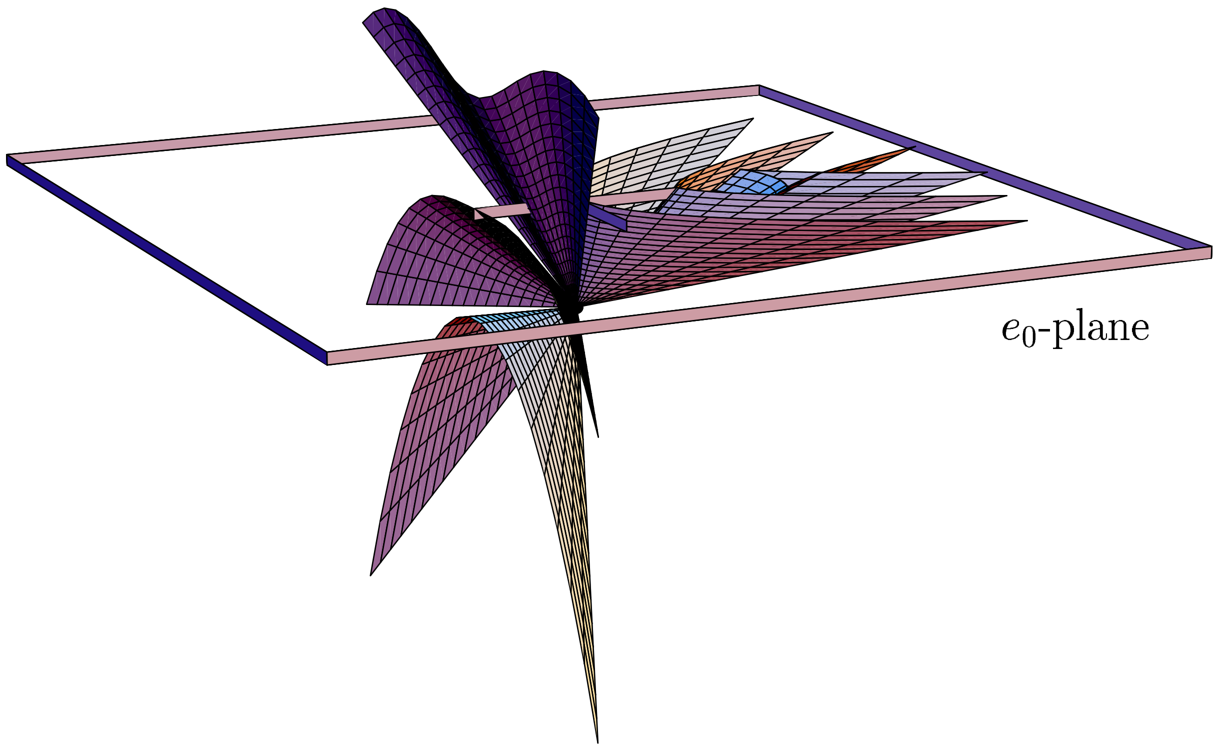


Propagation by a circle is equivalent to raising of the e^0-component by ρ, since for a circle $\mathcal{R}(\mathbf{n}) = \mathbf{n} + \rho\, e^0$.



(here the cone is shown in a cut by the plane $n_2 = -1$, where it is the *Legendre transform*).

16 Close-up: ‘irregularities’ regularized



Note that the ‘swallowtail’ of the boundary corresponds to a *concavity* in $\mathcal{R}(\mathbf{n})$: the representation is well-behaved!

The ‘cusps’ are represented by *inflection points*: second order sign changes, no discontinuities!

