

# HAMILTONIAN MECHANICS IN TERMS OF GEOMETRIC ALGEBRAS

Aleš Návrat and Petr Vašík<sup>a</sup>

<sup>a</sup>Institute of Mathematics, Faculty of Mechanical Engineering  
Brno University of Technology  
Technická 2896/2, 616 69 Brno, Czech Republic  
ales.navrat@vutbr.cz, petr.vasik@vutbr.cz

## Summary of the Abstract

*This presentation aims to formulate Hamilton's equations in terms of any GA for any Hamiltonian, not just the Hamiltonian for the rigid body motion. Actually, one generalizes this approach by replacing  $SE(3)$  by an arbitrary Lie group  $G$  and by defining Hamiltonian by an arbitrary function on its cotangent bundle. The form of the Hamilton's equations in such cases is well-known, and if  $G$  is formed by rotors of a geometric algebra, we can directly translate this general result for dynamics on Lie groups into the GA language. The case of the rigid body corresponds to the choice of geometric algebra PGA and  $G = SE(3)$ . Of course, we may choose  $SE(n)$  to obtain an  $n$ -dimensional version of the rigid body motion. But not just that, we may freely choose Hamiltonian to describe various interactions, and we may also change geometric algebra, for example, to CGA, which then leads to Lie group  $G = SO(n + 1, 1)$ . In particular, we get a description of the rigid body motion in CGA, and we also get its generalization to an "elastic body motion".*

## Extended abstract

It is well known that geometric algebra (GA) offers efficient representations for geometric transformations and operations, making it valuable for computer graphics tasks such as modeling, animation, and rendering. It also is widely used in robotics for kinematic modeling, control, and motion planning. It enables the representation and manipulation of robot poses and trajectories in a way that is both intuitive and computationally efficient. But in all these applications, GA was only treated as a tool for solving the kinematics or solving geometric problems in the usual mathematical sense.

Recently, S. De Keninck & L. Dorst, [5], used geometric algebra PGA to model the classical Newtonian mechanics. They solved the problem of Rigid Body Dynamics by an elegant treatment of Newton's laws that is valid in any number of dimensions and unifies linear and angular aspects completely. Namely, the motion is characterized by a *motor*  $M$ ; an invertible PGA element that represent a combination of rotation and translation. The kinematics equation reads

$$\dot{M} = -\frac{1}{2}MB_b, \quad (1)$$

where  $B_b$  is a bivector that represents a generator of the motion in the body frame. Its time evolution depends on the mass distribution and on the forces and torques acting on the body. Namely, the dynamics equation reads

$$\dot{B}_b = I_b^{-1}(B_b \times I_b(B_b) + F_b), \quad (2)$$

where  $\times$  is the geometric algebra commutator product defined by  $a \times b = 1/2(ab - ba)$  and where  $F_b = \widetilde{M}F_wM$  is an element in PGA that uniformly represents the total forces and torques in the body frame. For a rigid body consisting of a set of discrete points  $m_iX_i$ , the total body inertia map  $I_b$  is given by

$$I_b(B) = \sum_i m_i X_i \vee (X_i \times B) \quad (3)$$

and the Lagrangian of the system can be described as  $L = \frac{1}{2}B \vee I(B)$ . Alternatively, one can find the above equations of motion by taking this Lagrangian as an input and by using the Lagrangian or Hamiltonian formalism. Namely, the configuration space of a rigid body is the Euclidean Lie group

$$SE(3) = SO(3) \times \mathbb{R}^3,$$

and we may view the rigid body motion as a solution of the optimal control problem

$$\int L dt \rightarrow \min,$$

where  $L : TSE(3) \rightarrow \mathbb{R}$  is the Lagrangian function. This approach is known in physics as the principle of least action. By the Pontryagin maximum principle, the solution satisfies Hamilton's equations defined by the Hamiltonian associated with Lagrangian  $L$ . Indeed, as we shall see, we get equations equivalent to (1) and (2) in this way.

