FLUX QUANTIZATION IN TYPE II SUPERCONDUCTORS

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Summary. This presentation explores the physics of magnetic and electric flux tubes supported by current vortices in condensed matter having a superconducting state in which bosonic charge carriers flow without resistance. The starting point is that the boson wave function satisfies the Klein-Gordon equation of relativistic quantum mechanics. Next, the electromagnetic fields within the superconducting medium are assumed to obey the quasistatic Maxwell equations expressed with geometric algebra and calculus and incorporating either electric or hypothetical magnetic currents. Finally, the Fundamental Theorem of Calculus is utilized in two forms to examine flux tubes, first in electric superconductors and then in hypothetical magnetic superconductors. Geometric algebra and calculus enable a consistent treatment of both analyses and their extension from three to four spatial dimensions.

1. Introduction

The Meissner effect for electrical superconductors in three spatial dimensions (3-D) describes how a region of superconducting material excludes magnetic fields from its interior. Type I superconducting materials exhibit the Meissner effect as long as the field strength is below a critical threshold value. Above this threshold, the material loses it's bulk superconductivity. Type II superconductors are able to tolerate significantly higher magnetic fields by confining the loss of superconductivity to a tube of normally conducting material surrounded by a vortex of supercurrent. In this way magnetic flux is allowed to pass through the superconductor in vortex tubes without destroying the superconductivity around the tubes. Such flux tubes are named Abrikosov vortices honoring the work of the researcher who first explained the effect [1]. Importantly, the magnetic flux through such tubes is quantized in multiples of a basic unit.

Notation. A field F with four grade components in 3-D is expressed as $F = \chi + \bar{E} + \widehat{B} + \widetilde{T}$. Following the notation of Vold [11] used previously by this author [10], the embellishments over the variable symbols designate vectors with bars, bivectors with arcs, and trivectors with tildes. The grade of quantities without embellishments should be clear from context.

2. Needed Relationships from Geometric Calculus

2.1. **The Maxwell Equation with Magnetic Currents.** In his early work, David Hestenes [4] emphasized how the four traditional Maxwell equations can be expressed with geometric calculus (GC) as a single equation in spacetime algebra or in a laboratory inertial frame. This author took the latter approach [10], adding hypothetical magnetic charge and current densities. With the assumption of quasistatic conditions, the electric and magnetic fields and their source charge and current distributions satisfy the following four equations (separated by grade):

$$\partial_{\bar{x}} \cdot \bar{E} = 4\pi\rho_e \tag{1}$$

$$\partial_{\bar{x}} \cdot \widehat{B} = -(4\pi/c)\bar{J}_e \tag{2}$$

$$\partial_{\bar{x}} \wedge \bar{E} = -(4\pi/c)\widehat{J}_m \tag{3}$$

$$\partial_{\bar{x}} \wedge \widehat{B} = 4\pi\widetilde{\rho}_m \quad . \tag{4}$$

2.2. **Fundamental Theorem of Calculus.** Following the notation of Hestenes and Sobczyk [5], the Fundamental Theorem of Calculus using geometric algebra and geometric calculus is

$$\int_M \dot{L}(dV\dot{\partial}_{\bar{x}}) = \int_{\partial M} L(dS), \tag{5}$$

where the multivector-valued function $L(A)$ is a general linear function of its argument A. The over-dots indicate the action of the geometric derivative $\partial_{\bar{x}}$. The argument A is also multivector-valued and $L(A)$ may be a function of time t and spatial position $\bar{x}$. $L(A)$ is assumed well behaved over a region M bounded by the surface ∂M. dV is the pseudoscalar volume element in M and dS is the pseudovector surface element of the boundary ∂M.

The Fundamental Theorem of Calculus is sometimes referred to as the *boundary theorem* in various contexts. It is illustrated in detail both mathematically and graphically for selected spatial dimensions by Klausen [8]. For present purposes let the linear function in Eq. (5) be $L(A) = AF$ with a multivector-valued function $F = F(\bar{x}, t)$ and suppose that M is a 2-D surface S with surface element $d\widehat{\sigma}$ and that its boundary ∂M is a closed contour C with line element $d\bar{l}$. in this case, Eq. (5) becomes:

$$\iint_S d\widehat{\sigma}\partial_{\bar{x}}F = \oint_C d\bar{l}F. \tag{6}$$

If F is a vector field, taking the scalar part of this equation yields

$$\iint_S d\widehat{\sigma} \cdot (\partial_{\bar{x}} \wedge \bar{F}) = \oint_C d\bar{l} \cdot \bar{F}, \tag{7}$$

which is Stokes's theorem as used in traditional electrodynamics. If F is a bivector field, then taking the trivector part of Eq. (6) yields

$$\iint_S d\widehat{\sigma} \wedge (\partial_{\bar{x}} \cdot \widehat{F}) = \oint_C d\bar{l} \wedge \widehat{F}. \tag{8}$$

