Inverse of a Multivector

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Abstract

In this presentation, we will discuss the problem of finding multiplicative inverses of multivectors in non-degenerate Clifford algebras. We will start by considering well known examples of Clifford algebras such as complex numbers, quaternions and split complex numbers and notice a pattern which we get in the formulas for their inverses. This motivates the formulas for inverse of multivector in other Clifford algebras. We discuss known formulas for inverses in $\mathcal{C}\ell_{p,q}$ with $p+q \leq 6$ and their proofs using interesting techniques such as matrix representations and quaternion typification. We also discuss some special cases of inverses of paravectors, dual of a paravector and sum of a paravector and its dual which have a formula for their inverse independent of the values p and q. We will look at some isomorphisms between different Clifford algebras and use formula for inverse in a smaller dimension Clifford algebra to find inverses of multivectors in larger dimension Clifford algebras. We will also see why inverses of grade 4 elements are particularly important and existence of certain 'trouble causing' subalgebras which do not allow for the possibility of the formula of inverse of a multivector being a single product in $\mathcal{C}\ell_{p,q}$ with $p+q \geq 6$.

1 Recall the Clifford algebra

Let $p, q \in \mathbb{Z}_{\geq 0}$ and $n := p+q$. By $\mathcal{C}\ell_{p,q}$, we mean the usual thing, the (non-degenerate) Clifford algebra which is the $\mathbb{R}$ linear space

$$\mathrm{Span}_{\mathbb{R}}\{e_A | A \subseteq \{1, 2, \ldots, n\}\}$$

where $e_{\{i_1,i_2,\ldots,i_l\}} := e_{i_1 i_2,\ldots,i_l} := e_{i_1} e_{i_2} \cdots e_{i_l}$ for $1 \leq i_1 < i_2 < \ldots < i_l \leq n$ and $e_{\{\}} := 1$, the multiplicative identity. We call $e_{\{i_1 i_2,\ldots,i_k\}}$ as basis elements of length k and 1 as basis element of length 0. The generators $\{e_1, e_2, \ldots, e_n\}$ obey $e_i e_j + e_j e_i = 2\eta_{ij} 1$ where

$\eta = [\eta_{ij}]_{n\times n}$ is the diagonal matrix $\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ with the first p entries as $+1$, the last q entries as -1. A general element U of $\mathcal{C}\ell_{p,q}$ will be expressed in the form:

$$U = u + \sum_{1\le i\le n} u_i e_i + \sum_{1\le i_1<i_2\le n} u_{i_1 i_2} e_{e_{i_1} e_{i_2}} + \ldots + \sum_{1\le i_i<i_2<\ldots<i_{n-1}\le n} u_{i_1 i_2 \ldots i_{n-1}} e_{i_1 i_2 \ldots i_{n-1}} + u_{12\ldots n} e_{12\ldots n}$$

where $u, u_i, \ldots, u_{12\ldots n} \in \mathbb{R}$.

It is clear that $\mathcal{C}\ell_{p,q}$ as a $\mathbb{R}$-vector space has dimension 2^n. Let $k \in \{0,1,2,\ldots,n\}$. $\mathcal{C}\ell^k_{p,q}$ denotes the subspace of $\mathcal{C}\ell_{p,q}$ spanned by basis elements of length k and is called grade k subspace of $\mathcal{C}\ell_{p,q}$. The elements of $\mathcal{C}\ell^k_{p,q}$ are called elements of grade k. Note that $\mathcal{C}\ell^0_{p,q} \cong \mathbb{R}$, thus we will call elements of $\mathcal{C}\ell^0_{p,q}$ as real numbers. We denote projection of $U \in \mathcal{C}\ell_{p,q}$ onto a subspace of grade k by $\langle U\rangle_k$.

Let $I := e_{12\ldots n}$ be the pseudoscalar in $\mathcal{C}\ell_{p,q}$. Let $U \in \mathcal{C}\ell_{p,q}$. By the dual of U, we mean the multivector IU.

2 The problem

The problem is simple to state: Let $p, q \in \mathbb{Z}_{\ge 0}$. In general, every non-zero element $U \in \mathcal{C}\ell_{p,q}$ is not invertible (i.e., does it does not have a multiplicative inverse). Find out a way to check if a given $U \in \mathcal{C}\ell_{p,q}$ is invertible or not and if an element $U \in \mathcal{C}\ell_{p,q}$ is invertible, find its inverse.

3 Inspiration for the solution

We start by looking at some well known examples of Clifford algebras which have been studied independently.