17 Analysis of propagation

We can derive a differential property of the propagation operation:

The propagated boundary $\mathcal{C} = \mathcal{A}\check{\oplus}\mathcal{B}$ *obeys the 'velocity law':*

$$\underline{\mathbf{n}}_{\mathcal{C}}^{-1}(\mathbf{m}) = \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{m}) \oplus \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{m}),$$

where the quantities are to be evaluated at the corresponing points of $\mathcal{A}$*,* $\mathcal{B}$ *and* $\mathcal{A}\check{\oplus}\mathcal{B}$*. The result is* $\emptyset$ *for* $\mathbf{m}$ not *in the common range of* $\underline{\mathbf{n}}_{\mathcal{A}}[\mathbf{p}_{\mathcal{A}}](\cdot)$ *and* $\underline{\mathbf{n}}_{\mathcal{B}}[\mathbf{p}_{\mathcal{B}}](\cdot)$*.*

Proof: Introduce three tangent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, to measure the derivative on each of the surfaces, and use the chain rule to rewrite them in terms of derivatives of $\mathbf{p}[\mathbf{n}]$:

$$\mathbf{a} = \mathbf{P}_{\mathcal{A}}(\mathbf{a}) = (\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{A}} = (\underline{\mathbf{n}}_{\mathcal{A}}(\mathbf{a}) \cdot \boldsymbol{\partial}_{\mathbf{n}})\mathbf{p}_{\mathcal{A}}[\mathbf{n}]$$

$$\mathbf{b} = \mathbf{P}_{\mathcal{B}}(\mathbf{b}) = (\mathbf{b} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{B}} = (\underline{\mathbf{n}}_{\mathcal{B}}(\mathbf{b}) \cdot \boldsymbol{\partial}_{\mathbf{n}})\mathbf{p}_{\mathcal{B}}[\mathbf{n}]$$

$$\mathbf{c} = \mathbf{P}_{\mathcal{C}}(\mathbf{c}) = (\mathbf{c} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{C}} = (\underline{\mathbf{n}}_{\mathcal{C}}(\mathbf{c}) \cdot \boldsymbol{\partial}_{\mathbf{n}})\mathbf{p}_{\mathcal{C}}[\mathbf{n}]$$

Now select these such that $\underline{\mathbf{n}}_{\mathcal{A}}(\mathbf{a}) = \underline{\mathbf{n}}_{\mathcal{B}}(\mathbf{b}) = \underline{\mathbf{n}}_{\mathcal{C}}(\mathbf{c}) = \mathbf{m}$:

$$\mathbf{a} \in \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{m}), \quad \mathbf{b} \in \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{m}), \quad \mathbf{c} \in \underline{\mathbf{n}}_{\mathcal{C}}^{-1}(\mathbf{m}).$$

Then these tangents add as position vectors:

$$\mathbf{c} = (\mathbf{m} \cdot \boldsymbol{\partial}_{\mathbf{n}})\mathbf{p}_{\mathcal{C}}[\mathbf{n}] = (\mathbf{m} \cdot \boldsymbol{\partial}_{\mathbf{n}})\left(\mathbf{p}_{\mathcal{A}}[\mathbf{n}] + \mathbf{p}_{\mathcal{B}}[\mathbf{n}]\right) = \mathbf{a} + \mathbf{b},$$

and, over all possibilities of choosing $\mathbf{a}$ and $\mathbf{b}$ given $\mathbf{c}$:

$$\underline{\mathbf{n}}_{\mathcal{C}}^{-1}(\mathbf{m}) = \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{m}) \oplus \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{m}).$$

□

18 Directed Gaussian radii of curvature are additive

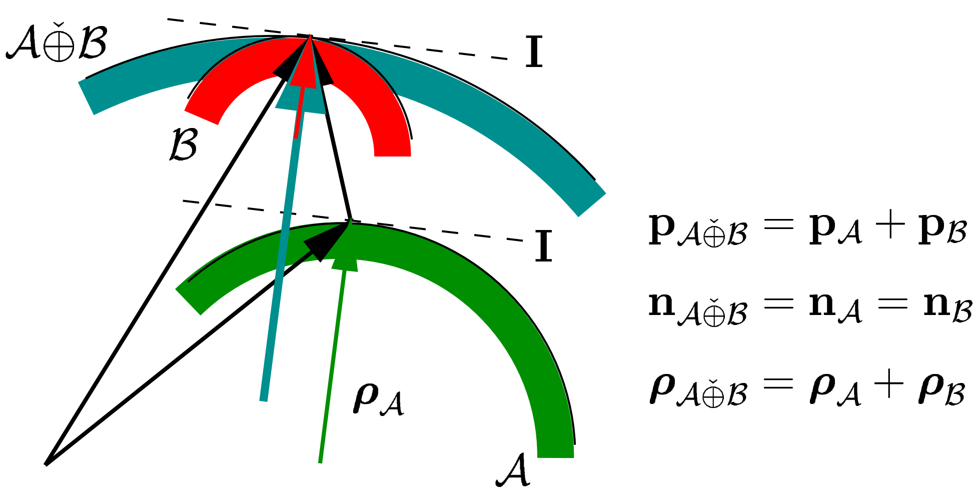
The interaction of the local differential geometries can produce involved results, especially for surfaces with torsion. Yet there is an simple property when we 'lump' over all tangent directions at $\mathbf{p}$:

