Chapter 2

The Inner Products of Geometric Algebra

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ABSTRACT Making derived products out of the geometric product requires care in consistency. We show how a split based on outer product and scalar product necessitates a slightly different inner product than usual. We demonstrate the use and geometric significance of this contraction*, and show how it simplifies treatment of* meet *and* join*. We also derive the sufficient condition for covariance of expressions involving outer, inner and scalar products.*

2.1 The Product Structure of Geometric Algebra

The geometric algebra of a metric vector space, like its Clifford algebra, is based on the geometric product, the associative linear product under which a vector squares to a particular scalar given by the bilinear form. Yet in much literature on geometric algebra, additional products are introduced to aid interpretation and derivation. These products are significant since they are computational consequences of *commutation* and *factorization*, which are related to the important geometric concepts of independence and perpendicularity of subspaces. It is convenient to have explicit computational representation of these concepts, and this is why the outer product and inner product occur so frequently in geometric algebra.

Historically, these useful additional products have been introduced in a slightly *ad hoc* manner. This has led to an inner product which sits slightly uncomfortably among the other products, leading to complications in the derivation of compact formulas describing the interaction of the products. An example is the expansion of the geometric product for a bivector **B** with an arbitrary multivector A (see [4],pg.15) which is mostly:

$$\mathbf{B}\,A = \mathbf{B}\cdot A + \mathbf{B}\times A + \mathbf{B}\wedge A \tag{2.1}$$

(where $\times$ is the commutator product, yet another derived product), but this equation is *not* valid when the second factor A is a vector $\mathbf{a}$; then we are supposed to use

$$\mathbf{B}\,\mathbf{a} = \mathbf{B}\cdot\mathbf{a} + \mathbf{B}\wedge\mathbf{a} \tag{2.2}$$

(The commutator term in eq.(2.1) would erroneously give an extra $\mathbf{B} \cdot \mathbf{a}$. Yet eq.(2.1) *does* hold for the a scalar second argument α, since $\mathbf{B} \cdot \alpha = 0 = \mathbf{B} \times \alpha$.) The fact that an exception needs to be made is not a consequence of some fundamental geometrical irregularity of bivectors; rather, it is an indication that the derived products have not quite been chosen properly, leading to grade-based exceptions when they interact. More examples of this can easily be found in [4], and the authors are clearly aware of the problem (see [4], pg.20). Of course such exceptions and special cases to basic formulas multiply to many cases when these formulas are combined, and compromise the usefulness of geometric algebra as a simple and universal grammar for geometric programming. This would be a great pity; but fortunately, the problem can be fixed.

In this paper, we will take our guidance in the decomposition of the geometric product not from outer product and inner product (as is common), but from outer product and scalar product, since these directly encode the important concepts of 'spanning' and 'metric'. When they interact, they lead automatically to a natural concept of inner product; this is the *contraction*, slightly different from the usual inner product in precisely the right way to resolve the issues above.

The contraction product is not new, Lounesto [5] introduced it as the adjoint of the outer product under the extended bilinear form – this is essentially the construction we will follow here. His suggestion has not been followed by many practitioners of geometric algebra, perhaps because he did not demonstrate the geometrical usage of this algebraically more elegant possibility. This paper provides those additional arguments for its adoption, in a language that is more familiar to the users of geometric algebra.

2.2 The Basic Products of Geometric Algebra

2.2.1 Geometric Product

As usual in geometric algebra (and Clifford algebra), we start with a vector space V^m over a scalar field (which in this paper will always be $\mathbb{R}$) with a bilinear form $\langle \cdot, \cdot \rangle$. We introduce the *geometric product* as a linear, associative product, distributive over $+$, and for two vectors $\mathbf{x}$ and $\mathbf{y}$ equal to the scalar given by their bilinear form $\mathbf{x}\,\mathbf{y} \equiv \langle \mathbf{x}, \mathbf{y} \rangle$.