This paper adopts this point of view and aims to formulate Hamilton's equations in terms of any GA for any Hamiltonian, not just the Hamiltonian for the rigid body motion. Actually, one generalizes this approach by replacing  $SE(3)$  by an arbitrary Lie group  $G$  and by defining Hamiltonian by an arbitrary function on its cotangent bundle. The form of the Hamilton's equations in such cases is well-known, and if  $G$  is formed by rotors of a geometric algebra, we can directly translate this general result for dynamics on Lie groups into the GA language. The case of the rigid body corresponds to the choice of geometric algebra PGA and  $G = SE(3)$ . Of course, we may choose  $SE(n)$  to obtain an  $n$ -dimensional version of the rigid body motion. But not just that, we may freely choose Hamiltonian to describe various interactions, and we may also change geometric algebra, for example, to CGA, which then leads to Lie group  $G = SO(n + 1, 1)$ . In particular, we get a description of the rigid body motion in CGA, and we also get its generalization to an "elastic body motion". We demonstrate our results in Ganja.js.

## 0.1 Classical Hamiltonian dynamics

The common practice in studying Lagrangian and Hamiltonian dynamics on a manifold is first to introduce local coordinates on the manifold and then formulate and analyze the dynamics in terms of these local coordinates. The classical Lagrangian dynamics

is formulated in terms of a function on this tangent bundle, the so-called Lagrangian function  $L : T\mathbb{R}^n \rightarrow \mathbb{R}$ , which may contain all physical information concerning the system and the forces acting on it. According to Hamilton's principle, the evolution of a physical system between two specified states  $q_0 = q(t_0)$  and  $q_1 = q(t_1)$  is then determined by a trajectory  $q(t)$  in the configuration space that is a stationary point of the action functional

$$\mathcal{S}[q] = \int_{t_0}^{t_1} L(q, \dot{q}) dt. \quad (4)$$

Computing the first variation, one finds that this requirement is equivalent to well-known Euler–Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \quad (5)$$

The Hamiltonian form of the equations of motion is given in terms of conjugate momentum covectors. The passage from the tangent bundle to the cotangent bundle is realized by the Legendre transformation  $T\mathbb{R}^n \rightarrow T^*\mathbb{R}^n : (q, \dot{q}) \mapsto (q, p)$ , where  $p \in T_q^*\mathbb{R}^n$  is the conjugate momentum defined by

$$p = \frac{\partial L}{\partial \dot{q}}. \quad (6)$$

Note that we assume throughout the article that the Legendre transformation is globally invertible. This property is usually referred to as the hyperregularity of the Lagrangian function in literature. The Hamiltonian dynamics is then equivalent to the Lagrangian and can be described by introducing the Hamiltonian function  $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$ :

$$H(q, p) = p \cdot \dot{q} - L(q, \dot{q}), \quad (7)$$

where  $\dot{q}$  is viewed as a function of  $(q, p)$  by inverting the Legendre transformation. Namely, the Euler-Lagrange equations (5) are equivalent to the well-known Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad (8)$$

$$\dot{p} = -\frac{\partial H}{\partial q}. \quad (9)$$

## 0.2 Hamiltonian Dynamics on Lie Groups

Although this local approach is common practice, there are many deficiencies. In the case that the configuration space is a Lie group  $G$ , it is possible to write down a global coordinate-free form of Hamilton's equations by using the global trivializations of the tangent bundle and the cotangent bundle. Let  $\mathfrak{g}$  be the associated Lie algebra. The tangent map (differential) at the identity to left translation map  $\ell_g : G \rightarrow G$ , defined as  $\ell_g(h) = gh$ , defines a map  $(\ell_g)_* : T_e G = \mathfrak{g} \rightarrow T_g G$  from the Lie algebra to a tangent space to  $G$ . It is easy to see that this map has the inverse  $(\ell_{g^{-1}})_* : T_g G \rightarrow \mathfrak{g}$  and thus is a linear isomorphism