In a propagation operation, Gaussian curvatures add reciprocally:

$$\boldsymbol{\kappa}^{-1}_{\mathcal{A}\check{\oplus}\mathcal{B}} = \boldsymbol{\kappa}^{-1}_{\mathcal{A}} + \boldsymbol{\kappa}^{-1}_{\mathcal{B}}$$

(locally, at every triple of corresponding points).

Proof: We extend $\underline{\mathbf{n}}$ to an outermorphism over all of $\mathbf{I}$. We know that $\underline{\mathbf{n}}_{\mathcal{A}}(\mathbf{I}) = (-1)^{m-1}\boldsymbol{\kappa}_{\mathcal{A}}\mathbf{I}$. Then $\underline{\mathbf{n}}^{-1}_{\mathcal{A}}(\mathbf{I}) = \overline{\mathbf{n}}_{\mathcal{A}}(\mathbf{I}^2)\mathbf{I}^{-1}/\det(\underline{\mathbf{n}}) = (-1)^{m-1}\mathbf{I}/\boldsymbol{\kappa}_{\mathcal{A}}$ (where $\overline{\mathbf{n}}_{\mathcal{A}}$ is the adjoint of $\mathbf{n}_{\mathcal{A}}$), and similarly for $\mathcal{B}$ and $\mathcal{A}\check{\oplus}\mathcal{B}$, and the result follows from the velocity law. □



19 Versor representation

The equation for the representation $\mathcal{R}(\mathbf{n})$ can be written in an interesting alternative form:

$$\begin{aligned}\mathcal{R}(\mathbf{n})[\mathbf{p}] &= \mathbf{n} - e^0(\mathbf{p}\cdot\mathbf{n}) = (1 - e^0\mathbf{p}/2)\,\mathbf{n}\,(1 + e^0\mathbf{p}/2) \\ &= e^{-e^0\mathbf{p}/2}\,\mathbf{n}\,e^{e^0\mathbf{p}/2}.\end{aligned}$$

Thus $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ can be constructed from a vector $\mathbf{n}$ via the general versor equation $\underline{U}(\mathbf{x}) = U\mathbf{x}\widehat{U}^{-1}$ using the *translational versor in* $\mathcal{C}\ell_{m+1,1}$:

$$T_{\mathbf{p}} \equiv e^{-e^0\mathbf{p}/2} = 1 - e^0\mathbf{p}/2,$$

which is a rotation over infinity.

For the representation p' of a *point at* $\mathbf{p}$ in $\mathcal{C}\ell_{m+1,1}$ [Li *et al.*] had:

$$p' = e_0 + \mathbf{p} - e^0\mathbf{p}^2/2 = T_{\mathbf{p}}\,e_0\,T_{\mathbf{p}}^{-1},$$

and now we find for a *boundary*:

$$\mathcal{R}(\mathbf{n})[\mathbf{p}] = T_{\mathbf{p}[\mathbf{n}]}\,\mathbf{n}\,T_{\mathbf{p}[\mathbf{n}]}^{-1}.$$