1) $\mathcal{C}\ell_{0,1}$ is isomorphic to complex numbers which is a field. Thus, every non-zero element $z = \mathrm{a} + \mathrm{b}i$ is invertible and its inverse is $\frac{\overline{z}}{z\overline{z}}$ where $\overline{z} = \mathrm{a} - \mathrm{b}i$ denotes the complex conjugate of z. As $z = 0 \iff z\overline{z} = 0$, the criteria for checking if an element is invertible is same as checking if $z\overline{z} \neq 0$.

2) $\mathcal{C}\ell_{0,2}$ is isomorphic to real Hamilton quaternions which is a division ring. Thus, every non-zero element $q = \mathrm{a} + \mathrm{b}i + \mathrm{c}j + \mathrm{d}k$ is invertible and its inverse is $\frac{\overline{q}}{q\overline{q}}$ where $\overline{q} = \mathrm{a} - \mathrm{b}i - \mathrm{c}j - \mathrm{d}k$ denotes the quaternion conjugate of q. As $q = 0 \iff q\overline{q} = 0$, the criteria for checking if an element is invertible is same as checking if $q\overline{q} \neq 0$.

3) $\mathcal{C}\ell_{1,0}$ is isomorphic to split complex numbers which unlike the above two examples, doesn't have inverse of all non-zero elements. But, the criteria to check if an element $U = \mathrm{a} + \mathrm{b}\eta$ is invertible or not is to check if $N(U) = a^2 - b^2 = U\overline{U}$ equals zero or not where $\overline{U} = \mathrm{a} - \mathrm{b}\eta$. If $N(U) \neq 0$, then $\frac{\overline{U}}{N(U)}$ is the inverse of U.

We see a general trend in the above examples of Clifford algebras. Given a Clifford algebra $\mathcal{C}\ell_{p,q}$ we have come up with a function which takes as input a multivector U and outputs a real number. Let us denote this 'norm' function as $N(U)$. These norm functions have a common thing in their structure, namely, they are a product of U with some other terms (which depend on U) which we denote by $f(U)$. $f(U)$ consists of products (and possibly sums) of 'conjugates' of U. Now, if for some U, $N(U) \neq 0$, then $\frac{f(U)}{N(U)}$ is inverse of as $N(U) = U \cdot f(U)$. It turns out that such formula exist for all Clifford algebras (See Theorem 3 in [1]).

So, idea is simple: start with an element $U \in \mathcal{C}\ell_{p,q}$. Multiply U with its suitably chosen 'conjugates' chosen in a manner such that the product is be a real number. This function of U which consists of sums and products of conjugates of U will be called a norm function for $\mathcal{C}\ell_{p,q}$. If for some multivector, the norm function evaluates to a non-zero real number, then that multivector is invertible.

4 Flow of the presentation.

1) First, we will introduce faithful representations of $\mathcal{C}\ell_{p,q}$ and the notion of trace and determinant for $\mathcal{C}\ell_{p,q}$, which are defined as trace and determinant of the corresponding matrices representing Clifford algebra elements. The notion of determinant of $\mathcal{C}\ell_{p,q}$ will replace the notion of 'norm' introduced in previous section and will make precise the discussion there. It also removes some ambiguities which are unclear from the discussion in the previous section such as the possibility of existence of one sided inverses.

2)Next, we will give the algebra $\mathcal{C}\ell_{p,q}$ different gradings. It is well known that $\mathcal{C}\ell_{p,q}$ is a superalgebra i.e., it is a $\mathbb{Z}_2$ graded algebra if we identify $\mathcal{C}\ell_{p,q}$ as $\mathcal{C}\ell^+_{p,q} \oplus \mathcal{C}\ell^-_{p,q}$ where $\mathcal{C}\ell^+_{p,q} := \mathcal{C}\ell^0_{p,q} \oplus \mathcal{C}\ell^2_{p,q} \oplus \mathcal{C}\ell^4_{p,q} \oplus \cdots$, the even subalgebra and $\mathcal{C}\ell^-_{p,q} := \mathcal{C}\ell^1_{p,q} \oplus \mathcal{C}\ell^3_{p,q} \oplus \mathcal{C}\ell^5_{p,q} \oplus \cdots$. Relabeling $\mathcal{C}\ell^+_{p,q}$ as R_0 and $\mathcal{C}\ell^-_{p,q}$ as R_1, we can represent the $\mathbb{Z}_2$ grading as