$$(\ell_{g^{-1}})_* : TG \cong G \times \mathfrak{g}. \quad (10)$$

In other words, by left trivialization, we can identify the tangent bundle  $TG$  of a Lie group  $G$  with  $G \times \mathfrak{g}$ . Similarly, we can use the cotangent map to left translation  $(\ell_g)^* : T_g^*G \rightarrow T_e^*G = \mathfrak{g}^*$ , defined by  $(\ell_g)^*(\mu)(\xi) = \mu((\ell_g)_*(\xi))$  for each  $\xi \in \mathfrak{g}$  and  $\mu \in T_g^*G$ , to get a global trivialization of the cotangent bundle  $T^*G$ :

$$(\ell_g)^* : T^*G \cong G \times \mathfrak{g}^*. \quad (11)$$

The next ingredient we need to formulate the Hamilton's equations on a Lie group  $G$  is the coadjoint operator, for each  $\mu \in \mathfrak{g}^*$  and  $\xi, \zeta \in \mathfrak{g}$ . defined by

$$\text{ad}_\xi^*(\mu)(\zeta) = \mu(\text{ad}_\xi(\zeta)) = \mu([\xi, \zeta]), \quad (12)$$

where  $[\cdot]$  denotes the Lie bracket in  $\mathfrak{g}$ . Now we are ready to formulate a coordinate-free Hamiltonian dynamics on Lie group  $G$ . Using the identification (11), the Hamiltonian function may be viewed as

$$H : G \times \mathfrak{g}^* \rightarrow \mathbb{R}.$$

Its derivative with respect to the second argument, which we write as  $\partial H / \partial \mu$ , is a linear map  $\mathfrak{g}^* \rightarrow \mathbb{R}$ , therefore it can be naturally considered as an element of  $(\mathfrak{g}^*)^* \cong \mathfrak{g}$ . The derivative of the Hamiltonian function with respect to the first (group) argument, which we write as  $\partial H / \partial g$ , may be seen as a directional derivative and thus as a vector in the cotangent space  $T_g^*G$ .

**Proposition 0.3** (*Hamilton's equations on a Lie group*) *Let  $G$  be a Lie group,  $\mathfrak{g}$  its associated Lie algebra, and let  $\xi = \partial H / \partial \mu \in \mathfrak{g}$ . The Hamilton's equations on  $G$  are a system on  $G \times \mathfrak{g}^*$  given by*

$$\dot{g} = (\ell_g)_*\xi, \quad (13)$$

$$\dot{\mu} = \text{ad}_\xi^* \mu - (\ell_g)^* \frac{\partial H}{\partial g}. \quad (14)$$

The Hamilton's equations have a particularly simple form in the case that the Hamiltonian function (or equivalently the Lagrangian function) is left-invariant, i.e. as a function on  $G \times \mathfrak{g}^*$  it depends only on the second coordinate and not on the group coordinate. In such a case, the Hamilton's equations simplify to a decoupled coordinate-free system on  $G$  and on  $\mathfrak{g}^*$  known as the *Lie-Poisson equation*

$$\begin{aligned} \dot{g} &= (\ell_g)_*\xi, \\ \dot{\mu} &= \text{ad}_\xi^* \mu. \end{aligned}$$

It means that one can solve first the dynamics part for  $\mu$ , and then substitute the result into the kinematics part, where  $\xi = \partial H / \partial \mu$  is given by the inverse Legendre transformation  $\mathfrak{g}^* \rightarrow \mathfrak{g}$ . The equivalent system in the Lagrange formalism is known as the *Euler-Poincaré equation*. It is obtained from (??) using the Legendre transformation  $\mu = \partial L / \partial \xi$ .