3. Superconductivity

The phenomena of electrical superconductivity in solids have been well studied and described in various approximations since its discovery in 1911 by the Dutch physicist Heike Kamerlingh Onnes. The approximations leading to the London equation [9] are sufficient for present purposes. For the analysis of flux quantization, we follow the development outlined in the solid state textbook by Kittel [7] but with the added allowance that the quantum mechanical fields leading to either electric or magnetic currents may require a relativistic description. A central assumption is that the conducting charges, either electric or magnetic, are bosons. To facilitate analysis herein, Kittel's development is recast in terms of geometric algebra and calculus.

In relativistic quantum mechanics, the field (wave function) of either a fermion or a boson satisfies the Klein-Gordon (K-G) equation. Unlike fermions (which must satisfy the Pauli exclusion principle), multiple bosons are allowed to occupy the same physical state. If sufficiently large numbers of bosons occupy the same physical state, their collective wave function acts as a classical field. A common example of this effect is the output of a laser wherein a very large number of photons of the same frequency are created in phase to generate the classical electromagnetic field of the laser beam.

If a pair of like-charged fermions are bound by some interaction into a boson state and if many of these pairs can be created in the same physical state, then the pairs will form a classical current. The explanation of electrical superconductivity grew from the realization that certain crystal lattices and other condensed matter systems can provide an attractive interaction binding two conduction electrons into a boson state (referred to as a Cooper pair) carrying twice the electric charge of a single electron. A sufficient number of such Cooper pairs can form a charged "beam" propagating as a supercurrent impervious to dissipation by scattering just as the photons of a laser beam propagate in vacuum.

4. Magnetic Flux Quanta in Three Spatial Dimensions

4.1. **Quasistatic Field and Current with Bosonic Conductors.** Assume that the quantum mechanical field of the bosonic charge carriers in an electric superconductor satisfy the K-G equation as does that of a charged, spin-zero meson [2, Sec 9.3]. This field ψ is complex in traditional quantum mechanics and represented in 3-D GA with scalar and trivector components using the unit trivector $I = e_1 e_2 e_3$ with (complex) conjugate $I^* = -I$. The electric current of this field derived from the K-G equation is

$$\bar{J}_e = \frac{q}{2m}[\psi^*(-I\hbar\partial_{\bar{x}} - \frac{q}{c}\bar{A})\psi - \psi(-I\hbar\partial_{\bar{x}} + \frac{q}{c}\bar{A})\psi^*], \tag{9}$$

where q and m (both scalars) are the electric charge and mass, respectively, of the bosons making up the current, $\partial_{\bar{x}}$ is the geometric derivative, c is the speed of light, and $\bar{A}$ is the vector potential of an external field. Finally, assume that the setup is quasistatic with only steady currents and no free charge density. In a uniform superconductor the field of a K-G plane wave solution is simply

$$\psi = \sqrt{n}e^{\pm I\varphi}, \tag{10}$$

where n is the number density of charge carriers (assumed constant) and $\varphi = \omega t - \bar{k} \cdot \bar{x}$ is the usual phase angle of the plane wave with constant values of ω and $\bar{k}$. In this case, $\partial_{\bar{x}}\psi = \mp I\bar{k}\psi$, $\partial_{\bar{x}}\psi^* = \pm I\bar{k}\psi^*$, and the current given by Eq. (9) reduces to

$$\bar{J}_e = \frac{qn}{m}[\mp\hbar\bar{k} - \frac{q}{c}\bar{A}]. \tag{11}$$

4.2. **London Equation and the Meissner Effect.** Given that $\bar{k}$ is constant and that the exterior derivative of $\bar{A}$ is the bivector magnetic field $\widehat{B}$, the exterior derivative of $\bar{J}_e$ from Eq. (11) is

$$\partial_{\bar{x}} \wedge \bar{J}_e = -\frac{q^2 n}{mc}\partial_{\bar{x}} \wedge \bar{A} = -\frac{q^2 n}{mc}\widehat{B}, \tag{12}$$

which is the well-known London equation of superconductivity expressed with geometric calculus. The traditional derivation of this result starts with the non-relativistic Schrodinger equation but this presentation shows that it follows equally well from the relativistic K-G equation.