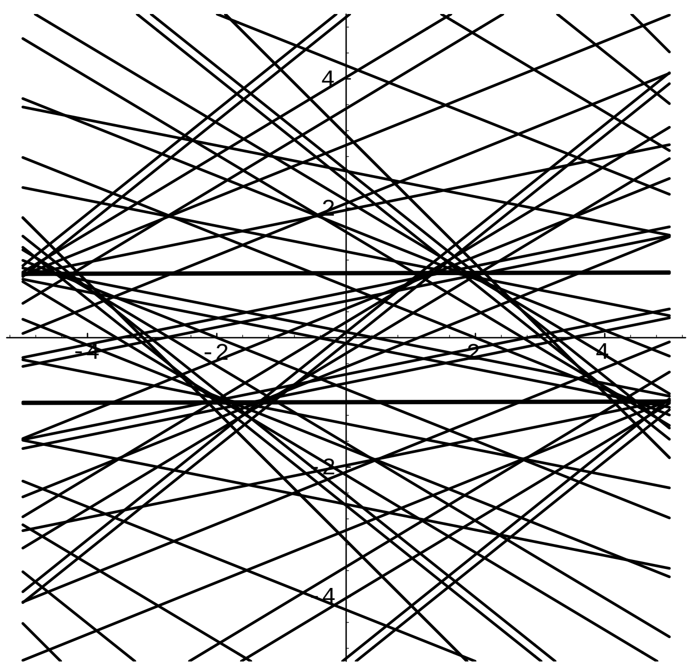
20 Object boundaries are versors

$$\boxed{\mathcal{R}(\mathbf{n})[\mathbf{p}] = T_{\mathbf{p}[\mathbf{n}]}\, \mathbf{n}\, T^{-1}_{\mathbf{p}[\mathbf{n}]}.}$$

So in this view, an object boundary (as represented by $\mathcal{R}(\mathbf{n})$) is an $\mathbf{n}$-dependent translation $\mathbf{p}[\mathbf{n}]$ applied to the unit normal vector $\mathbf{n}$. Since the latter is the representation of a *point object at the origin* as a (trivial) function of its orientation, this provides the view:

> *Any object boundary can be represented as a deformation by orientation-dependent translation of a point object at the origin.*

Non-convex objects may have a particular normal $\mathbf{n}$ at different points $\mathbf{p}$, so in general the function $\mathbf{p}[\mathbf{n}]$ is set-valued. Drawing this for $\mathbf{I}$ rather than $\mathbf{n}$ gives the boundary decomposed into *caustics*:



21 Boundary translation as versor

When the boundary translates over $\mathbf{t}$, the point $\mathbf{p}[\mathbf{n}]$ should become

$$\mathbf{p} \;\rightarrow\; \mathbf{p}+\mathbf{t}$$

and differentiation gives:

$$\mathbf{n} \;\rightarrow\; \mathbf{n}.$$

Both achieved by:

$$\begin{aligned} T_{\mathbf{p}'}\,\mathbf{n}'\,T_{\mathbf{p}'}^{-1} &= T_{\mathbf{p}+\mathbf{t}}\,\mathbf{n}\,T_{\mathbf{p}+\mathbf{t}}^{-1} \\ &= T_{\mathbf{t}}T_{\mathbf{p}}\,\mathbf{n}\,T_{\mathbf{p}}^{-1}T_{\mathbf{t}}^{-1} \\ &= (T_{\mathbf{t}}T_{\mathbf{p}})\;\mathbf{n}\;(T_{\mathbf{t}}T_{\mathbf{p}})^{-1} \end{aligned}$$

Thus the new versor for construction of the representation of the translated boundary is the boundary versor $T_{\mathbf{p}}$ pre-multiplied by $T_{\mathbf{t}}$.
Therefore:

$T_{\mathbf{t}}$ *represents boundary translation.*

22 Boundary rotation as versor

Rotation around $\mathbf{c}$ over rotor $\mathbf{R}$:

$$\mathbf{p} \;\rightarrow\; \mathbf{R}(\mathbf{p}-\mathbf{c})\mathbf{R}^{-1}+\mathbf{c},$$

and by differentiation

$$\mathbf{n} \;\rightarrow\; \mathbf{R}\mathbf{n}\mathbf{R}^{\mathbf{-1}}.$$

Both achieved by:

$$\begin{aligned} T_{\mathbf{p}'}\mathbf{n}'T_{\mathbf{p}'}^{-1} &= \left(T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}}T_{\mathbf{p}}\mathbf{R}^{\mathbf{-1}}\right)\mathbf{R}\mathbf{n}\mathbf{R}^{\mathbf{-1}}\left(\mathbf{T}_{\mathbf{c}}\mathbf{R}\mathbf{T}_{-\mathbf{c}}\mathbf{T}_{\mathbf{p}}\mathbf{R}^{\mathbf{-1}}\right)^{\mathbf{-1}} \\ &= (T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}})T_{\mathbf{p}}\mathbf{n}T_{\mathbf{p}}^{-1}(T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}})^{-1}, \end{aligned}$$

so the total result is the application of a new versor to $\mathbf{n}$ which is $T_{\mathbf{p}}$ left-multiplied by:

$$T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}} = \mathbf{R} - e^0(\mathbf{c}\cdot\mathbf{R}) \equiv R_{\mathbf{c},\mathbf{R}}.$$

