From Null Monomials to Versors in Conformal Geometry

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Abstract. Extended Abstract

In Conformal Geometric Algebra (CGA), a point (including the conformal point at infinity $\mathbf{e}_\infty$) is represented by a null vector, and the representation is unique up to scale. A versor is the geometric product of finitely many invertible vectors in $\mathbb{R}^{n+1,1}$. A conformal transformation acts on points via the (left) *adjoint action* of the corresponding versor $\mathbf{V}$ upon the null vectors representing the points:

$$Ad_{\mathbf{V}}(\mathbf{x}) = \mathbf{V}\mathbf{x}\mathbf{V}^{-1}, \tag{1}$$

and the versor inducing the conformal transformation is unique up to a nonzero scalar or pseudoscalar factor.

Although in conformal geometric constructions, starting from points one can construct lines and circles by 3-vectors, 2-planes and 2-spheres by 4-vectors, etc, which are outer products of null vectors representing the incident points involved in the construction, in symbolic geometric computing it is the geometric product of these null vectors that prove to be much more efficient than the graded parts of the geometric product, such as the outer product, the meet product, the inner product, etc. Hence, monomials that are the geometric product of null vectors, called *null monomials*, turn out to be the basic algebraic terms in symbolic computing. Their geometric interpretations and applications are an important topic of research.

In Grassmann algebra, a blade as the outer product of vectors, represents a linear subspace, and in CGA when focusing on the null vectors and positive vectors in the linear subspace, the blade can be interpreted geometrically as lines, circles, etc., or dual to such geometric entities. However, if a homogeneous multi-vector is not a blade, then its geometric interpretation is not clear in general.

In CGA, the geometric product of invertible vectors is a versor, and can be interpreted geometrically as the generator of a conformal transformation, the latter being the composition of the conformal transformations each being induced by one invertible vector factor of the versor. However, when some vector factors are not invertible, the geometric interpretation of such geometric products is still not clear.

In the first part of this talk, we address the problem of the geometric interpretation of null monomials in CGA, which is the the extreme case where in the geometric product all the vector factors are not invertible. We only highlight some guiding examples below.

In CGA, we do not distinguish between a non-zero entity and its nonzero scalings. We use $x \equiv y$ to denote that x, y are equal up to a nonzero scalar factor. Sometimes to take the sign into consideration, we use $x \equiv_+ y$ to denote that x, y are equal up to a positive scalar factor.

Example 1. Let null vector $\mathbf{n}$ represent an affine point. Then $\mathbf{e}_\infty\mathbf{n}$ is a null monomial of length (or degree) 2. Let there be another null monomial $\mathbf{n}_1\mathbf{n}_2$ of length 2 such that $\mathbf{n}_1\mathbf{n}_2 \equiv \mathbf{e}_\infty\mathbf{n}$. It is easy to see that $\mathbf{n}_1 \equiv \mathbf{n}$ and $\mathbf{n}_2 \equiv \mathbf{e}_\infty$. So null monomial $\mathbf{n}_1\mathbf{n}_2$ represents an ordered pair of points.

In contrast, $\mathbf{e}_\infty \wedge \mathbf{n}$ represents an unordered pair of points, because $\mathbf{e}_\infty \wedge \mathbf{n} \equiv \mathbf{n} \wedge \mathbf{e}_\infty$. When the sign is take into account, then $\mathbf{e}_\infty \wedge \mathbf{n}$ also represents the ordered pair of points $\mathbf{e}_\infty, \mathbf{n}$.

Example 2. Let null vectors $\mathbf{n}_1, \mathbf{n}_2$ represent two different affine points. Then $\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$ is a null monomial of length 3. If there is another null monomial $\mathbf{n}'_1\mathbf{n}'_2\mathbf{n}'_3$ of length 3 such that $\mathbf{n}'_1\mathbf{n}'_2\mathbf{n}'_3 \equiv \mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$, direct arguments show that $\mathbf{n}'_1 \equiv \mathbf{e}_\infty$, $\mathbf{n}'_3 \equiv \mathbf{n}_2$, and $\mathbf{n}'_2$ can be any affine point on line $\mathbf{n}_1\mathbf{n}_2$ other than $\mathbf{n}_2$. So $\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$ represents a pair of ordered points $\mathbf{e}_\infty, \mathbf{n}_2$, together with line $\mathbf{n}_1\mathbf{n}_2$.

