A SPACETIME ALGEBRA GUIDE TO OCTONIONS AND THE SYMMETRY GROUPS OF PARTICLE PHYSICS

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Abstract

In the context of a meeting bringing together people interested in applications of Geometric Algebra (GA) across wide areas of engineering and science, it is of interest to see if fairly elementary GA tools can help us understand an area of modern physics normally considered quite difficult conceptually, and which uses a whole set of advanced mathematical tools. Such a description could easily fit the topic of General Relativity, where the notions of curved manifolds, differential geometry and tensor analysis are often a barrier to people outside the field being able to feel they understand topics such as black holes and gravitational waves. These barriers are removed, or at least much reduced, if one instead takes a flat-space GA approach to gravity, as discussed in e.g. [1] and [2]. However, what we wish to talk about in this contribution are some topics in a different area, at the interface between mathematics and particle physics. This area concerns the use of octonions to motivate the symmetry groups arising in the standard model of particle physics, and also its extensions to 'unified theories' based upon groups with a larger set of symmetries and working in higher dimensions. Remarkably, we will be able to show that the octonions themselves, and the actions of these higher symmetry groups, can all be expressed using just the Spacetime Algebra (STA), i.e. the Geometric Algebra of 4d spacetime. This makes computations within e.g. the SU(3) group of quantum chromodynamics, and the representation of quark and gluon states, all accessible to someone with a computer algebra program able to work in the STA (or alternatively the $\mathrm{Cl}(1,3)$ space corresponding to the Conformal Geometric Algebra (CGA) of 2d (anti-)Euclidean space).

Recently, in [3], it was shown how we can indeed provide a faithful representation of octonions within the STA, and we argued there how the link with the *Dirac current* provides an interesting basis for why the STA is in fact a natural home for them.

Thus the aim for the first part of the current talk will be to briefly review, for a general GA audience, the STA and the way the Dirac algebra is expressed in it and then show how the non-associative octonion product can be expressed in terms of Dirac spinors.

Then we discuss how the particle physics SU(3) colour group can be represented by

double-sided multiplication by even-grade STA elements, and how this can be interpreted in terms of single-sided multiplication by octonions. The quantity which the SU(3) colour transformations leave invariant is the 'norm' of a general STA bivector F, defined as $\langle \gamma_0 F \gamma_0 F \rangle$. We show here, how this can be interpreted as a sub-part of more general requirement for the preservation of the norm of the Dirac current $\psi \gamma_0 \tilde{\psi}$ for a general Dirac spinor ψ, and not just for its bivector part, thus generalising from bivector norm to preservation of octonionic norm. This preservation is guaranteed for combinations of octonion multiplications, known as 'octonionic chains', which lie at the basis of ideas by Cohl Furey and others (e.g. [4]) about the link between octonions and the standard model of particle physics,. We can thus discuss the extent to which a reformulation of these can lead to a wholly STA-based version of the standard model. By this we mean a version where all group actions and quantities can be expressed in terms of elements of the STA, and since the STA is the geometric algebra of spacetime, can therefore be viewed as intrinsically geometric in nature.

This approach then naturally incorporates the question of whether larger symmetry groups than the SU(3), SU(2) and U(1) of the standard model can be accommodated within the approach based on octonions in the STA, and in particular whether the latter leads to particular candidates for a 'unification' symmetry group in particle physics, which can at some level 'explain' and link the symmetries we already know, and help relate various quantities within the standard model, the values of which are otherwise mysterious.

As part of this we link with recent work which has looked at the some exceptional Lie Groups as candidates for unification, such as for example in a series of papers by Wilson, Dray and Manogue (see e.g. [5]). These approaches, which use octonions, but in a matrix-based context, initially seem unlikely to be related to the STA at all, but in fact we have been able to translate the main aspects of the proofs into a wholly STA-based approach via extending our decoding of the octonions, to include also the *sedenions*. These are the next non-associative algebra one obtains after the octonions using the Cayley-Dickson process, and it turns out their properties can be reproduced by extending the octonion translation, which is in terms of Dirac spinors, to the entire 16-dimensional STA, i.e. by including odd as well as even elements. They do not form a division algebra, but nevertheless we have shown that using them we can now reach much larger symmetry groups in the STA, including an explicit version of the group Spin(10), the spinorial version of the orthogonal group SO(10), which has long been considered as a prime candidate for unification of the forces (see e.g. [6] and [7]).

In all these applications, the Cartan subalgebra (the maximal set of mutually commuting states) of the corresponding Lie group is very important, and we show how for the largest groups involved, for which the others mentioned above are subgroups, the Cartan subalgebra is picked out in the STA approach as corresponding to the set of *reflections* in STA elements, thus giving a wholly new insight into this aspect of Lie algebra.

We anticipate that the extension to sedenions will allow us to make contact (in STA terms) with the largest exceptional Lie group, namely E_8, which has also long-been proposed as the 'unification group' underlying the standard model, and this should hopefully be clear by the time of the talk. There is already a direct link with the octonions, and therefore the STA, in the fact that the root lattice of E_8 corresponds to norm-1 octonions with integer or half-integer coefficients for coordinates, which is again something we hope to have understood in STA terms in the near future.

Overall, although the individual topics mentioned above may sound somewhat esoteric or complicated for a general audience, there is a underlying aim of making some advanced aspects of particle physics approachable in a new way, and accessible to anyone who is familiar with the basics of GA and the STA.

References

[1] A. Lasenby, C. Doran, and S. Gull. Gravity, gauge theories and geometric algebra. *Royal Society of London Philosophical Transactions Series A*, 356:487, Mar. 1998, arXiv:gr-qc/0405033. doi:10.1098/rsta.1998.0178.

[2] A. N. Lasenby. Geometric Algebra, Gravity and Gravitational Waves. *Adv. Appl. Clifford Algebras*, 29(4):79, 2019, 1912.05960. doi:10.1007/s00006-019-0991-y.

[3] A. Lasenby. Some recent results for $SU(3)$ and octonions within the geometric algebra approach to the fundamental forces of nature. *Mathematical Methods in the Applied Sciences*, 47(3):1471–1491, 2024. doi:https://doi.org/10.1002/mma.8934.

[4] C. Furey. Charge quantization from a number operator. *Phys. Lett. B*, 742:195–199, 2015, 1603.04078. doi:10.1016/j.physletb.2015.01.023.

[5] R. A. Wilson, T. Dray, and C. A. Manogue. An octonionic construction of E8 and the Lie algebra magic square. *Innovations in Incidence Geometry: Algebraic, Topological and Combinatorial*, 20(2):611–634, 2023.

[6] F. Wilczek. SO(10) marshals the particles. *Nature*, 394(6688):15–15, 1998.

[7] A. J. Hamilton. The supergeometric algebra. *Advances in Applied Clifford Algebras*, 33(1):12, 2023.