Another expression for the exterior derivative of $\bar{J}_e$ results from taking the exterior derivative of the Maxwell equation given in Eq. (2):

$$\partial_{\bar{x}} \wedge (\partial_{\bar{x}} \cdot \widehat{B}) = -\frac{4\pi}{c}\partial_{\bar{x}} \wedge \bar{J}_e. \tag{13}$$

As with any multivector, the geometric derivative of the divergence $\partial_{\bar{x}} \cdot \widehat{B}$ may be expanded as

$$\partial_{\bar{x}}(\partial_{\bar{x}} \cdot \widehat{B}) = \partial_{\bar{x}} \cdot (\partial_{\bar{x}} \cdot \widehat{B}) + \partial_{\bar{x}} \wedge (\partial_{\bar{x}} \cdot \widehat{B}). \tag{14}$$

The first term on the right of this equation, being the divergence of a divergence, vanishes [3, Sec 6.1.3]. Then the combination of Eq. (13) and Eq. (14) yields

$$\partial_{\bar{x}} \wedge \bar{J}_e = -\frac{c}{4\pi}(\partial_{\bar{x}} \wedge (\partial_{\bar{x}} \cdot \widehat{B})) = -\frac{c}{4\pi}(\partial_{\bar{x}}(\partial_{\bar{x}} \cdot \widehat{B})). \tag{15}$$

Now $\partial_{\bar{x}}\widehat{B} = \partial_{\bar{x}} \cdot \widehat{B} + \partial_{\bar{x}} \wedge \widehat{B}$. Then according to Eq. (4) with no magnetic charge density, $\partial_{\bar{x}} \wedge \widehat{B} = 0$ giving $\partial_{\bar{x}} \cdot \widehat{B} = \partial_{\bar{x}}\widehat{B}$. This result in Eq. (15) yields

$$\partial_{\bar{x}} \wedge \bar{J}_e = -\frac{c}{4\pi}\partial_{\bar{x}}^2\widehat{B}. \tag{16}$$

Putting this result in Eq. (12) gives

$$\frac{c}{4\pi}\partial_{\bar{x}}^2 \widehat{B} = \frac{q^2 n}{mc}\widehat{B}.$$

This combination of the London and Maxwell equations for the interior of a superconductor shows that the second derivative of $\widehat{B}$ is proportional to $\widehat{B}$ itself:

$$\partial_{\bar{x}}^2 \widehat{B} = \widehat{B}/\lambda_L^2, \tag{17}$$

where $\lambda_L = (mc^2/4\pi nq^2)^{1/2}$. This equation prohibits a constant field $\widehat{B}_0$ within a superconductor unless $\widehat{B}_0 = 0$. On the other hand, if a halfspace for $z \geqslant 0$ is superconducting and the other half space for $z < 0$ has a finite-valued, constant field $\widehat{B}_0$ with axial vector $I\widehat{B}_0$ parallel to the superconductor surface, then the exponentially damped field $\widehat{B}_0 e^{-z/\lambda_L}$ will satisfy Eq. (17) for $z \geqslant 0$. For this reason, λ_L is referred to as the London penetration depth. Such damping explains the Meissner effect, that is, the experimental fact that $\widehat{B} \to 0$ inside a superconductor as first discovered by the German physicists W. Meissner and R. Ochsenfeld in 1933.

4.3. **Magnetic Flux Tubes.** Figure 1 provides context for analyzing a flux tube in a type II superconductor. It illustrates a cross section perpendicular to the axis of a tube. The inner cylinder is normal conductor with magnetic field $\widehat{B}$ represented by the small circles with a central dot. This bivector field is in the plane of the cross section. The surface of the tube consists of a sheet of superconducting material with thickness determined by the London penetration depth. This surface carries the Abrikosov vortex of supercurrent that isolates the flux tube.

Analysis will involve an integral along the contour C located in the superconductor well away from the thin sheet of material carrying the vortex current. Interestingly, Maxwell's Eq. (2) shows that the Meissner effect requiring $\widehat{B} = 0$ along the contour within the superconductor requires that the current $\bar{J}_e = 0$ along the contour, also.