In contrast, $\mathbf{e}_\infty \wedge \mathbf{n}_1 \wedge \mathbf{n}_2$ only represents the line passing through points $\mathbf{n}_1, \mathbf{n}_2$. To represent this line with the outer product, $\mathbf{e}_\infty, \mathbf{n}_1, \mathbf{n}_2$ can be replaced by any other three points on the line.

If the sign of $\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$ is taken into account, then any affine point $\mathbf{p}$ has a unique null-vector representation, denoted by the same symbol, that satisfies $\mathbf{e}_\infty \cdot \mathbf{p} = -1$. A point $\mathbf{n}_3$ on line $\mathbf{n}_1\mathbf{n}_2$ has the following unique null vector representation:

$$\mathbf{n}_3 = \lambda\mathbf{n}_1 + (1-\lambda)\mathbf{n}_2 + \epsilon\mathbf{e}_\infty, \tag{2}$$

for some scalar ϵ that ensures $\mathbf{n}_3^2 = 0$. In fact,

$$\epsilon = -\lambda(1-\lambda)\mathbf{n}_1 \cdot \mathbf{n}_2 = \frac{d^2_{\mathbf{n}_1\mathbf{n}_2}}{2}\lambda(1-\lambda). \tag{3}$$

So $\mathbf{e}_\infty\mathbf{n}_3\mathbf{n}_2 = \lambda\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$, and $\mathbf{e}_\infty\mathbf{n}_3\mathbf{n}_2 \equiv_+ \mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$ if and only if point $\mathbf{n}_3$ is on line $\mathbf{n}_1\mathbf{n}_2$ and on the same side with $\mathbf{n}_1$ relative to $\mathbf{n}_2$. So $\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2$ represents an ordered pair of points $\mathbf{e}_\infty, \mathbf{n}_2$, and the ray from $\mathbf{n}_2$ towards $\mathbf{n}_1$.

Similarly, $\mathbf{n}_2\mathbf{n}_1\mathbf{e}_\infty$ represents an ordered pair of points $\mathbf{n}_2, \mathbf{e}_\infty$, and line $\mathbf{n}_1\mathbf{n}_2$. When the sign is taken into account, then in the geometric interpretation, the line is replaced by the ray from $\mathbf{n}_2$ towards $\mathbf{n}_1$.

Formally, over a base numbers field $\mathbb{K}$, the *null monomial algebra* on an alphabet $\mathcal{A} = \{\mathbf{n}_i\}_{i\in I}$ is the quotient of the free associative algebra $\mathbb{K}[\mathcal{A}]$ modulo the two-sided ideal generated by elements of the form $\mathbf{n}_i\mathbf{n}_i$. Informally, we allow the alphabet $\mathcal{A}$ to take all null vectors (up to scale) of $\mathbb{R}^{n+1,1}$.

A null monomial has two ends. If the two ends are identical, then the null monomial is said to be *isotropic*, otherwise it is said to be *anisotropic*. The *degree* (or *length*) of a null monomial is the number of null vector factors in the sequence of the monomial. By definition, no two adjacent vector factors in the monomial are identical up to scale. An anisotropic monomial has degree > 1, while an isotropic monomial has degree ≥ 1.

In the first part of this talk, we present geometric interpretations for all null monomials in CGA, together with their normal forms leading to the geometric interpretations. The main conclusion is the following, called the *conformal split* of null monomials.

Theorem 1. In CGA, any isotropic null monomial $\mathbf{n}_1 \cdots \mathbf{n}_k\mathbf{n}_1$ equals up to scale $\mathbf{n}_1\mathbf{V} \equiv \mathbf{V}\mathbf{n}_1$, where $\mathbf{V}$ is a versor whose vector factors anticommute with $\mathbf{n}_1$, and can be chosen to anticommute with any fixed second null vector. Any anisotropic null monomial $\mathbf{n}_1 \cdots \mathbf{n}_k$ equals up to scale $\mathbf{n}_1\mathbf{n}_k\mathbf{V}_r$ (or $\mathbf{V}_l\mathbf{n}_1\mathbf{n}_k$), where $\mathbf{V}_r$ (or $\mathbf{V}_l$) is a versor whose vector factors anticommute with $\mathbf{n}_k$ (or $\mathbf{n}_1$), and can be chosen to anticommute with $\mathbf{n}_1$ (or $\mathbf{n}_k$) as well.

