Construction of Exceptional Lie Algebra G2
and Non-Associative Algebras using Clifford Algebra

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Summary of the Abstract

This article uses Geometric algebra to derive octonions and the Lie exceptional algebra G2 *from calibrations. This is simpler than the usual exterior algebra derivation and uncovers an invertible element using the calibrations that is used to classify six other algebras which are found to be related to the symmetries of* G2*. The 4-form calibration terms are a subalgebra of* Spin(7) *and provide a direct construction of* G2 *for each of the* 480 *representations of the octonions. This result is extended to 15 dimensions, deriving another 93 algebras including the sedenions.*

Introduction

My poster for AGACSE(2024) introduces a one-to-one relationship between simplices and Geometric algebras (GA) which are defined to be Clifford algebras of positive signature. In n-dimensions, GA(n) can be derived from the $(n-1)$-simplex which is the simplest way of connecting n points in a connecting space with one less dimension. Hence GA can be called the algebra of geometry. The poster also shows the relationship between Pfaffians, the Spin group and simplices. The Pfaffian uses the edges of the simplex to describe connections that are usually attributed to the anti-derivation in differential geometry. This paper translates results from differential geometry into GA which exposes the relationship to the Spin group and uncovers complete classifications of algebras related to the Cayley-Dickson series. In three dimensions the face of the triangle or 2-simplex is labeled e_{123} and translates to GA as $\pm e_{123} = \pm e_1 \wedge e_2 \wedge e_3$. This 3-form also represents the quaternions under the simplex construction as either a left-hand or right-hand cycle of three 90° rotations. The same concept extends to GA(7). Differential geometry in seven dimensions uses the Fano plane with arrows to represent a certain 3-form called a calibration that represents octonions, the next Calyey-Dickson algebra after quaternions. In GA(7), the Fano plane is interpreted as a projection of the 6-simplex with arrows describing independent 3-cycle rotations. GA allows all combinations of arrows to be analysed uncovering another 6 algebras alongside the octonions. This construction also works in GA(15) which derives sedenions and another 90 parallel algebras. The 3-form calibration in GA(7) has an associative 4-form, the terms of which generate a subalgebra of Spin(7) and uncover Lie algebra G2 as automorphisms that keep certain 3-forms invariant, not just the calibrations. They are easily visualised as symmetry operations of the 6-simplex, the Cartan root diagram for G2 and can be isolated from the associative 4-forms. Some of these concepts carry over to GA(15).
The fundamental structure equation for multivector products provides the mapping between GA and Grassmann or exterior algebga [1, 2],

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\ldots\mathbf{a}_n = \sum_{i=0}^{[\frac{n}{2}]}\sum_{\mu\in\mathcal{C}}(-1)^k \mathrm{pf}(\mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2},\ldots,\mathbf{a}_{\mu_{2i-1}}\cdot\mathbf{a}_{\mu_{2i}})\mathbf{a}_{\mu_{2i+1}}\wedge\cdots\wedge\mathbf{a}_{\mu_n}, \quad (1)$$

where $\text{pf}(A)$ is the Pfaffian of A and $\mathcal{C} = \binom{n}{2i}$ provides combinations, μ, of n indicies divided into $2i$ and $n-2i$ parts and k is the parity of the combination. The Spin and Pin groups are defined by Harvey [3] as even and odd numbers of unit length vectors, respectively. Note that it is the associativity provided by the Pfaffian co-factor expansion that proves GA multivectors are invertible. Associativity also allows pairs of unit multivectors to form bivectors as rotations equivalent to quaternions so can be called versors, $R_{ij}(\theta) = cos(\frac{\theta}{2}) + e_{\text{ij}} sin(\frac{\theta}{2})$. Even multivectors can be expressed as all combinations of pairs of points, or edges of the simplex, with equivalent generalised Euler angles. Denoting 90° rotations as $R_{ij} = \frac{1}{2}(1 + e_{\text{ij}})$ then (1) provides three combinations of what can be called birotations, $R_{ij}R_{kl}, R_{ik}R_{jl}, R_{il}R_{jl}$, i, j, k, l all distinct, which will all be denoted as R_{ijkl} since they all contain e_{ijkl}. This is used later for the derivation of Lie algebra G2. The Pin group represents reflections and can be taken as a single vector or any basis element after the rotations are applied. Automorphisms are conjugations, $A' = MAM^{-1}$, where M is a multivector, giving a double covering of the orthogonal group. Note that conjugation with a single basis, $M = e_{\text{i}}$, is a reflection through all other bases.