Assume now that the K-G wave function of the Cooper pairs at a given time along the contour around the flux tube is $\psi = \sqrt{n}e^{I\Theta(\bar{x})}$ with scalar phase angle $\Theta(\bar{x})$. Paralleling the development in Section 4.1, we have $\partial_{\bar{x}}\psi = I\partial_{\bar{x}}\Theta\psi$, $\partial_{\bar{x}}\psi^* = -I\partial_{\bar{x}}\Theta\psi^*$, and the electric current:

$$\bar{J}_e = \frac{qn}{m}[\hbar\partial_{\bar{x}}\Theta - \frac{q}{c}\bar{A}]. \tag{18}$$

Given that $\bar{J}_e = 0$, the two quantities within the brackets of this expression must be equal. Forming the line integral of this equality around the contour in Figure 1 yields:

$$\oint_C d\bar{l} \cdot \partial_{\bar{x}}\Theta = \frac{q}{\hbar c}\oint_C d\bar{l} \cdot \bar{A}. \tag{19}$$

The integral on the left is the phase change of the wave function ψ in going once around the loop C and must be an integer multiple of 2π for the wave function to be single-valued. Therefore, the integral must equal $2\pi s$ for some integer s. Applying Stokes's theorem from Eq. (7) and the fact that $\widehat{B} = \partial_{\bar{x}} \wedge \bar{A}$, Eq. (19) provides the magnetic flux enclosed by C:

$$2\pi s = \frac{q}{\hbar c}\oint_C d\bar{l} \cdot \bar{A} = \frac{q}{\hbar c}\iint_S d\widehat{\sigma} \cdot (\partial_{\bar{x}} \wedge \bar{A}) = \frac{q}{\hbar c}\iint_S d\widehat{\sigma} \cdot \widehat{B}. \tag{20}$$

Eq. (20) shows that the magnetic flux Φ carried by the Abrikosov vortex through the superconductor must have a quantized value

$$\Phi = \iint_S d\widehat{\sigma} \cdot \widehat{B} = \frac{2\pi\hbar c}{q}s. \tag{21}$$

Given that the Cooper pairs carry twice the charge e of a single electron, the quantum of magnetic flux (experimentally verified) for $s = 1$ is $\Phi_0 = 2\pi\hbar c/2e = 2.0678 \times 10^{-7}\,\text{gauss cm}^2$.

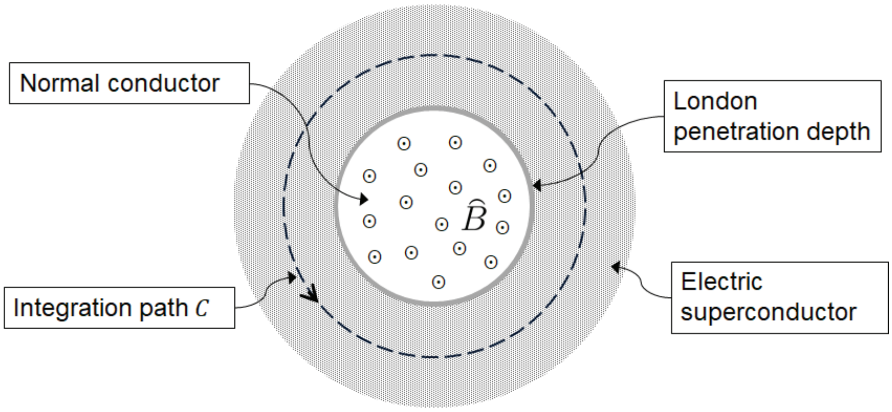

FIGURE 1. Cross section of a magnetic flux tube in a 3-D electric superconductor.

5. Electric Flux Quanta in Three Spatial Dimensions

With geometric calculus it is straightforward to repeat the above analysis for a hypothetical, magnetic superconductor of type II with bosonic, magnetic charge carriers giving rise to electric flux quanta within a vortex of magnetic current. The analysis is based on a bivector potential $\widehat{M}$ giving rise to an electric field via $\bar{E} = \partial_{\bar{x}} \cdot \widehat{M}$ in the quasistatic case analogous to how the vector potential gives rise to the magnetic field via $\widehat{B} = \partial_{\bar{x}} \wedge \bar{A}$. The bosonic magnetic charge carriers of density n have trivector charge $\tilde{q}_m = Iq_m$ of magnitude q_m.