Geometrically, if versor $\mathbf{V}$ has its vector factors anticommute with both $\mathbf{n}_1$ and $\mathbf{n}_k$, then it induces a conformal transformation fixing the two points represented by $\mathbf{n}_1, \mathbf{n}_k$ respectively. For example, when $\mathbf{n}_1 = \mathbf{e}_\infty$, if $\mathbf{V}$ is even, then it induces a rotation fixing point $\mathbf{n}_k$, else it induces the composition of a rotation fixing point $\mathbf{n}_k$ and a mirror reflection with respect to a hyperplane passing through the point.

Let there be two anisotropic null monomials $\mathbf{M} = \mathbf{n}_0\mathbf{n}_1 \cdots \mathbf{n}_k$ and $\mathbf{M}' = \mathbf{n}_0\mathbf{n}'_1 \cdots \mathbf{n}'_l$. Then $\mathbf{M} \equiv \mathbf{V}_\mathbf{M}\mathbf{n}_0\mathbf{n}_k$ and $\mathbf{M} \equiv \mathbf{n}_0\mathbf{V}_{\mathbf{M}'}\mathbf{n}'_l$ for some versors $\mathbf{V}_\mathbf{M}, \mathbf{V}_{\mathbf{M}'}$ fixing point $\mathbf{n}_0$. So

$$\mathbf{M}\mathbf{M}' \equiv \mathbf{V}_\mathbf{M}\mathbf{n}_0\mathbf{n}_k\mathbf{n}_0\mathbf{V}_{\mathbf{M}'}\mathbf{n}'_l \equiv \mathbf{V}_\mathbf{M}\mathbf{V}_{\mathbf{M}'}\mathbf{n}_0\mathbf{n}'_l. \tag{4}$$

The product of the two monomials is equivalent to the geometric product of the two versors, followed by the common leading vector factor $\mathbf{n}_0$, and the ending vector factor of $\mathbf{M}'$.

For an anisotropic null monomial ended by $\mathbf{n}_0$, say $\mathbf{x}_k\mathbf{x}_{k-1} \cdots \mathbf{x}_1\mathbf{n}_0$, we can also start from the right side of the monomial to insert copies of $\mathbf{n}_0$, turn it into the normal form $\mathbf{x}_k(\mathbf{n}_0\mathbf{x}_{i_{2l}}\mathbf{x}_{i_{2l-1}}) \cdots (\mathbf{n}_0\mathbf{x}_{i_2}\mathbf{x}_{i_1})\mathbf{n}_0$, and provide a geometric interpretation similar to that in Theorem 1.

For two anisotropic null monomials $\mathbf{M} = \mathbf{n}_0\mathbf{n}_1 \cdots \mathbf{n}_k$ and $\mathbf{M}' = \mathbf{n}'_1 \cdots \mathbf{n}'_l\mathbf{n}_0$, we have $\mathbf{M} \equiv \mathbf{V}_\mathbf{M}\mathbf{n}_0\mathbf{n}_k$ and $\mathbf{M} \equiv \mathbf{n}'_1\mathbf{n}_0\mathbf{V}_{\mathbf{M}'}$ for some versors $\mathbf{V}_\mathbf{M}, \mathbf{V}_{\mathbf{M}'}$ fixing point $\mathbf{n}_0$. So

$$\mathbf{M}\mathbf{M}' \equiv \mathbf{V}_\mathbf{M}\mathbf{n}_0\mathbf{n}_k\mathbf{n}'_1\mathbf{V}_{\mathbf{M}'}\mathbf{n}_0 \equiv \mathbf{V}_\mathbf{M}\,\overrightarrow{\mathbf{n}_k\mathbf{n}'_1}\,\mathbf{V}_{\mathbf{M}'}\mathbf{n}_0. \tag{5}$$

The product of the two monomials is equivalent to the geometric product of the first versor, the displacement vector between the two ends other than $\mathbf{n}_0$ in the two monomials, the second versor, and the common end $\mathbf{n}_0$.