Therefore:

$R_{\mathbf{c},\mathbf{R}}$ *represents boundary rotation.*

23 Propagation as (direction-dependent) versor

The versor of the wave propagation result $\mathcal{A}\check{\oplus}\mathcal{B}$ is the geometric product of the versors of $\mathcal{A}$ and $\mathcal{B}$ of wave front and propagator:

$$\boxed{T_{\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}[\mathbf{n}]} = T_{\mathbf{p}_{\mathcal{A}}[\mathbf{n}]}\, T_{\mathbf{p}_{\mathcal{B}}[\mathbf{n}]} = T_{\mathbf{p}_{\mathcal{B}}[\mathbf{n}]}\, T_{\mathbf{p}_{\mathcal{A}}[\mathbf{n}]}}$$

(where, for set-valued T, all combinations of products should be performed).

Proof: For single -valued $\mathbf{p}[\mathbf{n}]$, this is for each $\mathbf{n}$ a translation:

$$\begin{aligned} T_{\mathbf{p}_{\mathcal{B}}}\, T_{\mathbf{p}_{\mathcal{A}}} &= \big(1 - e^0\mathbf{p}_{\mathcal{B}}[\mathbf{n}]/2\big)\big(1 - e^0\mathbf{p}_{\mathcal{A}}[\mathbf{n}]/2\big) \\ &= \big(1 - e^0(\mathbf{p}_{\mathcal{B}}[\mathbf{n}] + \mathbf{p}_{\mathcal{A}}[\mathbf{n}])/2\big)\,. \end{aligned}$$

For set-valued $\mathbf{p}[\cdot]$s, we should combine all possibilities in the product, to produce:

$$1 - e^0(\mathbf{p}_{\mathcal{B}}[\mathbf{n}] \oplus \mathbf{p}_{\mathcal{A}}[\mathbf{n}])/2.$$

We include this in an overload of the geometric product notation for sets. □

Therefore:

$$T_{\mathbf{p}_{\mathcal{B}}[\mathbf{n}]} \textit{ represents propagation by } \mathcal{B}.$$

A boundary 'is' therefore a propagation operator!

The somewhat strange 'addition law' we found before ('add only the e^0-components, for the same $\mathbf{n}$') is just a disguised form of the geometric product of translational $\mathbf{n}$-dependent versors.

24 Versor representation of boundary operations

Many important operations (including wave propagation!) have a simple representation as a versor pre-multiplier.

boundary operation	**action on boundary versor**
identity	versor remains $T_{\mathbf{p}} = 1 - e^0\mathbf{p}/2$
translation over $\mathbf{t}$	left-multiply by $T_{\mathbf{t}} = 1 - e^0\mathbf{t}/2$
wave propagation by a boundary $T_{\mathbf{q}}$	left-multiply by $T_{\mathbf{q}[\mathbf{n}]}$ (same $\mathbf{n}$!)
rotation (center $\mathbf{c}$, spinor $\mathbf{R}$)	left-multiply by $R_{\mathbf{c},\mathbf{R}} = \mathbf{R} - e^0(\mathbf{c} \cdot \mathbf{R})$
mirror in hyperplane (support $\mathbf{d}$)	left-multiply by $M_{\mathbf{d}} = \mathbf{d} - e^0\mathbf{d}^2/2$
scaling by λ	replace by $(1 - e^0\mathbf{p}\lambda/2)$

Thus versors provide a framework to study the combination of such operations.