Calibrations in GA(7)

Since the number of edges and faces in GA(7) are both divisible by 3 and 7, then 7 independent 3-cycle faces can be selected so that each edge is selected once, called Φ_1 here. This defines a cross product, $\mathbf{a} \times \mathbf{b} = \Phi_1 \mathbf{a} \wedge \mathbf{b}$. A natural or primary cross product can be defined whereby the basis index of each 3-cycle only decreases once

$$\Phi_1 = e_{123} + e_{145} + e_{167} + e_{246} + e_{257} + e_{347} + e_{356}. \tag{2}$$

This is related to what are called forms of the associator calibration by Harvey and Lawson, [4], which are provided shortly. Firstly looking at the 36 projections of the 6-simplex there are 30 ways to select 7 independent faces, $\Phi_i, 1 \leq i \leq 30$, called primaries in natural order, which can be found by enumerating all combinations of 7 of the 35 triples of seven vertices, eliminating those with duplicate edges. Alternatively, applying 90° rotations and reflections to Φ_1 can be used, as will be carried through later.
Defining the compliment to Φ_i, Φ_i^* using the pseudoscalar, generates a 4-form, with will lead to the form of the coassociative calibration [4],

$$\Phi_i^* = -e_{1234567}\, \Phi_i, \quad 1 \leq i \leq 30.$$

This form is related to the Hodge operator acting on Φ as a contraction operator of the Grassmann algebra. In GA this operation is just multiplication by the pseudovector for any dimension. Note that the compliment of a single basis provides reflections through that basis under conjugation.

Lemma. *The terms of* $\{1+\Phi_i^*\}$ *and* $\{1+\Phi_i+\Phi_i^*+e_{1234567}\}$, $1 \leq i \leq 30$, *form commuting subalgebras of* $\text{Spin}(7)$ *and* $\text{Pin}(7)$, *respectively*

Proof. Since the terms of Φ_i along with the pseudoscalar are closed, the multiplication of the terms of Φ_1^2 contain only the terms of the 4-form coassociative calibration which has terms that are easily proved to be closed

$$\Phi_1^* = e_{1247} + e_{1256} + e_{1346} + e_{1357} + e_{2345} + e_{2367} + e_{4567}. \tag{3}$$

□

The terms of Φ_1^* as rotations leave Φ_1 invariant because they commute. The permutations of the indices change the signs and follow the symmetric group S_4 that has order 24. Since there are 7 terms this is the projective group of the cube $\mathrm{PSL}(2,\mathbb{Z}_7)$, isomorphic to $\mathrm{SL}(3,\mathbb{Z}_2)$, which has order $168 = 7 \times 24$. The permutations of the 7 vertices of the 6-simplex is S_7 which has order $7! = 30 \times 168$. Thus the primary forms provide complete coverage of S_7 and the 24 permutations of 4 indices surject to the three rotations R_{ijkl}, apart from sign. All sign combinations will be considered within $2^7 = 128$ sign combinations for each primary so the factor of 8 can be removed. Also the factor of 3 for each of the 7 3-form terms provide changes of arrow directions within the 3-cycle. By considering just changes of the arrows shown in the Fano plane in Figure 1, the factor of 21 can also be removed. Thus there are $(7!/168) \times 2^7 = 3,840$ possible label and arrow combinations for the Fano plane, which are now considered.

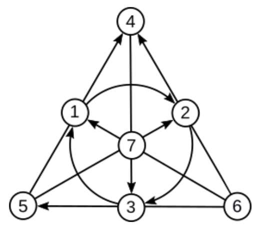

Figure 1: **Fano Plane Diagram**

The Fano plane diagram shows a projection of the 6-simplex with arrows defined by Φ_1. This diagram is not the usual representation that would define the product rules of octonions and represents another non-associative algebra that will be exposed shortly. Changing the sign of e_{146} in (2) means reversing the arrow on the right hand side of the diagram and this is now a calibration, representing the octonion multiplication rule. There are 16 representations of octonions in each primary and denoting any of the 16 as $\Phi_{i,O}$ and over all primaries as Φ_O then the GA statement of the well know exterior formulation, $\Phi \wedge \Phi^* = 7e_{1234567}$, [4], is

$$\Phi_O^2 + 7 = (-1)^\sigma 6e_{1234567}\Phi_O. \tag{4}$$

The exterior terms are the same and remaining contraction terms are related to Φ_O^* due to the Lemma. The σ factor in (4) is the number of minus signs applied to the primary and defines a sign parity for the octonions as 8 $\mathbb{O}^+$ and 8 $\mathbb{O}^-$ algebras for σ even or odd respectively.