The second part of the talk is the core content. It addresses the problem of generating versors by anisotropic null monomials. Let

$$\mathbf{M} = \mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_r \tag{6}$$

be an anisotropic null monomial. We shift the first vector factor $\mathbf{x}_1$ to the end of the monomial to get $\mathbf{x}_2 \cdots \mathbf{x}_r\mathbf{x}_1$, then we multiply the result by a nonzero scalar λ^{-1} and add it up with $\mathbf{M}$. The result is the following null binomial that we call a *shifted-scaled null binomial*:

$$\mathbf{V} := \mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_r + \lambda^{-1}\mathbf{x}_2\mathbf{x}_3\cdots\mathbf{x}_r\mathbf{x}_1. \tag{7}$$

The pair $(\mathbf{M}, \lambda)$ is called a *shifted-scaled pair*. The leading vector factor $\mathbf{x}_1$ of $\mathbf{M}$ is the shifted vector factor, called the *index* of the binomial. Vector factor $\mathbf{x}_2$ is called the *left end* of the binomial, and $\mathbf{x}_r$ is called the *right end*. λ is called the *scaling factor* of the binomial.

In binomial $\mathbf{V}$ there are all together 4 ends: vector $\mathbf{x}_1$ that occurs twice, and vectors $\mathbf{x}_2, \mathbf{x}_r$ each occur once. When $\mathbf{x}_2 = \mathbf{x}_r$, then vector $\mathbf{x}_2$ also occurs twice as ends. In this case, binomial $\mathbf{V}$ can be rewritten in a form where $\mathbf{x}_2$ serves as the index.

Proposition 1. If a shifted-scaled null binomial has two vector factors each occurring twice as ends of the binomial, then any of them can serve as the index of the binomial.

Proof. Let

$$\mathbf{W} = \mathbf{x}_2\mathbf{x}_1\mathbf{x}_3\cdots\mathbf{x}_{r-1}\mathbf{x}_1 + \lambda^{-1}\mathbf{x}_1\mathbf{x}_3\cdots\mathbf{x}_{r-1}\mathbf{x}_1\mathbf{x}_2. \tag{8}$$

be a shifted-scaled null binomial indexed by $\mathbf{x}_2$, where $\mathbf{x}_1$ also occurs twice as ends. Write $\mathbf{W}$ as

$$\mathbf{W} \equiv \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3\cdots\mathbf{x}_{r-1}\mathbf{x}_1\mathbf{x}_2 + \lambda\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3\cdots\mathbf{x}_{r-1}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1. \tag{9}$$

The right side is a binomial generated by shifting $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3\cdots\mathbf{x}_{r-1}\mathbf{x}_1\mathbf{x}_2$ and then scaling by λ. Q.E.D.

Theorem 2. Every shifted-scaled null binomial is a versor in CGA. Conversely, up to a nonzero scalar or pseudoscalar factor, every versor in CGA equals a shifted-scaled null binomial. Because of this, when a versor takes the form of a shifted-scaled null binomial, we call it a *null versor*.

We raise some examples below.

Example 3. Let

$$\mathbf{V} = \mathbf{e}_\infty\mathbf{n} + \lambda^{-1}\mathbf{n}\mathbf{e}_\infty. \tag{10}$$

Then

$$\mathbf{V}\mathbf{V}^\dagger = \lambda^{-1}(\mathbf{e}_\infty\mathbf{n}\mathbf{e}_\infty\mathbf{n} + \mathbf{n}\mathbf{e}_\infty\mathbf{n}\mathbf{e}_\infty) = 4\lambda^{-1}(\mathbf{e}_\infty\cdot\mathbf{n})^2 \neq 0, \tag{11}$$

where we have used the *contraction identity*

$$\mathbf{n}_1\mathbf{n}_2\mathbf{n}_1 = 2(\mathbf{n}_1\cdot\mathbf{n}_2)\mathbf{n}_1, \tag{12}$$

for any null vector $\mathbf{n}_1$. So $\mathbf{V}$ is a versor in the Clifford algebra over the 2-space $\mathbf{e}_\infty \wedge \mathbf{n}$, and generates a dilation centered at affine point $\mathbf{n}$.