## 0.4 GA formulation of Hamilton's equations

The general form of Hamilton's equations on a Lie group given above allows a direct translation into the GA language if  $G = \mathbb{G}(V)$  is the Lie group of rotors in a geometric

algebra induced on a vector space  $V$  equipped with a quadratic form. In the case of a non-degenerate form of signature  $(p, q)$ , the Lie group  $G$  is a covering of the orthogonal Lie group  $SO(p, q)$ . If we allow a degenerate quadratic form with a kernel of dimension  $r$ , the resulting geometric algebra can be embedded into the geometric algebra induced by a non-degenerate quadratic form of signature  $(p + r, q + r)$ . Hence  $G$  is a covering of a Lie subgroup of  $SO(p + r, q + r)$ . In any case, the associated Lie algebra is the algebra of bivectors,

$$\mathfrak{g} \cong \Lambda^2 V.$$

The dual Lie algebra  $\mathfrak{g}^* \cong \Lambda^2 V^*$  can be identified with  $\mathfrak{g}$  in the non-degenerate case. Namely, the defining bilinear form extends then extends to a non-degenerate bilinear form on bivectors via  $\langle A, B \rangle = \langle A\tilde{B} \rangle_0$ , where  $\langle \rangle_0$  denotes the projection to the scalar part and  $\tilde{B}$  is the reversion of  $B$ . Another possibility of viewing the dual algebra  $\mathfrak{g}^*$  in GA is via a pseudoscalar  $I$  ( or volume form). Namely, if we associate a  $\mu^\sharp \in \Lambda^{n-2} V$  to each  $\mu \in \mathfrak{g}^* \cong \Lambda^2 V^*$  such that  $\mu^\sharp \wedge B = \mu(B)I$  for each bivector  $B \in \Lambda^2 V$ , we get an isomorphism of Lie algebras

$$\mathfrak{g}^* \cong \Lambda^{n-2} V,$$

where the Lie bracket on the right-hand side is given by the commutator with respect to the geometric product in  $\mathbb{G}(V)$ . It is easy to show that, in this identification, the coadjoint action corresponds to a commutator in GA of a bivector and an element of grade  $n - 2$ ,

$$(\text{ad}_\xi^* \mu)^\sharp = [\mu^\sharp, \xi].$$

From now on, we will omit writing the sharp symbol, and we will consider an element of  $\mathfrak{g}^*$  as an element of  $\Lambda^{n-2} V$ . Then the Hamiltonian is a function

$$H : G \times \Lambda^{n-2} V \rightarrow \mathbb{R},$$

the inverse Legendre transformation gives a bivector

$$B = \frac{\partial H}{\partial \mu} \in \Lambda^2 V \tag{15}$$

while the derivative of the Hamiltonian function with respect to the group variable, expressed in the right trivialization is an element

$$F_s = \frac{\partial H}{\partial M} \tilde{M} \in \Lambda^{n-2} V.$$

This term corresponds in the Newtonian picture to forces that are expressed in the world frame. Then, the Hamiltonian dynamics on the Lie group  $G$  in GA language look as follows.

**Proposition 0.5** *Let  $\mathbb{G}(V)$  be a geometric algebra over a quadratic space  $V$  of dimension  $n$  and of any signature, and let  $G$  be the Lie group of its invertible elements. The Hamilton's equations on  $G$  read*

$$\begin{aligned} \dot{M} &= MB \\ \dot{\mu} &= [\mu, B] - \tilde{M} F_s M \end{aligned}$$

where  $M \in G$ ,  $\mu \in \Lambda^{n-2} V$ ,  $B = \partial H / \partial \mu$  is the bivector obtained by the inverse Legendre transformation, and where  $[\cdot, \cdot]$  is the commutator with respect to the geometric product in  $\mathbb{G}(V)$ .

## 0.6 Examples

Our basic example is the rigid body motion. A suitable geometric algebra for its description is PGA since its rotors form the Lie group  $G = SE(3)$ . Indeed, PGA is generated by vector space  $V$  and quadratic form of degenerate signature  $(3, 0, 1)$ , i.e. with basis  $(e_0, e_1, e_2, e_3)$  such that

$$e_1^2 = e_2^2 = e_3^2 = 1 \text{ and } e_0^2 = 0.$$

Due to the one-dimensional kernel generated by  $e_0$ , the bivectors form Lie algebra  $\mathfrak{g} = \mathfrak{se}(3) = \text{span}\{e_0 \wedge e_i, e_i \wedge e_j\}$ . Hence any bivector is of a form

