The Algebra of Geometry

Abstract *Pascal's triangle is known to relate to both simplices and Clifford algebra with the later providing a complete representation. This relationship allows each Clifford algebra to be derived from the associated simplex which exposes the connection to geometry justifying the terminology geometric algebra. Simplices are also related to Pfaffians and this close connection can be used to explain the Pfaffian structure of the algebra. An example is the grad operator acting on a product of two vectors to bring all three primary vector calculus identities into one equation with geometric identification of each component.*

1 Simplices

Table 1: Pascal's Triangle

N	**V**	**E**	**F**	$\cdots$				
1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The 3-simplex is described by the fifth row of Pascal's triangle with 4 vertices, 6 edges and 4 faces. We can go further by putting a point in the middle of the tetrahedron and connecting all vertices to this point. This is a 4-simplex shown in Figure 1(a) with the internal connections shown as dotted lines. This is described by the $\mathbf{V} = 5$ row of Table 1, which also indicates that there are 5 tetrahedrons in the next column of the table.

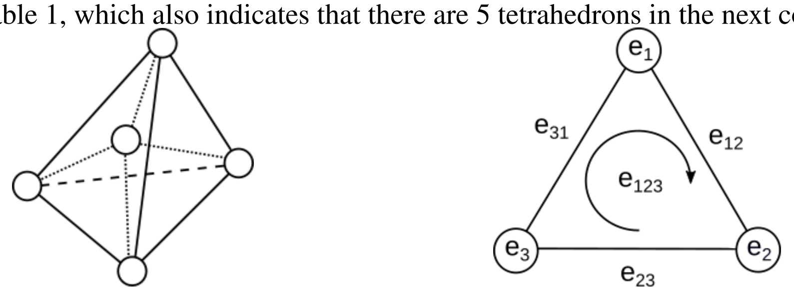

Figure 1: (a) The 4-simplex and (b) labelled 2-simplex with clockwise ordering

Names can be introduced for all these geometric elements in order to make it easier to proceed to higher dimensions. A single point is designated e_1 and two points as e_1 and e_2 with connecting edge labelled as e_{12}. With a third vertex, e_3, the extra edges are e_{23} and e_{31} and the face is e_{123}. This is shown in Figure 1(b) but it is also important to specify the ordering of points. The face e_{123} shown in Figure 1(b) designates a clockwise ordering of the points while e_{321} would show an anticlockwise order. Such an arrow defines a 3-cycle because starting from e_1 there are three steps to get back to e_1.

Starting with a triangle with face e_{123} we can add a point, e_4, in the middle and connect this to the starting three points. Figure 2 shows a tetrahedron looking from the top or a projection of the tetrahedron onto the plane of e_{123}. For clarity only the point labels are shown. The 3-simplex, by Pascal's triangle, has another three edges labelled e_{14}, e_{24} and e_{34} and four faces, e_{123}, e_{124}, e_{134} and e_{234}. Later the last element on each row will be called the pseudoscalar, e_{1234}, in this case, which corresponds to the whole tetrahedron. This is a good example to show pseudo-symmetry. Removing each vertex from e_{1234} gives four triangles, e_{123}, e_{124}, e_{134} and e_{234}. These are called pseudo-vertices since they match the symmetry shown in Pascal's triangle.

Returning to the 3-simplex, it can not have four 3-cycles, one for each face, because then the labelling would be inconsistent. Only two faces can be consistent, so choose the anti-clockwise one from below on the outside face of Figure 2, e_{123}, and, for example, the side, e_{143}, taken clockwise as marked. The shared edge has the same direction for the arrows so both 3-cycles can be labelled using e_{31}. No other 3-cycle can be chosen and keep this consistency on the shared edges. But this scheme provides a constraint on the 3-cycles. Changing e_{123} to e_{321} means e_{143} must also change to e_{134}. When this is applied to the Fano plane in 7 dimensions we find that keeping the sense of the 3-cycles independent is an important concept. In fact leaving them undefined generates Clifford algebras and constraining them generates different algebras such as octonions for the 6-simplex. This shows the tetrahedron can only have one independent 3-cycle and the notation is used to show a 6-simplex has 7 independent 3-cycles.