$$R_k R_l \subseteq R_{k+l}$$

where $k, l \in \mathbb{Z}_2$. The 'product' $R_k R_l$ of the sets R_k and R_l is defined to be the set of products of the form ab where $a \in R_k$ and $b \in R_l$.
The $\mathbb{Z}_2$ grading discussed above is not the only grading one can give to $\mathcal{C}\ell_{p,q}$. $\mathcal{C}\ell_{p,q}$ is also a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra with respect to commutator and anti-commutator. This is the idea of quaternion typification presented in [4]. It might seem out of the blue that $\mathcal{C}\ell_{p,q}$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading but it is a direct consequence of the multiplication in $\mathcal{C}\ell_{p,q}$. Let

$$\bar{0} := \mathcal{C}\ell^0_{p,q} \oplus \mathcal{C}\ell^4_{p,q} \oplus \mathcal{C}\ell^8_{p,q} \oplus \cdots,$$

$$\bar{1} := \mathcal{C}\ell^1_{p,q} \oplus \mathcal{C}\ell^5_{p,q} \oplus \mathcal{C}\ell^9_{p,q} \oplus \cdots,$$

$$\bar{2} := \mathcal{C}\ell^2_{p,q} \oplus \mathcal{C}\ell^6_{p,q} \oplus \mathcal{C}\ell^{10}_{p,q} \oplus \cdots$$

$$\bar{3} := \mathcal{C}\ell^3_{p,q} \oplus \mathcal{C}\ell^7_{p,q} \oplus \mathcal{C}\ell^{11}_{p,q} \oplus \cdots.$$

and

$$\{\bar{A}, \bar{B}\} := \{ab + ba \mid a \in \bar{A}, b \in \bar{B}\},$$

$$[\bar{A}, \bar{B}] := \{ab - ba \mid a \in \bar{A}, b \in \bar{B}\} \ \text{ for } \ A, B \in \{0, 1, 2, 3\}.$$

Then

$$\{\bar{A}, \bar{A}\} \subseteq \bar{0} \ \text{ for } \ A \in \{0, 1, 2, 3\} \qquad \text{(A1)}$$

$$\{\bar{0}, \bar{A}\} \subseteq \bar{A} \ \text{ for } \ A \in \{0,1,2,3\} \qquad \text{(A2)}$$
$$\{\bar{1}, \bar{2}\} \subseteq \bar{3} \qquad \text{(A3)}$$
$$\{\bar{2}, \bar{3}\} \subseteq \bar{1} \qquad \text{(A4)}$$
$$\{\bar{3}, \bar{1}\} \subseteq \bar{2} \qquad \text{(A5)}$$

and

$$[\bar{A}, \bar{A}] \subseteq \bar{2} \ \text{ for } \ A \in \{0,1,2,3\} \qquad \text{(C1)}$$
$$[\bar{2}, \bar{A}] \subseteq \bar{A} \ \text{ for } \ A \in \{0,1,2,3\} \qquad \text{(C2)}$$
$$[\bar{3}, \bar{0}] \subseteq \bar{1} \qquad \text{(C3)}$$
$$[\bar{0}, \bar{1}] \subseteq \bar{3} \qquad \text{(C4)}$$
$$[\bar{1}, \bar{3}] \subseteq \bar{0} \qquad \text{(C5)}$$

If we denote $\bar{0}$ by $R_{(0,0)}$, $\bar{1}$ by $R_{(0,1)}$, $\bar{2}$ by $R_{(1,0)}$ and $\bar{3}$ by $R_{(1,1)}$, then one can express the equations A1, A2, A3, A4, A5 as

$$R_k R_l \subseteq R_{k+l}$$

where $k, l \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and the 'product' $R_k R_l$ of the sets R_k and R_l is defined to be the set of anti-commutators $ab + ba$ where $a \in R_k$ and $b \in R_l$.
If we denote $\bar{2}$ by $R_{(0,0)}$, $\bar{3}$ by $R_{(0,1)}$, $\bar{0}$ by $R_{(1,0)}$ and $\bar{1}$ by $R_{(1,1)}$, then one can express the equations C1, C2, C3, C4, C5 as

$$R_k R_l \subseteq R_{k+l}$$

where $k, l \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and the 'product' $R_k R_l$ of the sets R_k and R_l is defined to be the set of commutators $ab - ba$ where $a \in R_k$ and $b \in R_l$. Thus making $\mathcal{C}\ell_{p,q}$ a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra with respect to the commutator and anti-commutator.
This is very insightful and helps us comment about grades of products of multivectors. As our approach to finding inverses of multivectors is by eliminating elements of all non-zero grades by multiplying the element with its conjugates until only real numbers remain, the idea of quaternion typification helps tremendously.