25 Rotations of boundaries

Taking central rotation, we find that a rotated boundary represented relative to the *original* $\mathbf{n}$ has versor:

$$1 - e^0 \mathbf{R}\mathbf{p}[\mathbf{R}^{-1}\mathbf{n}\mathbf{R}]\mathbf{R}^{-1}/\mathbf{2}.$$

Since $\mathbf{p}$ is an arbitrary function (it is capable of characterizing arbitrary boundaries $\mathcal{A}$), this reparametrization can not be simplified in general and related to the original $\mathbf{p}$ in any simple manner.

For *small rotations*, we can linearize $\mathbf{p}$ and study the local effects. We set $\mathbf{R} = e^{-\mathbf{i}\phi/2} = 1 - \mathbf{i}\phi/2$ in first order in ϕ:

$$\begin{aligned}
\mathbf{R}\mathbf{p}[\mathbf{R}^{-1}\mathbf{n}\mathbf{R}]\mathbf{R}^{-1} &= (1-\mathbf{i}\phi/2)\mathbf{p}\left[(1+\mathbf{i}\phi/2)\mathbf{n}(1-\mathbf{i}\phi/2)\right](1+\mathbf{i}\phi/2) \\
&= (1-\mathbf{i}\phi/2)\mathbf{p}\left[\mathbf{n} - \mathbf{n}\cdot\mathbf{i}\phi\right](1+\mathbf{i}\phi/2) \\
&= (1-\mathbf{i}\phi/2)\Big(\mathbf{p}[\mathbf{n}] - \underline{\mathbf{n}}^{-1}(\mathbf{n}\cdot\mathbf{i}\phi)\Big)(1+\mathbf{i}\phi/2) \\
&= \mathbf{p}[\mathbf{n}] + \mathbf{p}[\mathbf{n}]\cdot\mathbf{i}\phi - \underline{\mathbf{n}}^{-1}(\mathbf{n}\cdot\mathbf{i}\phi) \quad \text{(1st order in } \phi)
\end{aligned}$$

Therefore the versor for the rotated boundary is, to first order in ϕ:

$$1 - e^0\Big(\mathbf{p}[\mathbf{n}] + \mathbf{p}[\mathbf{n}]\cdot\mathbf{i}\phi - \underline{\mathbf{n}}^{-1}(\mathbf{n}\cdot\mathbf{i}\phi)\Big)/2,$$

and the versor product then gives the locally displaced boundary:

$$\mathcal{R}(\mathbf{n}) - e^0(\mathbf{n}\wedge\mathbf{p})\cdot(\mathbf{i}\phi).$$

The tangent $\mathbf{I}$ thus shifts over a perpendicular distance $(\mathbf{n}\wedge\mathbf{p})\cdot(\mathbf{i}\phi)$. Note: does not involve derivatives of $\underline{\mathbf{n}}$!

26 Conclusions

- Boundaries can be represented as geometric objects:
  1. hyper-surfaces $\mathcal{R}(\mathbf{n})$ in $\mathcal{G}^1(e_0\mathbf{I}_m)$ or $\mathcal{G}^1(E\mathbf{I}_m)$, or
  2. direction-dependent translation versors $T_{\mathbf{p}[\mathbf{n}]}$ in $\mathcal{C}\ell_{m+1,1}$

  Both decompose boundaries *per tangent direction.*
- The involved boundary interaction of wave propagation (or collision, or milling, or growing) becomes simply:
  1. addition of e^0-components, or
  2. versor multiplication

  These perform boundary interaction per direction component: addition/multiplication of *'direction spectra'*.

Now *implementation.* Note analogy with Fourier transform (convolution becomes spectral multiplication). Algebraically similar: is there a 'Fast Boundary Transform'?