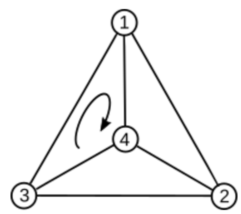

Figure 2: The 3-simplex projection

The forms e_{13}, e_{12}, e_{23} can easily be seen to generate a 3-cycle of 2-forms as each rotates the next in turn, which is the same as quaternions. The standard way of diagrammatically showing quaternion multiplication is shown in Figure 3 which demonstrates the 3-cycle. The diagram for $-i, -k, -j$ is also shown to expose the left-hand screw rule. Quaternions actually generate a negative cross product but this makes sense since these are "imaginary". Imaginary is not the appropriate term for these operators since they do not commute and the term "pure" is used to distinguish such operators and also specifies time-like operations in space-time.

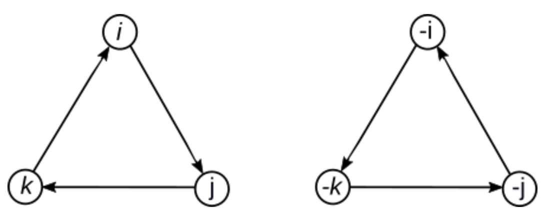

Figure 3: Standard Quaternion Geometry

This shows that the quaternions when operating on quaternions do not form a 3-cycle because the triple product is $ijk = -1$. To return a quaternion vector back to the start via a 3-cycle, the identification $e_1 = i' = -i$ is needed so that $i'jk = 1$. This provides the transformation to a right-hand screw rule shown in Figure 4. Hence the sense of rotation is fixed for quaternions demonstrating no independence whereas in Clifford algebra the two sets e_{13}, e_{32}, e_{21} and e_{12}, e_{23}, e_{31} are both 3-cycles, describing the two senses of rotation.

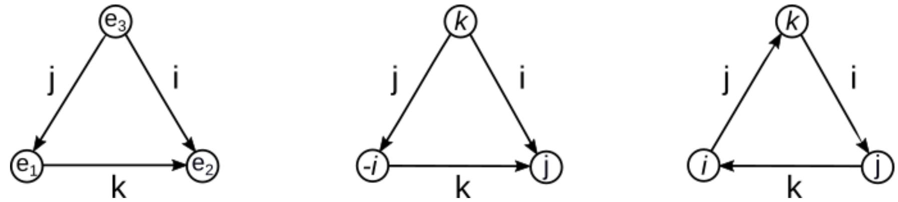

Figure 4: Quaternion 2-simplex Geometry

2 Pfaffians

The Pfaffian can be defined using a recurrence rule that is only applicable for odd simplices. Here, de Bruijn shows how to generate an expansion for even simplices and this is later shown to be inherent in Clifford algebra. It is easy to see that if we add another column to a 4-simplex then the five new terms connect each 4-simplex vertex to the new vertex and a shorthand notation is introduced showing just the lower diagonal which for the 5-simplex is

$$\begin{vmatrix} e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ & e_{23} & e_{24} & e_{25} & e_{26} \\ & & e_{34} & e_{35} & e_{36} \\ & & & e_{45} & e_{46} \\ & & & & e_{56} \end{vmatrix} = \backslash e_{12}, e_{23}, e_{34}, e_{45}, e_{56}|.$$

The recurrence relation here expands each signed term of the newly added column with the smaller Pfaffian without these indices. This is called the cofactor expansion, for n even,

$$\begin{aligned} \backslash e_{12}, e_{23}, \dots e_{(n-1)n}| = \; & e_{1n} \; \backslash e_{23}, e_{34}, \dots, e_{(n-2)(n-1)}| \\ + \; & \sum_{j=2}^{n-2} (-1)^{(j-1)} e_{jn} \; \backslash \dots, e_{(j-1)(j+1)}, \dots| \\ + \; & e_{(n-1)n} \; \backslash e_{12}, e_{23}, \dots, e_{(n-3)(n-2)}|. \end{aligned} \tag{1}$$

The first and last rows cover j missing in the Pfaffian as the first or last index otherwise j is missing from the middle of the pairs of indicies. So the criteria is that as j sequences over 1 though $(n-1)$ then j and n indices are removed from the original Pfaffian by removing the last column, the row with j as the first index as well as the column with j as the second index, if it exists. For the 5-simplex above this is

$$e_{16} \; \backslash e_{23}, e_{34}, e_{45} \; | - e_{26} \; \backslash e_{13}, e_{34}, e_{45} \; | + e_{36} \; \backslash e_{12}, e_{24}, e_{45} \; | - e_{46} \; \backslash e_{12}, e_{23}, e_{35} \; | + e_{56} \; \backslash e_{12}, e_{23}, e_{34}|.$$

The last row of the cofactor expansion applied repeatedly gives the last term of the Pfaffian expansion as

$$e_{12} e_{34} \dots e_{(n-1)n} = \; \backslash e_{12}, 0, e_{34}, 0, \dots, 0, e_{(n-1)n}|, \tag{2}$$

where all the other terms are zeros. This is an important result when extended to antisymmetric matrices and can be used to show that the determinant applied to simplices defines the square of the hyper-volume defined by an $(n-1)$-simplex.