Using the geometric product to generate new elements (called *multivectors*) from the scalars and the vectors, we eventually generate the Clifford algebra of the vector space V^m. Each element generated may be decomposed using the $+$ into parts of distinct *grades* (or *steps*), the number of non-cancelling vectors in each term. It is therefore convenient to introduce the *grade* operator denoted $\langle \cdot \rangle_r$ which gives us the r-th grade of a general multivector. No multivector has components of negative grade.

2.2.2 Outer Product; Blades

In a space of unspecified dimension, the geometric product of two multivectors A_r and B_s of grade r and s has a highest possible term of grade $r+s$. This term can be viewed as a product in its own right. Perhaps surprisingly, it has nice algebraic properties: it is linear, associative, and skew-symmetric for vectors. We extend its construction to arbitrary multivectors by linearity. This gives the familiar *outer product* of two general multivectors A and B, denoted $A \wedge B$, as:

$$A \wedge B = \sum_{r,s} \langle \langle A \rangle_r \, \langle B \rangle_s \rangle_{r+s}. \tag{2.3}$$

The outer product is used to define *blades*, which are the useful computational representation of subspace elements in geometric algebra. A blade is a multivector which can be factored as an outer product of vectors. It therefore has a single specific grade r and is often called an r-blade; vectors are 1-blades and it is convenient to consider scalars as 0-blades (this is consistent since they can be factored by zero vector factors). We will denote a blade by a bold upper case symbol, such as $\mathbf{A}$; though vectors will be denoted by bold lower case (such as $\mathbf{a}$) and scalars by Greek symbols (such as α). Often we will denote the grade as a subscript, as in the r-blade $\mathbf{A}_r$. By definition, blades are closed under the outer product.

Two monadic operators are convenient to have, the reversion and the grade involution. The *reversion* $\widetilde{\mathbf{A}}_r$ of an r-blade $\mathbf{A}_r$ is the blade obtained by writing its vector factors in the reverse order. By skew-symmetry of the outer product, this can be re-arranged to the equivalent definition $\widetilde{\mathbf{A}}_r \equiv (-1)^{r(r-1)/2} \mathbf{A}_r$. We extend this definition to general multivectors by linearity, so: $\widetilde{A} \equiv \sum_r \widetilde{\langle A \rangle}_r$. By definition, $(A \wedge B)^\sim = \widetilde{B} \wedge \widetilde{A}$.

The other elementary operation is the *grade involution* $\widehat{\cdot}$, which reverses the sign of odd blades: $\widehat{\mathbf{A}}_r \equiv (-1)^r \mathbf{A}_r$ and similarly extends to general multivectors: $\widehat{A} \equiv \sum_r \widehat{\langle A \rangle}_r$.

2.2.3 Scalar Product: Metric Properties

Another important derived product is the scalar product of two multivectors, defined as:

$$A * B = \sum_{r,s} \langle \langle A \rangle_r \, \langle B \rangle_s \rangle_0. \tag{2.4}$$

The scalar product is symmetrical and reversible:

$$A * B = B * A \quad \text{and} \quad A * B = \widetilde{B} * \widetilde{A} \quad (= \widetilde{A} * \widetilde{B})$$

When applied to blades $\mathbf{A}$ and $\mathbf{B}$, the scalar product is only non-zero when $\mathbf{A}$ and $\mathbf{B}$ have the same grade, and then produces a scalar $\widetilde{\mathbf{A}} * \mathbf{B}$ which can be interpreted as the extension of the metric (i.e. the bilinear form) to blades: between blades of different grade, it is zero, and $\widetilde{\mathbf{A}} * \mathbf{A}$ can be used as 'norm squared' of $\mathbf{A}$.

The useful orthogonality property of the scalar product is

$$(A * B = A * C, \quad \forall A) \iff B = C \tag{2.5}$$

In degenerate Clifford algebras, the implication to the right does not necessarily hold and A needs to be chosen with more care. To avoid such distracting details, we will focus on non-degenerate algebras in this paper.