Definition. $\Phi_{i,j}$, for $1 \le i \le 30$ and $1 \le j \le 128$ is all signed combinations of the 7 terms of each Φ_i, with the sequence startng with all positive terms, followed by 7 single minus terms, etc. Thus $\Phi_{1,5} = \Phi_O$ denotes the $\mathbb{O}^-$ algebra with e_{246} negated in Φ_1, as discussed above. It is only necessary to consider the first half of 3-forms because the second half is just the negation of the first half which swaps the $\mathbb{O}^+$ and $\mathbb{O}^-$ classes, $\Phi_{i,129-j} = -\Phi_{i,j}$ and $\sigma_{129-j} = 7 - \sigma_j$ for $1 \le j \le 64$.

Definition. $\rho_{i,j} = \frac{1}{4}(3e_{1234567} - (-1)^\sigma \Phi_{i,j})$, where, if $\Phi_{i,j} = \Phi_O$ then $\rho_{i,j} = \rho_O$.
This is a better way to represent (4) since ρ_O is invertible, $\rho_O^{-1} = -\rho_O$ because $\rho_O^2 = -1$, and it allows all $\Phi_{i,j}$ to be classified as one of 6 algebras, $\mathbb{S}_2, \mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_7$ or $\mathbb{S}_8$, with 4, 8, 10, 12, 14 or 16 unique non-associative triple products, respectively. These are called the sub-octonion algebras for reasons discussed later.

Theorem 1. *Classification Theorem*

$$2e_{1234567}(\rho_{i,j}^2+1)=\begin{cases}0, & or\\ \pm\phi_k+(-1)^\sigma\Phi_{i,j}\end{cases} \tag{5}$$

where ϕ_k, $1\le k\le 7$, called the remainder, is one of the terms of Φ_i. For convenience the 0 is also called the remainder and represents octonions. The classification scheme for non-octonion algebras is shown in Table 1.

Table 1: Primary Table and Classification Map

i	$\mathbf{\Phi_i}$	**Classes**	**Remainder**
1	$e_{123}+e_{145}+e_{167}+e_{246}+e_{257}+e_{347}+e_{356}$	$(\mathbb{S}_4,2\mathbb{S}_{12},4\mathbb{S}_{14})$	$-e_{246}$
2	$e_{123}+e_{145}+e_{167}+e_{247}+e_{256}+e_{346}+e_{357}$	$(\mathbb{S}_4,2\mathbb{S}_{12},4\mathbb{S}_{14})$	$-e_{357}$
3	$e_{123}+e_{146}+e_{157}+e_{245}+e_{267}+e_{347}+e_{356}$	$(\mathbb{S}_4,2\mathbb{S}_{14},2\mathbb{S}_{12},2\mathbb{S}_{14})$	$-e_{157}$
4	$e_{123}+e_{146}+e_{157}+e_{247}+e_{256}+e_{345}+e_{367}$	$(\mathbb{S}_4,4\mathbb{S}_{14},2\mathbb{S}_{12})$	$-e_{146}$
5	$e_{123}+e_{147}+e_{156}+e_{245}+e_{267}+e_{346}+e_{357}$	$(\mathbb{S}_4,2\mathbb{S}_{14},2\mathbb{S}_{12},2\mathbb{S}_{14})$	$-e_{346}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Proof. The proof does not show a class equivalence between the remainder and the sub-octonion algebras but demonstrates how the progression over the first 8 $\Phi_{1,j}$, $1\le j\le 8$, using automorphisms from the derived Cayley table permutations, steps through each remainder and verifies the number of non-associative triple products. This is repeated for each primary to uncover 6 unique algebras. It is straight forward to prove complete coverage over all $\Phi_{i,j}$, $9\le j\le 128$, by finding series with the same remainder, and hence the same non-associativity, including octonions. □

The reason why $\rho_{1,5}$ is invertible is because $\Phi^*_{1,5}=1-8\sigma$ where $\sigma=\frac{1}{8}(1-e_{1247})(1+e_{1357})(1-e_{4567})$. These are 3 independent idempotents defining the projection operator $\sigma^2=\sigma$ which means

$$\begin{aligned}(\rho^*_{1,5})^2 &= (\tfrac{1}{4}(3+\Phi^*_{1,5}))^2\\ &= (1-2\sigma)^2\\ &= 1.\end{aligned}$$

This works for most of the 35 combinations of 3 terms from $\Phi^*_{i,O}$ for all calibrations. It does not work for the sub-octonion algebra representations nor for positions $(1,2,7)$, $(1,3,6)$, $(1,4,5)$, $(2,3,5)$, $(2,4,6)$, $(3,4,7)$ or $(5,6,7)$.