A convenient notation is introduced to represent subsets of the full permutation group which uses the square backets to enumerated all $n!$ permutations of $\mathbb{N}_1^n$ as $\mathcal{P}_n = [1, 2, \dots, n-1, n]$. Any partition into m and $n-m$ parts can then be represented using round brackets that fix increasing order within the brackets so that an enumeration of combinations is defined as $\mathcal{C}^n_m = [(1,2,\dots,m)(m+1,\dots,n)]$. The multiple partition into pairs for n even is then denoted $\mathcal{P}^n_{2,2,\dots} = \mathcal{P}_{n,n} = [(1,2),(3,4),\dots,(n-1,n)]$. The second index is used later used to help define Clifford algebra. Pfaffians can be defined using the subset with all the first indices of each pair also being ordered using extra round brackets as

$$\mathcal{P}'_{n,n} = [((1,2),(3,4),\dots,(n-1,n))].$$

This is because the simplex edges not only represent rotation operations but also the area covered by the two vectors. Since these are orthonormal the volume is the same no matter which order is chosen, $e_{12}e_{34} = e_{13}e_{24} = e_{14}e_{23}$, because these define the same hyper-volume. This is extended to arbitrary dimensions and the cardinality of terms gives $\mathcal{P}_{n,n} = \frac{n}{2}!\mathcal{P}'_{n,n}$.

The anaylsis of Pfaffians used basis vectors which were assumed orthonormal. This is extended to arbitrary vectors using the exterior algebra which is usually defined as the determinant of n vectors as

$$\mathbf{a}\wedge\mathbf{b}\wedge\cdots\wedge\mathbf{n} = |\mathbf{a},\mathbf{b},\dots,\mathbf{n}|\; e_{123\dots n}.$$

This is called an n-form and for two vectors $\mathbf{a}$ and $\mathbf{b}$ in the plane e_{12} with internal angle θ then the 2-form is $\mathbf{a}\wedge\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)e_{12}$. Since the determinant of n vectors defines the volume then for n even the Pfaffian for the simplex for non-orthonormal vectors also defines the volume

$$\begin{aligned}
&|\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\dots,\mathbf{m},\mathbf{n}|\; e_{12\dots n}\\
&\quad = \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\cdots\wedge\mathbf{m}\wedge\mathbf{n}\\
&\quad = \tfrac{1}{n!}\sum_{\mu\in\mathcal{P}_n}(-1)^\sigma a_{\mu_1}e_{\mu_1}b_{\mu_2}e_{\mu_2}c_{\mu_3}e_{\mu_3}d_{\mu_4}e_{\mu_4}\dots m_{\mu_m}e_{\mu_m}n_{\mu_n}e_{\mu_n}\\
&\quad = \tfrac{1}{(n/2)!}\sum_{\mu\in\mathcal{P}_{n,n}}(-1)^\sigma(a_{\mu_1}e_{\mu_1}\wedge b_{\mu_2}e_{\mu_2})(c_{\mu_3}e_{\mu_3}\wedge d_{\mu_4}e_{\mu_4})\dots(m_{\mu_m}e_{\mu_m}\wedge n_{\mu_n}e_{\mu_n})\\
&\quad = \sum_{\mu\in\mathcal{P}'_{n,n}}(-1)^\sigma(a_{\mu_1}e_{\mu_1}\wedge b_{\mu_2}e_{\mu_2})(c_{\mu_3}e_{\mu_3}\wedge d_{\mu_4}e_{\mu_4})\dots(m_{\mu_m}e_{\mu_m}\wedge n_{\mu_n}e_{\mu_n})\\
&\quad = \Big\backslash|\mathbf{a},\mathbf{b}|,|\mathbf{c},\mathbf{d}|,\dots,|\mathbf{m},\mathbf{n}|\Big|\; e_{12\dots n}\\
&\quad = \pm V e_{12\dots n}.
\end{aligned}$$

Another important result is that the cofactor expansion can be expressed more generally as a division of not just of the last column but as a division between any number of terms[2]. This is analogous to the Laplace expansion for determinants.