2.2.4 Contractions

Combination of the outer product and the scalar product leads to expressions such as $(A \wedge B) * C$. This calls for a distributive law specifying how $\wedge$ and $*$ interact. We define two new products, the *left and right contraction* (denoted $\rfloor$ and $\lfloor$) by:

$$(A \wedge B) * C = A * (B \rfloor C), \quad \forall A, B, C$$

and

$$C * (B \wedge A) = (C \lfloor B) * A, \quad \forall A, B, C$$

This defines them fully and leads to the concrete expressions for the *left contraction* (or 'contraction onto'):

$$A \rfloor B \equiv \sum_{r,s} \langle \langle A \rangle_r \, \langle B \rangle_s \rangle_{s-r}, \tag{2.6}$$

and for the *right contraction* (or 'contraction by')

$$A \lfloor B \equiv \sum_{r,s} \langle \langle A \rangle_r \, \langle B \rangle_s \rangle_{r-s}. \tag{2.7}$$

It is an easy exercise in grade manipulation to show that these formulas are correct. Using A_k as a shorthand for $\langle A \rangle_k$, etc., we obtain for the left contraction: $(A \wedge B) * C = \sum_{k,\ell,m} (A_k \wedge B_\ell) * C_m = \sum_{k,\ell} \langle A_k B_\ell C_{k+\ell} \rangle_0 = \sum_{k,\ell} \langle A_k \langle B_\ell \rfloor C_{k+\ell} \rangle_k \rangle_0 = A * (B \rfloor C)$, and similarly for the right contraction.

There is a close relationship between the two contractions through the reversion operation, which is immediate from their definitions:

$$(A \lfloor B)^\sim = \widetilde{B} \rfloor \widetilde{A}. \tag{2.8}$$

Since we have reversion available as a standard procedure in our geometric algebra, there is no absolute need for two different contractions.

2.2.5 Relationship of the Contraction to the Inner Product

Obviously, the contractions are very similar to the commonly used inner product introduced in [4]. This product is defined in terms of grades as:

$$A \cdot B \equiv \sum_{r\neq 0, s\neq 0} \langle \langle A\rangle_r \langle B\rangle_s\rangle_{|s-r|} \tag{2.9}$$

Note the exclusion of the sum over the scalar parts, the exceptional definition from [4] which sets the parts of inner products involving scalars explicitly to zero. This causes some problems in [4] itself for the universality of equalities, and partly explains the trouble with the bivector expansion which we mentioned in the introduction. To simplify later considerations, it is convenient to introduce a variation which does not treat the scalar parts in an exceptional manner. Let us call it the 'dot product' and denote it by a fat dot '$\bullet$':

$$A \bullet B \equiv \sum_{r,s} \langle \langle A\rangle_r \langle B\rangle_s\rangle_{|s-r|} \tag{2.10}$$

The similarity of the definitions of dot product and contractions gives a simple equation relating them [6]:

$$A \rfloor B + A \lfloor B = A * B + A \bullet B. \tag{2.11}$$

This shows that the contractions and the scalar and dot products are different ways of encoding the same kind of quantitative information.

Let us write the consequences of eq.(2.11) more explicitly for blades $\mathbf{A}_k$ and $\mathbf{B}_\ell$ of grades k and ℓ:

$$\begin{array}{ll} k < \ell: & \mathbf{A}_k \rfloor \mathbf{B}_\ell = \mathbf{A}_k \bullet \mathbf{B}_\ell, \quad \mathbf{A}_k \lfloor \mathbf{B}_\ell = \mathbf{A}_k * \mathbf{B}_\ell = 0 \\ k = \ell: & \mathbf{A}_k \rfloor \mathbf{B}_\ell = \mathbf{A}_k \lfloor \mathbf{B}_\ell = \mathbf{A}_k * \mathbf{B}_\ell = \mathbf{A}_k \bullet \mathbf{B}_\ell \\ k > \ell: & \mathbf{A}_k \lfloor \mathbf{B}_\ell = \mathbf{A}_k \bullet \mathbf{B}_\ell, \quad \mathbf{A}_k \rfloor \mathbf{B}_\ell = \mathbf{A}_k * \mathbf{B}_\ell = 0 \end{array}$$