An analysis like that of Sections 4.1 and 4.2 shows, similar to the Meissner effect, that electric fields would be exponentially damped entering the surface of a magnetic superconductor leading to an interior field $\bar{E} \to 0$. Paralleling Section 4.3, Eq. (18) becomes

$$\widehat{J}_m = -\frac{\tilde{q}_m n}{m}[\hbar \partial_{\bar{x}} \Theta - \frac{\tilde{q}_m}{c}\widehat{M}] \tag{22}$$

giving the magnetic current $\widehat{J}_m$ in the superconductor in terms of the boson charge $\tilde{q}_m$ and the bivector potential $\widehat{M}$. Maxwell's Eq. (3) shows that a Meissner-like effect requiring $\bar{E} = 0$ deep in the superconductor would require that the current $\widehat{J}_m$ vanish, also. Then Eq. (19) becomes

$$\oint_C d\bar{l} \cdot \partial_{\bar{x}} \Theta = \frac{1}{\hbar c} \oint_C d\bar{l} \cdot (\tilde{q}_m \widehat{M}). \tag{23}$$

The inner product of the integrand on the right side of this equation picks out the component of $\tilde{q}_m \widehat{M}$ that is parallel to the contour C. Given that $\tilde{q}_m = Iq_m$ is proportional to the 3-D pseudoscalar, the integrand can be expressed as $d\bar{l} \cdot (\tilde{q}_m \widehat{M}) = \tilde{q}_m (d\bar{l} \wedge \widehat{M}))$. As with electrical superconductors, the integral on the left must be an integral multiple of 2π. Using this value with Eq. (8) and $\bar{E} = \partial_{\bar{x}} \cdot \widehat{M}$, Eq. (23) shows that

$$2\pi s = \frac{\tilde{q}_m}{\hbar c} \oint_C d\bar{l} \wedge \widehat{M} = \frac{\tilde{q}_m}{\hbar c} \iint_S d\widehat{\sigma} \wedge (\partial_{\bar{x}} \cdot \widehat{M}) = \frac{\tilde{q}_m}{\hbar c} \iint_S d\widehat{\sigma} \wedge \bar{E} = \frac{\tilde{q}_m}{\hbar c} \tilde{\Phi}_v, \tag{24}$$

where $\tilde{\Phi}_v$ is the total electric flux through the vortex of magnetic current. Thus, similar to Eq. (21), the absolute value of the electric flux in the vortex is

$$|\tilde{\Phi}_v| = |\iint_S d\widehat{\sigma} \wedge \bar{E}| = \frac{2\pi \hbar c}{q_m} s. \tag{25}$$

Though no magnetic charge has been discovered as of yet on which to base q_m, Dirac formulated a quantum mechanical rationale (see, for example, [6, Sec 6.12]) involving the angular momentum of the combined fields of an electron and a single magnetic monopole $\tilde{g} = Ig$ requiring that the magnitude g satisfy the relationship:

$$\frac{ge}{\hbar c} = \frac{n_d}{2}, \tag{26}$$

where n_d can be any positive or negative integer. Likewise, there is no information on whether the monopole would be a boson or fermion so we introduce a positive integer n_b giving the number of Dirac monopoles in a boson of the magnetic supercurrent. With these assumptions, the magnitude of the bosonic magnetic charge is

$$q_m = n_b g = n_b n_d \frac{\hbar c}{2e} \tag{27}$$

so that Eq. (25) becomes

$$|\tilde{\Phi}_v| = \frac{s}{n_b n_d} 4\pi e = \frac{s}{n_b n_d} \Phi_e, \tag{28}$$

where $\Phi_e = 4\pi e$ is the magnitude of the total electric flux from an electron. In the simplest case, having $s = 1$, $n_d = 1$ and the monopole being a fermion so that $n_b = 2$ like a Cooper pair, the resulting vortex flux is $1/2$ that of an electron. Another possibility is that the monopole is a boson, allowing $n_b = 1$. In this case, with $s = 1$ and $n_d = 3$, the vortex flux is $1/3$ that of the electron, bringing to mind the 1/3 charges of quarks in the Standard Model of particle physics.

6. Flux Hypertubes in Four Spatial Dimensions

The 3-D flux tube pictured in Figure 1 may be thought of as a circular hole in a 2-D plane repeated in parallel planes to form a tube in 3-D space. Similarly, a 4-D flux hypertube may be thought of as spherical hole in a 3-D volume repeated in parallel volumes to form a hypertube in 4-D space. Equation (5) for this situation yields a boundary theorem analogous to Eq. (6):

$$\iiint_V d\widetilde{V} \partial_{\widetilde{x}} F = \oint\!\!\oint_S d\widehat{\sigma} F.$$

The extension of the analysis of Section 4 to magnetic flux hypertubes is reasonably straightforward given the scalar nature of electric charge. A key element is to switch from a simple exponential wave function around a circular boundary to a spherical harmonic wave function on a spherical boundary. The extension of the analysis of Section 5 to electric flux hypertubes is complicated by the possibility of four orthogonal, trivector magnetic charges in 4-D rather than the single trivector charge corresponding to the pseudoscalar in 3-D. This presentation will describe progress on these analyses.

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