In details, let $\mathbf{n} = \mathbf{e}_0$ be the origin, and let $\mathbf{n}_\mathbf{x} = \mathbf{e}_0 + \mathbf{x} + \mathbf{e}_\infty\mathbf{x}^2/2$ be the null vector representation of point $\mathbf{x} \in (\mathbf{e}_\infty \wedge \mathbf{n})^\perp = \mathbb{R}^n$, then with $\mathbf{e}_0\cdot\mathbf{e}_\infty = -1$, we have

$$\begin{aligned} Ad_\mathbf{V}(\mathbf{n}_\mathbf{x}) &\equiv (\mathbf{e}_\infty\mathbf{e}_0 + \lambda^{-1}\mathbf{e}_0\mathbf{e}_\infty)(\mathbf{e}_0 + \mathbf{x} + \mathbf{e}_\infty\mathbf{x}^2/2)(\mathbf{e}_0\mathbf{e}_\infty + \lambda^{-1}\mathbf{e}_\infty\mathbf{e}_0) \\ &= 4(\lambda^{-2}\mathbf{e}_0 + \lambda^{-1}\mathbf{x} + \mathbf{e}_\infty\mathbf{x}^2/2) \\ &\equiv \mathbf{e}_0 + \lambda\mathbf{x} + \mathbf{e}_\infty(\lambda\mathbf{x})^2/2. \end{aligned} \tag{13}$$

So λ is the dilation ratio. In particular, if $\lambda = 1$, then $\mathbf{e}_\infty\mathbf{n} + \mathbf{n}\mathbf{e}_\infty \equiv 1$ induces the identity transformation.

Example 4. Let

$$\mathbf{V} = \mathbf{e}_\infty\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_1 + \lambda^{-1}\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_1\mathbf{e}_\infty \equiv \frac{\mathbf{e}_\infty\mathbf{n}_1}{\mathbf{e}_\infty\cdot\mathbf{n}_1} + \lambda^{-1}\frac{\mathbf{n}_2\mathbf{e}_\infty}{\mathbf{e}_\infty\cdot\mathbf{n}_2}. \tag{14}$$

Then

$$\mathbf{V}\mathbf{V}^\dagger \equiv \mathbf{e}_\infty\mathbf{n}_1\mathbf{e}_\infty\mathbf{n}_2 + \mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_1\mathbf{e}_\infty = 4(\mathbf{e}_\infty\cdot\mathbf{n}_1)(\mathbf{e}_\infty\cdot\mathbf{n}_2) \neq 0. \tag{15}$$

Expanding $\mathbf{V}$ into graded terms, we get

$$\mathbf{V} \equiv (1 + \lambda^{-1}) + \mathbf{e}_\infty \wedge \left(\frac{\mathbf{n}_1}{\mathbf{e}_\infty\cdot\mathbf{n}_1} - \lambda^{-1}\frac{\mathbf{n}_2}{\mathbf{e}_\infty\cdot\mathbf{n}_2}\right). \tag{16}$$

By

$$\mathbf{e}_\infty \cdot \left(\frac{\mathbf{n}_1}{\mathbf{e}_\infty\cdot\mathbf{n}_1} - \lambda^{-1}\frac{\mathbf{n}_2}{\mathbf{e}_\infty\cdot\mathbf{n}_2}\right) = 1 - \lambda^{-1}, \tag{17}$$

if $\lambda = 1$, then $\mathbf{V}$ induces a translation, else it induces a dilation.

In details, if $\lambda = 1$, by $\mathbf{e}_\infty \cdot \mathbf{n}_i = -1$, we have

$$\mathbf{V}\mathbf{n}_1\mathbf{V}^{-1} \equiv (\mathbf{e}_\infty\mathbf{n}_1 + \mathbf{n}_2\mathbf{e}_\infty)\mathbf{n}_1(\mathbf{n}_1\mathbf{e}_\infty + \mathbf{e}_\infty\mathbf{n}_2) = \mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_1\mathbf{e}_\infty\mathbf{n}_1 \equiv \mathbf{n}_2, \tag{18}$$

so $\mathbf{V}$ induces the translation from point $\mathbf{n}_1$ to point $\mathbf{n}_2$.