Disregarding the scalar case (so that the dot product equals the standard inner product), the left and right contraction are therefore like the inner product with a sensitivity to the relative grade of the arguments. In a sense, the conditional statement on the relative size of k and ℓ (which is often required in statements about the properties of the inner product and the dot product) has been incorporated into the products, rather than having to be stated separately. This seems a trivial change, hardly worth the trouble: we would think that we could easily keep track of the grades of the arguments ourselves (and indeed, we have done so for years). However, the conditional rewriting of a series of formulas in which the grades 'nest' leads to a large set of conditions on validity of rewriting. Some of that is visible in [4]. If we can prevent that by the contractions which 'switch themselves off' (i.e. become zero) as soon as grade conditions do not hold, this provides a simpler computational structure, better suited to recursive nesting.

2.3 Understanding the Contraction

The contraction is clearly very similar in many of its properties to the inner product, but the differences are important to understand.

2.3.1 Defining Axioms

1. **Scalars:** For a 0-blade α, we get $\alpha \rfloor B = \alpha B$; if B has no scalar part, then $B \rfloor \alpha = 0$. By contrast, the inner product is explicitly zero for any scalar argument.
2. **Vectors:** Let $\mathbf{a}$ and $\mathbf{b}$ be vectors, then $\mathbf{a} \rfloor \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle$, the value of their bilinear form; this is also the result for the standard inner product.
3. **First argument a vector:** The expansion formula for a vector contracted onto an outer product is simply a product rule
$$\mathbf{a} \rfloor (B \wedge C) = (\mathbf{a} \rfloor B) \wedge C + \widehat{B} \wedge (\mathbf{a} \rfloor C),$$
which means that (algebraically speaking) the contraction is a *derivation* (see [5]). This then leads to a full expansion formula identical to that of the standard inner product.
4. **First argument an outer product:** The contraction has a universally valid rewriting rule (which the inner product does *not* have):
$$(A \wedge B) \rfloor C = A \rfloor (B \rfloor C). \tag{2.12}$$
The proof shows how well the products intertwine: $D * ((A \wedge B) \rfloor C) = (D \wedge (A \wedge B)) * C = ((D \wedge A) \wedge B) * C = (D \wedge A) * (B \rfloor C) = D * (A \rfloor (B \rfloor C))$. This is valid for arbitrary D; by eq.(2.5) we have eq.(2.12).

Using these properties actually *defines* the contraction of blades, since repeated use, together with linearity, reduces all contractions of multivectors to products of vectors and/or scalars. Similar rules for the inner product are hard to give, the fourth rule breaks down into many grade-dependent cases. This is cumbersome, for an expression like $(A \wedge B) \cdot C$ arises very frequently in computations, and one cannot always know the grades of A, B and C beforehand.

2.3.2 Famous Formulas

Similar to the standard inner product, we can derive the contraction of a vector and a multivector from the equations $\mathbf{x} A = \mathbf{x} \rfloor A + \mathbf{x} \wedge A$ and $A \mathbf{x} = A \lfloor \mathbf{x} + A \wedge \mathbf{x}$, together with the easily derived $A \lfloor \mathbf{x} = -\mathbf{x} \rfloor \widehat{A}$. This gives:

$$\mathbf{x} \rfloor A = \tfrac{1}{2}(\mathbf{x} A - \widehat{A} \mathbf{x}) \quad \text{and} \quad A \lfloor \mathbf{x} = \tfrac{1}{2}(A \mathbf{x} - \mathbf{x} \widehat{A})$$

which may be compared to the familiar outer product relationships:

$$\mathbf{x} \wedge A = \tfrac{1}{2}(\mathbf{x}\,A + \widehat{A}\,\mathbf{x}) \quad \text{and} \quad A \wedge \mathbf{x} = \tfrac{1}{2}(A\,\mathbf{x} + \mathbf{x}\,\widehat{A})$$

The subtlety of distinguishing two contractions is thus nicely reflected in the asymmetry in the definition of the inner product due to the minus sign.