3 Geometric Algebra

The cofactor expansion of Pfaffians shows that simplices, like Pfaffians, have terms describing connections between each dimension. The recursion relation of Pascal's triangle shows each row contains the r-forms of the previous row plus all these as $(r+1)$-forms. This is described by the Pfaffian with $\mathbf{a}\wedge\mathbf{b}$ components and the expansion defining the hyper-volume covered by the vectors and we have seen that this is defined by exterior algebra. Clifford algebra contains this exterior part but also contains a contraction part, as show by the 2-vector or versor[1] expanded using the fundamental product of vectors, $\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$. Each product in Pascal's recurrance relation involves a contraction giving $(r-1)$-forms in the example above. This is again a Pfaffian but consisting of dot products of all the vectors which is related to the metric tensor and for an orthonormal basis is zero.

[1]The versor is Hamilton's term for quaternion rotations. But since the multiplication of two vectors defines a rotation and the rotation plane is independent of other dimensions then it is appropriate to call it a versor

It is shown in [1, 2] that adding contractions to the external products introduces a metric on the structure of simplices described by the fundamental expansion of Clifford multivectors

$$\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3\ldots\mathbf{a}_n = \sum_{i=0}^{[\frac{n}{2}]}\sum_{\mu\in\mathcal{C}}(-1)^\sigma \setminus \mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2},\ldots,\mathbf{a}_{\mu_{2i-1}}\cdot\mathbf{a}_{\mu_{2i}}|\;\mathbf{a}_{\mu_{2i+1}}\wedge\cdots\wedge\mathbf{a}_{\mu_n} \tag{3}$$

where $\mathcal{C} = \binom{n}{2i}$ provides combinations, μ, of n indicies divided into $2i$ and $n-2i$ parts and σ is the parity of the combination. This naturally incorporates the Pfaffian trick used by de Bruijn for even simplices.

Importantly, [2] uses the general Laplace-like Pfaffian expansion to prove that the product of two multivectors expands to the same expansion of the whole multivector,

$$\mathbf{a}_1\,\mathbf{a}_2\ldots\mathbf{a}_{r+s} = (\mathbf{a}_1\,\mathbf{a}_2\ldots\mathbf{a}_r)(\mathbf{a}_{r+1}\,\mathbf{a}_{r+2}\ldots\mathbf{a}_{r+s}). \tag{4}$$

This result proves the fundamental operation of multivectors, $\tilde{A} = rMr^{-1}$, which operates on a multivector in the same way as it operators on each vector.

The Pfaffian expansion also works for the differential operator called the grad vector,

$$\nabla = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + \cdots + e_n\frac{\partial}{\partial x_n}.$$

This extends the usual vector identities to arbitrary dimensions and the Jacobian matrix becomes

$$\nabla\mathbf{a} = (\nabla\cdot\mathbf{a} + \nabla\wedge\mathbf{a}) = \frac{\partial a_j}{\partial x_i}e_ie_j,$$

which is more appropriately called the Jacobian tensor since is contains co-variant and contra-variant components. Applying this to a shifted vector $\mathbf{a}' = \mathbf{a} + \Delta\mathbf{a}$ generates an operator that reproduces the shifted vector $\mathbf{a}' = (\nabla\mathbf{a}')\mathbf{a}$.

The divergence, $\nabla\cdot\mathbf{a} = \frac{\partial a_1}{\partial x_1} + \cdots + \frac{\partial a_n}{\partial x_n}$, is the change of length of the stream vectors in all dimensions. The curl $\nabla\times\mathbf{a} = e_{321}\nabla\wedge\mathbf{a}$ generalises to the 2-form curl as all the vector product cross terms

$$\nabla\wedge\mathbf{a} = e_{12}(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}) + \cdots + e_{(n-1)n}(\frac{\partial a_n}{\partial x_{n-1}} - \frac{\partial a_{n-1}}{\partial x_n}),$$

which is seen to be the rotation of a stream as $\mathbf{a}'$ moves away from the origin.