If $\lambda \neq 1$, by (10), let $\mathbf{V} = (\mathbf{e}_\infty\mathbf{x} + \mu^{-1}\mathbf{x}\mathbf{e}_\infty)/\mathbf{e}_\infty \cdot \mathbf{x}$ for some unknown null vector $\mathbf{x}$ and scalar μ, so that $\mathbf{x}$ is the dilation center and λ is the dilation ratio. The 0-graded part and 2-graded part of the equality respectively give $\mu = \lambda$ and

$$\mathbf{x} = \frac{\lambda\mathbf{n}_1/\mathbf{e}_\infty \cdot \mathbf{n}_1 - \mathbf{n}_2/\mathbf{e}_\infty \cdot \mathbf{n}_2}{\lambda - 1} + \epsilon\mathbf{e}_\infty, \tag{19}$$

where scalar ϵ is chosen to make $\mathbf{x}^2 = 0$. Indeed,

$$\epsilon = \frac{\lambda\mathbf{n}_1 \cdot \mathbf{n}_2}{(\lambda-1)^2(\mathbf{e}_\infty \cdot \mathbf{n}_1)(\mathbf{e}_\infty \cdot \mathbf{n}_2)}. \tag{20}$$

Geometrically, $\mathbf{x}$ is the point on affine line $\mathbf{n}_1\mathbf{n}_2$ with affine ratio $\overrightarrow{\mathbf{n}_2\mathbf{x}} : \overrightarrow{\mathbf{n}_1\mathbf{x}} = \lambda$.

Example 5. Let

$$\mathbf{V} = \mathbf{e}_\infty\mathbf{n}_0\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_0 + \lambda^{-1}\mathbf{n}_0\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_0\mathbf{e}_\infty \equiv \mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_0 + \lambda^{-1}\mathbf{n}_0\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty. \tag{21}$$

It is easy to verify that $\mathbf{V}\mathbf{V}^\dagger \equiv 1$. Take $\mathbf{n}_0$ as the origin of $\mathbb{R}^n$. Then $\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty \equiv \overrightarrow{\mathbf{n}_1\mathbf{n}_2}\mathbf{e}$, where $\overrightarrow{\mathbf{n}_1\mathbf{n}_2} \in \mathbb{R}^n$ is the displacement vector from point $\mathbf{n}_1$ to point $\mathbf{n}_2$, and as a vector in $\mathbb{R}^{n+1,1}$, $\overrightarrow{\mathbf{n}_1\mathbf{n}_2}$ is orthogonal to both $\mathbf{e}_\infty, \mathbf{n}_0$. So

$$\mathbf{V} \equiv \overrightarrow{\mathbf{n}_1\mathbf{n}_2}\mathbf{e}_\infty\mathbf{n}_0 + \lambda^{-1}\mathbf{n}_0\overrightarrow{\mathbf{n}_1\mathbf{n}_2}\mathbf{e}_\infty = \overrightarrow{\mathbf{n}_1\mathbf{n}_2}(\mathbf{e}_\infty\mathbf{n}_0 - \lambda^{-1}\mathbf{n}_0\mathbf{e}_\infty) = (\mathbf{e}_\infty\mathbf{n}_0 - \lambda^{-1}\mathbf{n}_0\mathbf{e}_\infty)\overrightarrow{\mathbf{n}_1\mathbf{n}_2}. \tag{22}$$

By (10), $\mathbf{e}_\infty\mathbf{n}_0 - \lambda^{-1}\mathbf{n}_0\mathbf{e}_\infty$ induces the dilation of ratio $-\lambda$ centered at $\mathbf{n}_0$. So when $\lambda = -1$,

$$\mathbf{e}_\infty\mathbf{n}_0\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_0 - \mathbf{n}_0\mathbf{e}_\infty\mathbf{n}_1\mathbf{n}_2\mathbf{e}_\infty\mathbf{n}_0\mathbf{e}_\infty \equiv \overrightarrow{\mathbf{n}_1\mathbf{n}_2} \tag{23}$$

induces the reflection with respect to the affine hyperplane normal to $\overrightarrow{\mathbf{n}_1\mathbf{n}_2}$ and passing through point $\mathbf{n}_0$.