27 Analytically propagating waves

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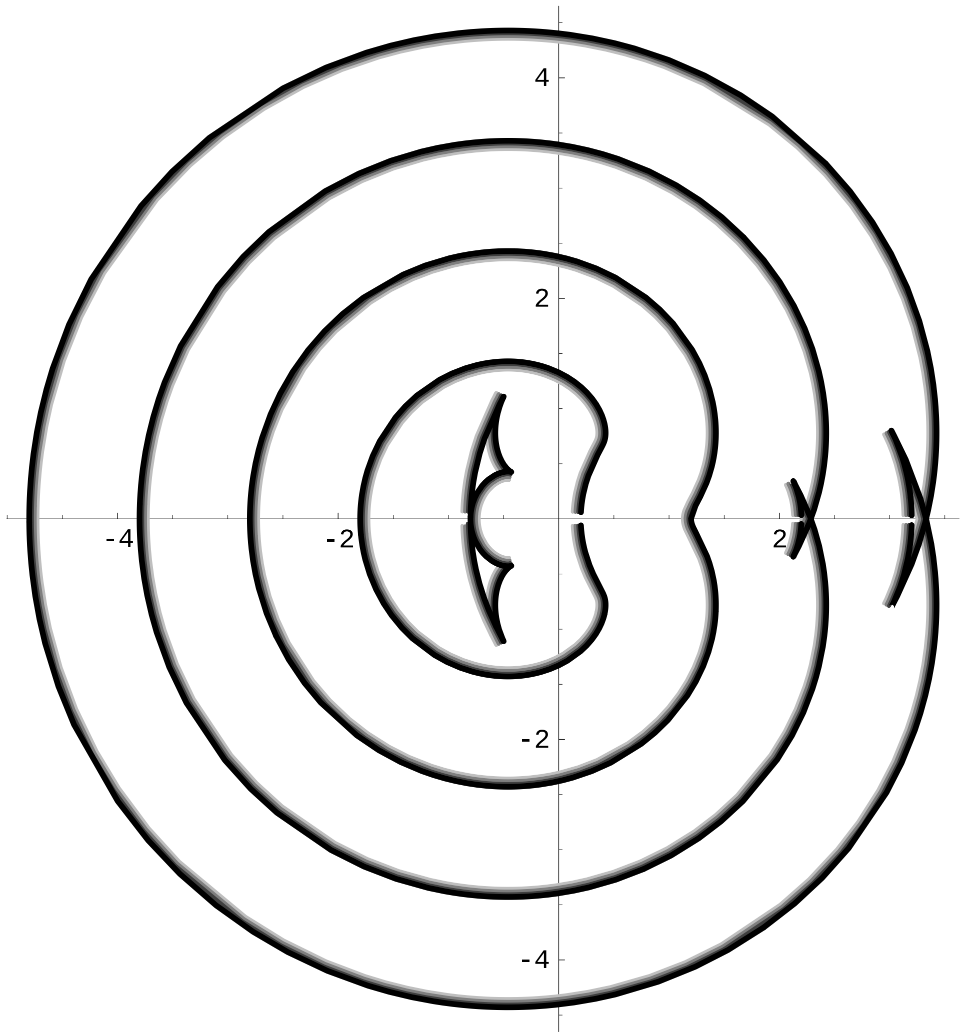

Propagation of circular waves from a *cardioid* shape (both inwards and outwards), computed in Mathematica by addition of their $\mathcal{R}(\cdot)$ representations, and inverse representation of the result. 'Inside' indicated by shading.

28 Related work

This table is a classification of literature on orientation-based representations of curves and surfaces. Not all can be used for *boundaries*, and few combine well with propagation and collision.

approach by	inside explicit	collision/ propagation	convexity waived	self-intersect	geom. prop.	m-D	algorithms given
class. diff. geom.	$-$	$-$	$+$	$\sim$	$+$	m	$-$
Horn 84	$-$	$-$	$\sim$	$-$	$+$	3	$\sim$
Osher 88	$\sim$	$+$	$+$	$-$	$-$	m	$+$
Nalwa 89	$\sim$	$-$	$+$	$-$	$\sim$	$3(m)$	$-$
Stolfi 91	$\sim$	$-$	$-$	$-$	$-$	m	$+$
Ghosh 93	$-$	$+$	$\sim$	$\sim$	$-$	2(3)	$+$
Liang 94	$-$	$-$	$+$	$-$	$+$	3	$\sim$
Dorst 94	$-$	$+$	$\sim$	$+$	$+$	2	$-$
Schmitt 96	$-$	$+$	$+$	$+$	$-$	2	$\sim$
Dorst 98	$+$	$+$	$+$	$+$	$+$	2	$\sim$
Dorst 99	$+$	$+$	$+$	$+$	$+$	m	coming!

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