$$B = \sum v_i e_0 \wedge e_i + \sum \omega_k e_i \wedge e_j,$$

where we assume that  $k$  is the complementary index to indices  $i, j$ . For the dual Lie algebra we have  $\mathfrak{g}^* = \text{span}\{e_i \wedge e_j, e_0 \wedge e_i\}$ . Actually, it has the same structure as  $\mathfrak{g}$ , and a dual bivector can be written as

$$\mu = \sum p_i (e_0 \wedge e_i)^* + \sum \ell_k (e_i \wedge e_j)^*,$$

where the star denotes the usual PGA duality. The dependence of rigid body Lagrangian  $L(M, B)$  on the fiber variable is given by

$$L(, B) = \frac{1}{2} m v^2 + \frac{1}{2} \sum J_k \omega_k^2$$

and thus the Legendre transformation  $\mathfrak{g} \rightarrow \mathfrak{g}^*$  is given by linear function

$$I(B) = \frac{\partial L}{\partial B} = m v_i (e_0 \wedge e_i)^* + J_k \omega_k (e_i \wedge e_j)^*.$$

The corresponding Hamiltonian reads

$$H(, \mu) = p_i \frac{p_i}{m} - \frac{1}{2} m \left(\frac{p_i}{m}\right)^2 + \frac{1}{2} \sum J_k \left(\frac{\ell_k}{J_k}\right)^2 = \frac{p^2}{2m} - \sum \frac{\ell_k^2}{2J_k}$$

and the inverse Legendre transformation is given by the (also linear) function

$$I^{-1}(\mu) = \frac{\partial H}{\partial \mu} = \frac{p_i}{m} e_0 \wedge e_i + \frac{\ell_k}{J_k} e_i \wedge e_j.$$

Due to the linearity, the Hamilton's equations can be written as

$$\begin{aligned} \dot{M} &= MB \\ I(\dot{B}) &= [I(B), B] + \tilde{M} \frac{\partial H}{\partial M} M \end{aligned}$$

which is equivalent to the equations in [5] up to conventions concerning signum and factor 1/2. If we view the Euclidean group  $SE(3)$  as a subgroup of  $SO(4, 1)$ , and we take the same Hamiltonian, we get a description of rigid body motion in terms of geometric algebra CGA. Such conformal description also allows a generalization to the motion of an "elastic body." Due to the existence of two null vectors  $e_0^2 = e_\infty^2 = 0$ , the lie algebra has extra elements. In particular, it is  $e_0 \wedge e_\infty$  that generates scaling. We will discuss this case and show some implementations in Ganja.js (A versatile and multiplatform Algebra generator with a focus on education and visualization available at [BiVector.net](http://BiVector.net))

## References

- [1] A. Agrachev, D. Barilari, U. Boscain, *A comprehensive introduction to sub-Riemannian geometry. From the Hamiltonian viewpoint*, Cambridge Studies in Advanced Mathematics, Vol. 181, Cambridge University Press, (2020)
- [2] T. Lee, M. Leok, N. H. McClamroch *Global Formulations of Lagrangian and Hamiltonian Dynamics on Manifolds A Geometric Approach to Modeling and Analysis* , Interaction of Mechanics and Mathematics, Springer, 1860-6245 (2017)
- [3] A. M. Bloch, *Nonholonomic Mechanics and Control*, Interdisciplinary Applied Mathematics, Springer, 0939-6047 (2010)
- [4] Leo Dorst and Steven De Keninck. *Guided tour to the plane-based geometric algebra PGA* (version 2.0), February 2022. Available at [bivector.net/PGA4CS.html](https://bivector.net/PGA4CS.html)
- [5] L. Dorst, S. De Keninck, *May the Forque Be with You*, <https://bivector.net/PGADYN.html>
- [6] Lasenby, Anthony. (2011). *Rigid Body Dynamics in a Constant Curvature Space and the '1D-up' Approach to Conformal Geometric Algebra*. Guide to Geometric Algebra in Practice. Springer, London.