Theorem 3. All shifted-scaled pairs with the same index form a group under the component-wise product, called the *connecting group*, and the product is called the *connecting product*. For two pairs $(\mathbf{n}_0\mathbf{n}_1\cdots\mathbf{n}_r, \lambda)$, $(\mathbf{n}_0\mathbf{n}'_1\cdots\mathbf{n}'_s, \lambda')$, their connecting product $\big((\mathbf{n}_0\mathbf{n}_1\cdots\mathbf{n}_r)(\mathbf{n}_0\mathbf{n}'_1\cdots\mathbf{n}'_s),\ \lambda\lambda'\big)$ generates a null versor that equals the geometric product of three versors up to scale: the versor generated by the first pair, the versor generated by $(\mathbf{n}_0\mathbf{n}_r\mathbf{n}_0\mathbf{n}'_1, 1)$, and the versor generated by the second pair. Geometrically, the connecting group of shifted-scaled pairs with the same two indices is surjectively homomorphic to the group of conformal transformations fixing the two points represented by the two indices respectively.

For two shifted-scaled pairs $(\mathbf{n}_0\mathbf{n}_1\cdots\mathbf{n}_r, \lambda)$ and $(\mathbf{n}_0\mathbf{n}'_1\cdots\mathbf{n}'_s, \lambda')$ indexed by $\mathbf{n}_0$, their *prepending product* is of the form

$$\Big(\mathbf{n}_0\mathbf{x}(\mathbf{n}_0\mathbf{n}_1\cdots\mathbf{n}_r)(\mathbf{n}_0\mathbf{n}'_1\cdots\mathbf{n}'_s), \lambda\lambda'\Big), \tag{24}$$

where $\mathbf{x}$ is a null vector determined by the two pairs, whose explicit expression will not be presented here.

Theorem 4. All shifted-scaled pairs with the same index form a group under the prepending product, called the *prepending group* of scaled null monomials. This group is homomorphic to the group of null versors each being generated by such a pair. Geometrically, the prepending group of shifted-scaled pairs with the same index is surjectively homomorphic to the group of conformal transformations fixing the point represented by the index.

Finally, consider the geometric product of a sequence of invertible vectors and null vectors. Let $\mathbf{V}_1, \ldots, \mathbf{V}_{k+1}$ be versors, and let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be null vectors. Then

$$\begin{aligned}\mathbf{W} &= \mathbf{V}_1\mathbf{x}_1\mathbf{V}_2\mathbf{x}_2\cdots\mathbf{V}_k\mathbf{x}_k\mathbf{V}_{k+1}\\ &= (\mathbf{V}_1\mathbf{x}_1\mathbf{V}_1^{-1})\mathbf{V}_1\mathbf{V}_2\mathbf{x}_2(\mathbf{V}_1\mathbf{V}_2)^{-1}\cdots(\mathbf{V}_1\cdots\mathbf{V}_k)\mathbf{x}_k(\mathbf{V}_1\cdots\mathbf{V}_k)^{-1}\mathbf{V}_1\cdots\mathbf{V}_k\mathbf{V}_{k+1}\\ &= \mathbf{x}'_1\mathbf{x}'_2\cdots\mathbf{x}'_k\mathbf{V}_1\cdots\mathbf{V}_k\mathbf{V}_{k+1},\end{aligned} \tag{25}$$

where $\mathbf{x}'_i = (\mathbf{V}_1\cdots\mathbf{V}_i)\mathbf{x}_i(\mathbf{V}_1\cdots\mathbf{V}_i)^{-1}$ is a null vector, for all $1 \leq i \leq k$. By this and Theorem 1, we get

Theorem 5. The geometric product $\mathbf{W} = \mathbf{V}_1\mathbf{x}_1\mathbf{V}_2\mathbf{x}_2\cdots\mathbf{V}_k\mathbf{x}_k\mathbf{V}_{k+1}$ equals $\mathbf{x}'_1\mathbf{x}'_k\mathbf{V}'$, where $\mathbf{V}'$ is a versor. By shifting and scaling the subsequence of null vectors, the following versor can be generated from $\mathbf{W}$:

$$\mathbf{V}_1\mathbf{x}_1\mathbf{V}_2\mathbf{x}_2\cdots\mathbf{V}_k\mathbf{x}_k\mathbf{V}_{k+1} + \lambda^{-1}\mathbf{V}_1(\mathbf{V}_2\mathbf{x}_2\cdots\mathbf{V}_k\mathbf{x}_k)\mathbf{x}''_1\mathbf{V}_{k+1}, \tag{26}$$

where $\mathbf{x}''_1 = (\mathbf{V}_2\cdots\mathbf{V}_k)^{-1}\mathbf{x}_1(\mathbf{V}_2\cdots\mathbf{V}_k)$.