Eq.(2.12) gives the distribution of an inner product over an outer product; here is the converse for a vector argument:

$$\mathbf{a} \wedge (B \rfloor C) = (\mathbf{a} \rfloor B) \rfloor C + \widehat{B} \rfloor (\mathbf{a} \wedge C)$$

This can be lifted to the level of multivectors, but only under certain conditions: we need to restrict the statement to the case where $\mathbf{a} \wedge C = 0$. The resulting rule inherits (in a manner similar to an outermorphism), and gives the partial duality:

$$(A \rfloor B) \rfloor C = A \wedge (B \rfloor C) \quad \text{if } A \subseteq C \tag{2.13}$$

where '$A \subseteq C$' should be interpreted as '$\mathbf{x} \wedge A = 0 \;\Rightarrow\; \mathbf{x} \wedge C = 0$'. So even when using the contraction, some equations become conditional. Yet the condition is not on the grades of the arguments (as in [4], (1-1.25b)), but on their geometrical relationship – which is more fundamental.

2.3.3 Geometric Interpretation

The geometric interpretation of the contraction is straightforward, and involves the important geometrical concepts of *perpendicularity* (as related to the metric of the vector space) and *containment* (of subspaces viewed as sets of points). It is simplest to state the geometric interpretation of the contraction for the (clearly geometrical) blades:

> The contraction $\mathbf{A} \rfloor \mathbf{B}$ of an a-blade $\mathbf{A}$ onto a b-blade $\mathbf{B}$ is a sub-blade of $\mathbf{B}$ of grade $b - a$ which is perpendicular to $\mathbf{A}$, and linear in both arguments.

Since blades of negative grades do not exist, the result is zero when $b < a$. This is in contrast to the standard inner product, in which the roles of $\mathbf{A}$ and $\mathbf{B}$ then reverse (the result is then a sub-blade of $\mathbf{A}$). This reversal has a rather different geometrical feeling to it, and this is indeed reflected in a change of algebraic properties when the reversal happens. This is the geometrical reason behind the need for the conditions on many statements using the inner product: it has become a different object.

We may recognize the rewriting of eq.(2.12) (as restricted to blades) $(\mathbf{A} \wedge \mathbf{B}) \rfloor \mathbf{C} = \mathbf{A} \rfloor (\mathbf{B} \rfloor \mathbf{C})$ as an important statement on the inheritance of containment of subspaces: anything in $\mathbf{C}$ perpendicular to the blade $\mathbf{A} \wedge \mathbf{B}$ can be obtained by considering the blade $\mathbf{B} \rfloor \mathbf{C}$ which is in $\mathbf{C}$ perpendicular to $\mathbf{B}$; and then taking of that the part perpendicular to $\mathbf{A}$. Through the contraction, this property of 'containment' has been made an intrinsic part of geometric algebra, as it should be. In contrast, when one uses the standard inner product, it needs to be superimposed on geometric algebra through the application of conditional tests on the relative grades, which I find much less satisfactory (since those grades do not really reflect the containment relationship in all its aspects).

2.4 Projection

In geometric algebra, one would like to have a *projection operator*, especially for blades (since they represent subspaces). Apart from being linear and idempotent, we would obviously want the projection operator to have the property that a blade $\mathbf{A}$ projected into $\mathbf{B}$ would yield a result totally in $\mathbf{B}$.

Hestenes (in [4], pg.20) defines $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{B}^{-1}$ as the projection operator of $\mathbf{A}$ onto $\mathbf{B}$, but this causes problems; it gives incorrect answers for $\mathbf{A}$ a scalar and in the case $\mathbf{A}=\mathbf{B}$, requiring rather inelegant exceptions to be made. The formula can actually be made generic by replacing the inner product by the dot product. The authors of [4] realize this but reject it, since other important equations then require exceptions (such as the fundamental equation $\mathbf{x}\,A = \mathbf{x}\cdot A + \mathbf{x}\wedge A$, in which the first term should be zero when A is a scalar, but would not be if the inner product is replaced by a dot product).