3) Next we will generalise the well known operations of conjugation in Clifford algebras, namely, the grade involution and the grade reversion (see section 2 in [2]). Let $p+q := n$. We define $m := 1 + \lfloor log_2(n) \rfloor$ operations of conjugation, $(.)^{\triangle_j}$; $j \in \{1, 2, \ldots, m\}$ by

$$U^{\triangle_j} = \sum_{0 \leq k \leq n} (-1)^{C^k_{2^{j-1}}} \langle U \rangle_k$$

where C^y_x is the binomial coefficient $\frac{x!}{y!(x-y)!}$. Note that $\triangle_1$ is same as grade involution $\widehat{}$ and $\triangle_2$ is same as grade reversion $\widetilde{}$. Table 1 below shows how $\triangle_j$'s and their superpositions act on elements of fixed grades:
The table has been truncated to include only first 16 grades and first 4 operations of conjugation defined above. The action of these operations is periodic on the grades i.e., after a point the pattern of +s and −s repeats in the rows of the table. It turns out that one can express any operation of conjugation as a linear combination of these m operations of conjugations.
We will see that these operations of conjugates have nice properties and allow us to

grade k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
id	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\triangle_1 = \widehat{\ }$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$\triangle_2 = \widetilde{\ }$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
$\triangle_1\triangle_2$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$\triangle_3$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\triangle_1\triangle_3$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
$\triangle_2\triangle_3$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$\triangle_1\triangle_2\triangle_3$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
$\triangle_4$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\triangle_1\triangle_4$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$
$\triangle_2\triangle_4$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$
$\triangle_1\triangle_2\triangle_4$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$
$\triangle_3\triangle_4$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
$\triangle_1\triangle_3\triangle_4$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$
$\triangle_2\triangle_3\triangle_4$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$
$\triangle_1\triangle_2\triangle_3\triangle_4$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$

Table 1: $+$ and $-$ signs denote the sign of element of a grade after the operation of conjugation is applied.

express the determinant of a Clifford algebra element as products of its conjugates. This gives us nice explicit formulas for inverse of a multivector without invoking the faithful representation we used do define determinant for Clifford algebra elements. We will prove the formulas for inverses of multivectors presented in section 5 for $n \leq 5$. The proofs requires us to combine the idea of quaternion typification and the nice properties of operations of conjugations.

4) We will also look at why suddenly for $n = 6$, the formula of norm is a linear combination. It turns out that there exists a (trouble causing) subalgebra $S := \mathrm{Span}_{\mathbb{R}}\{1, e_{1256}, e_{2345}, e_{1346}\}$ (discovered by A. Acus and A. Dargys using Mathematica in [5]) which has the property that product of any two elements of S with non-zero grade 4 part is another element of S with non-zero grade 4 part and this one cannot eliminate grade 4 part of a multivector by just multiplying it with its conjugates, one needs to carry a sum.

5) We will use the analogue of Faddeev–LeVerrier algorithm in Clifford algebras introduced in [2] to show that there always exists a norm function, namely, the determinant which is in the form $Uf(U)$ where $f(U)$ is a linear combination of products of conjugates of U (check Theorem 3 in [1]). This is what we had proposed in section 3 of this abstract. The algorithm also gives a recursive method to find determinant of a Clifford algebra element and in general all characteristic polynomial coefficients of a multivector.

6) We will also see some isomorphism theorems between different Clifford algebras and see how they can simplify our job significantly by allowing us to use formulas for inverses for a Clifford algebra of smaller dimension for Clifford algebra of larger dimension. In particular, we will use the isomorphism between even subalgebra of a Clifford algebra

and another Clifford algebra (check [3]).

7) Till now, explicit formulas for inverses of multivectors are known only for $\mathcal{C}\ell_{p,q}$ with $n \leq 6$. The problem for finding a explicit formula for inverses in $\mathcal{C}\ell_{p,q}$ with $n \geq 7$ stands open. However, there are a few cases where one can get a explicit formula for inverse of special multivectors like paravectors (sum of scalar and a vector) and sum of a paravector and its dual which are independent of n. We will discuss why getting explicit formulas for inverses of grade higher grade elements is hard and likely depends on n. Inverses of grade $4, 8, 12, \ldots$ are particularly important because square of any multivector is an element of $\bar{0} = \mathcal{C}\ell^0_{p,q} \oplus \mathcal{C}\ell^4_{p,q} \oplus \mathcal{C}\ell^8_{p,q} \oplus \cdots$.