In GA(15) using hexadecimal so e_{F} is the 15$^{\text{th}}$ dimension and $e_{123456789\mathrm{ABCDEF}}$ is the imaginary pseudoscalar then the first primary is

$$\begin{aligned}\Phi &= e_{123}+e_{145}+e_{167}+e_{189}+e_{1\mathrm{AB}}+e_{1\mathrm{CD}}+e_{1\mathrm{EF}}+e_{246}+e_{257}+e_{28\mathrm{A}}+e_{29\mathrm{B}}+e_{2\mathrm{CE}}\\ &\quad +e_{2\mathrm{DF}}+e_{347}+e_{356}+e_{38\mathrm{B}}+e_{39\mathrm{A}}+e_{3\mathrm{CF}}+e_{3\mathrm{DE}}+e_{48\mathrm{C}}+e_{49\mathrm{D}}+e_{4\mathrm{AE}}+e_{4\mathrm{BF}}+e_{58\mathrm{D}}\\ &\quad +e_{59\mathrm{C}}+e_{5\mathrm{AF}}+e_{5\mathrm{BE}}+e_{68\mathrm{E}}+e_{69\mathrm{F}}+e_{6\mathrm{AC}}+e_{6\mathrm{BD}}+e_{78\mathrm{F}}+e_{79\mathrm{E}}+e_{7\mathrm{AD}}+e_{7\mathrm{BC}}.\end{aligned}$$

Following the procedure above generates 93 unique algebras and one with a maximal 252 non-associative triple products. This corresponds to the middle term, $e_{48\mathrm{C}}$, being negated which is analogous to $\Phi_{1,5}$ and in this case generates the sedenion algebra, $\mathbb{S}$, which has 252 non-associative triple products. Sedenions are power-associative so have zero divisors. The split octonions have 24 zero divisors related to the idempotents and 12 of these do not involve the number one. The sub-octonion algebras are also power associative and, although having different numbers of non-associative triples, each have 12 overlapping, zero divisors. Hence the name sub-octonion. The 90 algebras alongside the sedonions could be called sub-sedenion algebras.

Construction of G2

The first birotation from R_{1234} is $R_{12}R_{34} = \frac{1}{2}(1 + e_{12} + e_{34} + e_{1234})$. This separates into $\alpha = \frac{1}{2}(1+e_{1234})$ and $\beta = \frac{1}{2}(e_{12}+e_{34})$ so that $R_{12}R_{34} = \alpha+\beta$. Note that R_{12} and R_{47} are symmetric in the Fano diagram. Applying one generates another primary but applying the next returns to the previous primary. This can be seen as a symmetry of the Fano plane. Alternatively, they both have the same action so that $\beta = (e_{12} + e_{47})/2$ acts identically. This is true in general otherwise R_{1234} would be inconsistent. It is easy to see that for any R_{ijkl}, i, j, k, l all distinct and e_{ijkl} a term of Φ_i^*, then α commutes with Φ_i and $\beta\Phi_i = \Phi'\beta$ where $\Phi'_i = \Phi_{i,j}$ for some j. This provides the following identities for any R_{ijkl} with distinct indices, $\alpha^2 = \alpha$, $\beta^2 = \alpha - 1$, $\alpha\beta = \beta\alpha = 0$. We can also prove that $\beta\Phi'_{i,O} = \beta\Phi_{i,O}$.