We can resolve all issues by using the contraction throughout (for which the equation just mentioned holds since $\mathbf{x}\rfloor\alpha = 0$). This gives as definition of the projection:

$$\mathrm{P}_{\mathbf{B}}(\mathbf{A}) \equiv (\mathbf{A}\rfloor\mathbf{B}^{-1})\rfloor\mathbf{B} \tag{2.14}$$

where we have preferred to have the inverse inside the bracket, to show more explicitly that the result is indeed in $\mathbf{B}$, by the geometrical properties of the contraction. [4] shows under which circumstances the final inner product in his definition $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{B}^{-1}$ can be replaced by a geometric product – which 'makes it much easier to use'. This requires $\mathrm{grade}(\mathbf{A}) \leq \mathrm{grade}(\mathbf{B})$, a demand on the relative grades, the projection is zero otherwise. When using the contraction in non-degenerate algebras we *always* have:

$$(\mathbf{A}\rfloor\mathbf{B}^{-1})\rfloor\mathbf{B} = (\mathbf{A}\rfloor\mathbf{B}^{-1})\,\mathbf{B}, \tag{2.15}$$

so the rewriting to a geometric product is always permitted. This makes the projection based on the contraction universally 'much easier to use', independent of grade considerations.

2.5 Meet and Join

When two blades $\mathbf{A}$ and $\mathbf{B}$ have a common factor, their outer product $\mathbf{A}\wedge\mathbf{B}$ equals zero. In that case, the terms of other grades in their geometric product $\mathbf{A}\,\mathbf{B}$ are required to describe their geometrical relationship. The meet is the 'largest common divisor' of the blades $\mathbf{A}$ and $\mathbf{B}$, the join their 'least common multiple'. These dyadic 'products' meet and join are essentially limited to blades, they have no meaning for general multivectors.

The definition of meet and join is based on a factorization. If $\mathbf{M}$ is the highest grade blade in common to $\mathbf{A}$ and $\mathbf{B}$, we can factor each of them orthogonally, defining the parts $\mathbf{A}'$ and $\mathbf{B}'$ through: $\mathbf{A} = \mathbf{A}' \wedge \mathbf{M} = \mathbf{A}'\,\mathbf{M}$ and $\mathbf{B} = \mathbf{M} \wedge \mathbf{B}' = \mathbf{M}\,\mathbf{B}'$, and the join $\mathbf{J}$ is then $\mathbf{J} = \mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}'$. Grade considerations yield that join and meet are related through:

$$\langle \mathbf{A}\,\mathbf{M}^{-1}\,\mathbf{B}\,\mathbf{J}^{-1}\rangle_0 = 1$$

as well as the relative maximal containment demands. This equation can be rewritten using the derived products in various ways,

$$\mathbf{J} = \mathbf{A} \wedge (\mathbf{M}^{-1}\rfloor\mathbf{B}) = (\mathbf{A}\lfloor\mathbf{M}^{-1}) \wedge \mathbf{B},$$

$$\mathbf{M} = (\mathbf{B}\rfloor\mathbf{J}^{-1})\rfloor\mathbf{A} = \mathbf{B}\lfloor(\mathbf{J}^{-1}\lfloor\mathbf{A})$$

(notice the apparent but consistent reversal of arguments as in [7]), and

$$(\mathbf{A}\lfloor\mathbf{M}^{-1}) * (\mathbf{B}\rfloor\mathbf{J}^{-1}) = 1 = (\mathbf{M}^{-1}\rfloor\mathbf{B}) * (\mathbf{J}^{-1}\lfloor\mathbf{A}).$$

In all these computations with meet and join, we should take care to use the correct inner product. If $\mathbf{X}$ is a blade, and $\mathbf{I}$ a blade containing $\mathbf{X}$ (so that any vector in $\mathbf{X}$ is also in $\mathbf{I}$), then

$$\mathbf{X}\rfloor\mathbf{I} = \mathbf{X} \bullet \mathbf{I} \quad \text{and} \quad \mathbf{I}\lfloor\mathbf{X} = \mathbf{I} \bullet \mathbf{X} \quad \text{if} \quad \mathbf{X} \subseteq \mathbf{I}$$