The Pfaffian expansion (3) for a three multivector product is

$$\mathbf{abc} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}. \tag{5}$$

This can be separated as $\mathbf{abc} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}(\mathbf{b}\wedge\mathbf{c})$. The wedge product from exterior algeba is well defined as being the exterior part of any multivector product but the dot product has no equivalent since the multivector product is semi-graded with the Pfaffian expansion in (3) providing terms of grade $n, n-2, n-4, \ldots, 0$ or 1. These are divided in half by symmetric and anti-symmetric products of (4). Here we limit ourselves to expansion and contraction of a vector product with a term M of single grade m,

$$\mathbf{a}\wedge\mathbf{M} = \frac{1}{2}(\mathbf{aM} + (-1)^m\mathbf{Ma}) \quad \text{and} \quad \mathbf{a}\cdot\mathbf{M} = \frac{1}{2}(\mathbf{aM} - (-1)^m\mathbf{Ma}).$$

This means the product with scalars are external, $\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) = \mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{c})$, and since $\mathbf{a}\times\mathbf{b} = e_{321}\mathbf{a}\wedge\mathbf{b}$ then

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\wedge(e_{123}\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})e_{123}, \tag{6}$$

because when the e_{123} term is removed from the wedge product it changes from a 2-form to a 1-form thus changing the symmetry to generate a dot product. Another vector identity is $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ so the 3 multivector becomes

$$\mathbf{abc} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})e_{123}.$$

Now we can proceed to the gradient operator applied to a versor

$$\nabla\mathbf{ab} = \nabla(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}) = \nabla(|\mathbf{a}||\mathbf{b}|\cos(\phi) + |\mathbf{a}||\mathbf{b}|I_{\mathbf{ab}}\sin(\phi)), \tag{7}$$

where $I_{\mathbf{ab}} = \frac{\mathbf{a}\wedge\mathbf{b}}{\mathbf{ab}_\perp}$ and $\mathbf{b}_\perp$ is perpendicular to $\mathbf{a}$ so that $I_{\mathbf{ab}}^2 = -1$. This can be compared to the Pfaffian expansion (5) with ∇ giving change terms applicable within the plane and those exterior to it that move the plane by operating on $I_{\mathbf{ab}}$. Taking the last term of (5) then differential geometry has $\nabla\wedge\mathbf{a}\wedge\mathbf{b} = (\nabla\wedge\mathbf{a})\wedge\mathbf{b} - \mathbf{a}\wedge(\nabla\wedge\mathbf{b})$. This is called an anti-derivation because the operator must apply to both components but under the exterior product the sign changes when it is applied to $\mathbf{b}$. This is the same as the pseudoscalar (or complex) vector calculus identity $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = (\nabla\times\mathbf{a})\cdot\mathbf{b} - \mathbf{a}\cdot(\nabla\times\mathbf{b})$ using (6).

The middle two terms of (5) again use an anti-derivation which is the same as the vector calculus identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}.$$

Hence the Pffafian expansion in (5) as a derivation provides

$$\nabla\mathbf{ab} = \nabla(\mathbf{a}\cdot\mathbf{b}) - \nabla\times(\mathbf{a}\times\mathbf{b}) + \nabla\cdot(\mathbf{a}\times\mathbf{b})e_{123},$$

where the first term is the change of the $\cos$ term in (7), the middle term is the change to the $\sin$ term within the plane and the last term is the change to $\mathbf{a}\wedge\mathbf{b}$ external to the plane. This equation is equivalent to the tensor formulation with the advantage of using vector notation to expose the geometry and make the concepts more accessible to students.

The geometric interpretation is more that just a teaching aid. It is well known that Maxwell's four equations of electro-magnetism can be represented as one equation in geometric algebra. This can be done using scalars in 3-D or in space-time with the introduction of a time dimension, e_0, where $e_0^2 = -1$. This provides more information because the pseudoscalar in space-time, e_{0123}, can be used to represent the four Maxwell's equations for monopoles. The equation has the symmetry that multiplying by the space-time pseudoscalar swaps the roles of the monopole and normal 4-current and swaps the electric and magnetic fields. This leads to an equation with a solution which has a gauge transformation between the 4-current and monopole term.

Adding another space dimension, e_4, to space-time generates a commuting, imaginary pseudoscalar, e_{01234}, that can be used to provide solutions to Dirac's equation. Normally, i is used, but is there geometric information being lost in this assignment? The positive signature 7-D geometric algebra also has an imaginary, commuting pseudoscalar and this algebra has properties related to the geometry of octonions. Each time two dimensions, with one positive and one negative signature, are added to such algebras, another commuting, imaginary pseudoscalar is obtained. There are many algebras that naturally incorporate i and the geometric interpretation is not being considered. Although e_{123} is not commuting within seven dimensions, distinguishing it from $e_{1234567}$ is necessary and easily done in geometric algebra and may provide insights not easily seen in the tensor formulation.

References

[1] E. R. Cannello, Combinatorics and Renormalisation in Quantum Field Theory. Benjamin Reading, Mass., 1973.

[2] G. P. Wilmot, The structure of Clifford algebra. Journal of Mathematical Physics, **29**, 1988, pp. 2338–2345.