Theorem 2. *Automorphism Theorem*

$$\begin{aligned} R_{jklm}\Phi_{i,O}R_{mlkj} &= \Phi'_{i,O}\,, \quad \textit{where } e_{\mathrm{jklm}} \textit{ is a term of } \Phi_i^*, \textit{ and} \\ \Phi'_{i,O} &= \Phi_{i,O}, \quad \textit{if } e_{\mathrm{jklm}} \textit{ is a term of } \Phi_{i,O}^* \end{aligned}$$

Proof.

$$\begin{aligned} R_{jk}R_{lm}\Phi_{i,O}R_{ml}R_{kj} &= (\alpha+\beta)\Phi_{i,O}(\alpha-\beta) \\ &= \alpha\Phi_{i,O} - \beta^2\Phi'_{i,O} \\ &= \alpha\Phi_{i,O} - (\alpha-1)\Phi'_{i,O} \\ &= \Phi'_{i,O} \end{aligned}$$

With β having the correct parity then $\beta\Phi_{i,O} = \Phi_{i,O}\beta$. □

This means α plays no part in the transformation and β can be isolated. Defining a parity for β leads to the construction of Lie algreba G2 shown in Table 2 for $\Phi_{1,64}$. Such a construction leads to a unique identification of the terms of G2 apart from overall sign showing that the GA construction is a sufficient definition of G2.

Table 2: G2 Relationship to Φ_1^*

$\mathbf{\Phi^*_{1,64}}$ term	**Normal Rotations**	**Mixed Rotations**	**Outer Rotations**
e_{1247}	$C = (e_{12} + e_{47})/2$	$E = (e_{14} - e_{27})/2$	$F = (e_{17} + e_{24})/2$
e_{1256}	$C + J = (e_{12} + e_{56})/2$	$-D = (e_{15} - e_{26})/2$	$-G = (e_{16} + e_{25})/2$
e_{1346}	$-B = (e_{13} + e_{46})/2$	$E - L = (e_{14} - e_{36})/2$	$-G - N = (e_{16} + e_{34})/2$
$-e_{1357}$	$-B - I = (e_{13} - e_{57})/2$	$-D - K = (e_{15} + e_{37})/2$	$-M = (e_{17} - e_{35})/2$
$-e_{2345}$	$A = (e_{23} - e_{45})/2$	$F + M = (e_{24} + e_{35})/2$	$N = (e_{25} - e_{34})/2$
$-e_{2367}$	$A + H = (e_{23} - e_{67})/2$	$-K = (e_{26} + e_{37})/2$	$-L = (e_{27} - e_{36})/2$
$-e_{4567}$	$H = (e_{45} - e_{67})/2$	$I = (e_{46} + e_{57})/2$	$-J = (e_{47} - e_{56})/2$

The Bryant form for the representation of the octonions, [5], can be partially mapped to the Cartan form via matrix representations to give, for example,

$$E - L = X_1 - Y_1, D + K = X_2 - Y_2, H = Y_3 - X_3.$$

These partially match the symmetry of the roots of the Cartan Root diagram for G2. From the classification theorem, if there is a remainder then only this remainder commutes with the invertible form and hence the sub-octonion algebra. This represents one of the six axes on the root diagram. But if there is no remainder then all Φ_i^* terms commute and we get octonions and the full symmetry of the G2 root diagram.

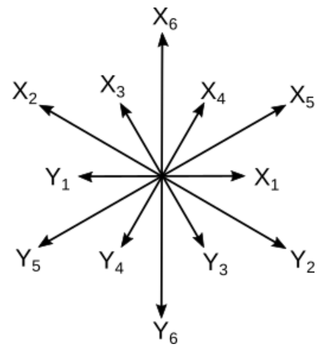

Figure 2: **Cartan Root Diagram for** G2

Summary

The diffential geometry notation and formulations can be expressed by GA and its connection to the geometry of simplices provides further insights. It is further enhanced by the invertible form related to the calibrations that uncovers a relationship to Spin(7) and a connection to three factor spinors. This extends the Pauli and Dirac spinors to a relationship with octonions. There are intriguing extensions of these results to GA(15) and sedenions and potentially GA(31), which also has an imaginary pseudoscalar.
The Pfaffian validates the formulation of the Spin group and leads to a derivation of Lie algebra G2. Again the geometric insights are apparent and the connection to the sub-octonion algebras as partial G2 symmetries indicates that there is much more to be learnt. This includes an understanding of zero divisors which are related to ideals of the algebra for split octonions but this is not the case for the sub-octonions.
This work was verified with the use of geometric algebra, quaternion and octonion/sedenion calculators written in Python. Quaternions were used to verify the Clifford calculator using Euler angles and these generalise to 15 dimensions and have been compared to matrix expansions of rotations for up to 7 dimensions. The github URL for the calculators is https://github.com/GPWilmot/geoalg.

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