For the meet and join relationships above, whenever we use a contraction the condition actually applies; so for our work in this section, either contraction or dot product work equally well. But the standard inner product $\mathbf{A}\cdot\mathbf{B}$ does *not*: a 0-blade $\mathbf{A}$ or $\mathbf{B}$ would cause the right hand side to be zero. Since a 0-blade is a scalar which represents a 0-dimensional space at the origin, this is a legitimate geometrical situation for which meet and join should give the appropriate answer; so we should reject the standard inner product for this application. To keep the structure of geometric algebra clean and minimal, we should therefore reject it altogether, possibly preferring the dot product. But as we saw in the previous section, the desired and universal validity of the vector expansion $\mathbf{x}\,A = \mathbf{x} \bullet A + \mathbf{x} \wedge A$ then leads inexorably to the adoption of the contraction.

2.6 Linear Transformations as ‘Innermorphisms’

2.6.1 Outermorphisms and Adjoint Transformations

If we have a linear transformation f on vectors, we naturally extend it as an outermorphism to $\underline{f}$ (see [4]), by the recursive definition $\underline{f}(\mathbf{a} \wedge A) = f(\mathbf{a}) \wedge \underline{f}(A)$ and linearity. It is also natural to define the *adjoint* $\overline{f}$ of the extended linear transformation $\underline{f}$ by (see [4], pg.55): $\overline{f}(A) * B = A * \underline{f}(B)$. This adjoint function of an outermorphism then combines naturally with the adjoint of the outer product. For instance, assuming $\underline{f}$ and therefore $\overline{f}$ invertible, we can easily show how a contraction transforms under a linear mapping:

$$\underline{f}(A \rfloor B) = \overline{f}^{\,-1}(A) \rfloor \underline{f}(B) \tag{2.16}$$

Proof: $C * \underline{f}(A \rfloor B) = \overline{f}(C) * (A \rfloor B) = (\overline{f}(C) \wedge A) * B = \big(\overline{f}(C) \wedge \overline{f}(\overline{f}^{\,-1}(A))\big) * B = \overline{f}(C \wedge \overline{f}^{\,-1}(A)) * B = \big(C \wedge \overline{f}^{\,-1}(A)\big) * \underline{f}(B) = C * \big(\overline{f}^{\,-1}(A) \rfloor \underline{f}(B)\big)\ , \forall C$.

This important result from [4] for the standard inner product therefore also holds for the contraction.

2.6.2 Covariance of Inner Product Formulas

We know that the outer product transforms covariantly under a linear transformation, in the sense that $\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B)$. It is possible to construct inner product formulas which transform naturally (i.e. covariantly) under a linear transformation; however, this appears only doable *for arguments which are blades.* Such covariant constructions involve the combinations $\mathbf{A}^{-1} \rfloor \mathbf{B}$ and $\mathbf{B} \lfloor \mathbf{A}^{-1}$. Both the projection and the meet/join formulas can be rewritten to contain these as ‘macros’ from which the total formula is constructed, for instance:

$$\mathrm{P}_{\mathbf{B}}(\mathbf{A}) = (\mathbf{A} \rfloor \mathbf{B}^{-1}) \rfloor \mathbf{B} = (\mathbf{B} \lfloor \mathbf{A}^{-1})^{-1} \rfloor \mathbf{B} \tag{2.17}$$

The covariance of this formula follows from the following proposition:

> The recipe for testing the covariance of an expression on blades involving $\wedge$, $\rfloor$ and $\lfloor$ is: rewrite the expression using only permitted constructions of the form $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A}^{-1} \rfloor \mathbf{B}$ and $\mathbf{B} \lfloor \mathbf{A}^{-1}$. If it succeeds, replace ‘ $^{-1}$’ by ‘ $^{\perp}$’ throughout and verify that the expression thus obtained is identically equal to zero; then the original expression is covariant.

Here $\mathbf{X}^{\perp}$ a convenient shorthand for *any blade of the same grade as* $\mathbf{X}$ *and satisfying* $\mathbf{X}^{\perp} \rfloor \mathbf{X} = 0$.