5 Formulas for inverses of multivectors in small dimensions

Let $\widehat{\ }$, $\widetilde{\ }$, $\triangle$ denote the operations of conjugation $\triangle_1$, $\triangle_2$, $\triangle_3$ respectively. There exist the following norm functions $N : \mathcal{C}\ell_{p,q} \to \mathbb{R}$:

$$
\begin{aligned}
N(U) &= U\widehat{U}, \quad n=1;\\
N(U) &= U\widehat{\widetilde{U}}, \quad n=2;\\
N(U) &= U\widetilde{U}\widehat{U}\widehat{\widetilde{U}}, \quad n=3;\\
N(U) &= U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}, \quad n=4;\\
N(U) &= U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle})^{\triangle}, \quad n=5,\\
N(U) &= U\left(\frac{1}{3}\widetilde{U}\widehat{U}\widehat{\widetilde{U}}(\widehat{U}\widehat{\widetilde{U}}U\widetilde{U})^{\triangle} + \frac{2}{3}\widetilde{U}((\widehat{U}\widehat{\widetilde{U}})^{\triangle}((\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U})^{\triangle})^{\triangle})^{\triangle}\right), \quad n=6.
\end{aligned}
$$

The formulas are not unique. There are multiple ways to express the norm functions in terms of the multivector and its conjugates, see Theorem 3 in [2] for details. The norm function for $n = 6$ case was first presented in [5], we have put their result in our notation (see Lemma 5 in [2]). These norm functions directly give the formulas for inverses of multivectors:

$$
\begin{aligned}
U^{-1} &= \frac{\widehat{U}}{N(U)}, \quad n=1;\\
U^{-1} &= \frac{\widehat{\widetilde{U}}}{N(U)}, \quad n=2;\\
U^{-1} &= \frac{\widetilde{U}\widehat{U}\widehat{\widetilde{U}}}{N(U)}, \quad n=3;\\
U^{-1} &= \frac{\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle})^{\triangle}}{N(U)}, \quad n=5,\\
U^{-1} &= \frac{\left(\frac{1}{3}\widetilde{U}\widehat{U}\widehat{\widetilde{U}}(\widehat{U}\widehat{\widetilde{U}}U\widetilde{U})^{\triangle} + \frac{2}{3}\widetilde{U}((\widehat{U}\widehat{\widetilde{U}})^{\triangle}((\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U})^{\triangle})^{\triangle})^{\triangle}\right)}{N(U)}, \quad n=6.
\end{aligned}
$$

A striking thing is that the formulas for inverse of a multivector in some $\mathcal{C}\ell_{p,q}$ do not depend on the individual values of p and q but on their sum n i.e., the the formulas for inverses of multivectors are independent of the signature of the Clifford algebra under consideration.

6 Applications for formulas of inverses

Explicit formulas for inverses of multivectors give us explicit solutions to linear equations in a Clifford algebra setting which are widely used in image and signal processing, control theory, etc for example the Sylvester equation in which one looks for multivectors X satisfying $AX + XB = C$ for given $A, B, C \in \mathcal{C}\ell_{p,q}$. The recursive method obtained from Faddeev–LeVerrier algorithm to compute determinant of a multivector can be employed in symbolic computation involving Clifford algebra.

References

[1] Dmitry Shirokov, *On Noncommutative Vieta Theorem in Geometric Algebras.* In *Mathematical Methods in the Applied Sciences, 2023, 16 pp, arXiv:2301.06848v2.*

[2] Dmitry Shirokov, *On computing the determinant, other characteristic polynomial coefficients, and inverse in Clifford algebras of arbitrary dimension* In *Computational and Applied Mathematics 40(2021), 173, 29 pp, arXiv:2005.04015v4.*

[3] Eckhard Hitzer, Stephen J. Sangwine, *Construction of Multivector Inverse for Clif Ford Algebras Over 2m+1-Dimensional Vector Spaces from Multivector Inverse for Clifford Algebras Over 2m-Dimensional Vector Spaces* In *Advances in Applied Clifford Algebras (AACA), 29, 29 (2019), viXra:1901.0246.*

[4] Dmitry Shirokov, *Quaternion typification of Clifford algebra elements* In *Advances in Applied Clifford Algebras (AACA), 22:1, 2012, 243-256, arXiv:0806.4299v3.*

[5] Arturas Acus, Adolfas Dargys, *Inverse of multivector: Beyond p+q=5 threshold* In *Advances in Applied Clifford Algebras (AACA), 28, 65 (2018), arXiv:1712.05204v2.*