Derivation: From $1 = \underline{f}(1) = \underline{f}(\mathbf{A}^{-1} \rfloor \mathbf{A}) = \overline{f}^{\,-1}(\mathbf{A}^{-1}) \rfloor \underline{f}(\mathbf{A})$ it follows that $\overline{f}^{\,-1}(\mathbf{A}^{-1}) = \underline{f}(\mathbf{A})^{-1} + \underline{f}(\mathbf{A})^{\perp}$. Using this, we obtain $\underline{f}(\mathbf{A}^{-1} \rfloor \mathbf{B}) =$

$\overline{f}^{-1}(\mathbf{A}^{-1})\rfloor\underline{f}(\mathbf{B}) = \underline{f}(\mathbf{A})^{-1}\rfloor\underline{f}(\mathbf{B}) + \underline{f}(\mathbf{A})^{\perp}\rfloor\underline{f}(\mathbf{B})$. So covariance of $\mathbf{A}^{-1}\rfloor\mathbf{B}$ under $\underline{f}$ requires that $\underline{f}(\mathbf{A})^{\perp}\rfloor\underline{f}(\mathbf{B})$ be zero. Invertibility of $\underline{f}$ now demands that $\mathbf{A}^{\perp}\rfloor\mathbf{B}$ be zero. A similar result can be derived for the right contraction $\lfloor$; and these properties inherit under nesting. QED

In showing the covariance of the projection $(\mathbf{A}\rfloor\mathbf{B}^{-1})\rfloor\mathbf{B}$, eq.(2.17) is the rewriting. For non-scalar $\mathbf{A}$, we need to show that the equality $(\mathbf{A}\rfloor\mathbf{B}^{\perp})\rfloor\mathbf{B} = 0$ holds. This is obvious: any non-scalar both in $\mathbf{B}$ and in $\mathbf{B}^{\perp}$ must be zero. For a scalar $\mathbf{A}$, the equation is obviously covariant. Therefore the projection is covariant. One can show in a similar manner that meet and join are.

In [1], the construction $\mathbf{A}^{-1}\rfloor\mathbf{B}$ is identified as the natural counterpart in geometric algebra of the set-difference operation on subspaces; its relevance to linear algebra is therefore not too surprising.

2.7 Replacing the Inner Product

The derived products in the standard formulation of geometric algebra in [4] have relationships which are somewhat opaque, and which lead to identities whose validity is often conditional on the grade of their variables. While the numerous applications of geometric algebra in physics and engineering have shown that this is not an impediment to its use, any extended algorithm containing several of those relationships would fracture into many different grade-dependent cases. This could hamper the impact of geometric algebra on fields like computer graphics and geometric modeling, where the grade of the objects constructed is often dependent on the data itself, and are not foreseeable.

Fortunately, this aspect of geometric algebra can be cleaned up, by the simple expedient of replacing the inner product by a very similar construction, the contraction. This is less drastic than it sounds, since the contraction has the same geometric meaning (though subtly more precise), and obeys very similar equations, though now with universal validity. We have shown how well the contraction meshes with the other two important derived products in geometric algebra (outer product and scalar product), and how it makes treatment of geometrically significant operations such as projection and meet unconditional and unexceptional. These operations are at the basis of computer graphics, so this version of geometric algebra should find good use there.

But of course we should not really have two versions of geometric algebra, so subtly different in this one aspect – it would lead to confusion. Therefore I would strongly advocate the usage of the contraction in physics and engineering as well. Knowing the importance of bivector fields in electromagnetism, it should be of advantage that one can universally write:

$$\mathbf{B}\,A = \mathbf{B}\rfloor A + \mathbf{B} \times A + \mathbf{B} \wedge A$$

rather than eq.(2.1) with its messy exception rule. The universal duality relationship $(A \wedge B)\rfloor C = A\rfloor(B\rfloor C)$ is also too good not to have available in one's computations and derivations. In all cases, the contraction $\rfloor$ takes care of its own exceptions, internally, invisibly and consistently, and without any sacrifice in geometric expressibility.

Trying the Contraction

If you would like to become familiar with the properties of the contraction at a computational level, including visualization, you might try GABLE [3], a Matlab package for numerical computations in geometric algebra. It permits changing the inner product you use from the contraction (its default) to the standard inner product and the dot product.

References

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