Math. Appl. **8** (2019), 1–6

PENCILS AND SET OPERATORS IN 3D CGA

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Abstract. Geometric Algebra can be considered as a language that unifies mathematics, physics and computer sciences etc. Among other, CGA is of special interest for its powerful transformations and its ability to represent any hypersphere or hyperplane. Moreover, CGA is an algebra capable of representing pencils of spheres. This paper presents a reinterpretation of every objects of 3D CGA as pencils of spheres and introduces set operators on its elements (i.e. union, intersection, complement, etc). As an application, these operators are used to find the smallest tangent sphere of two skew lines.

1. **Introduction**

Clifford Algebras, also known as Geometric Algebras (GA), form a powerful and flexible mathematical framework, ideally tailored for investigating geometry and its associated fields. Most geometric algebras share the possibility to construct their objects with some of their points or by intersection [7, 1, 5]. This common property comes from the fact that these algebras allow the manipulation of pencils of objects, as established by preexisting work [8, 3, 2]. This paper presents a way to manipulate CGA objects through their pencils and defines set operators that applies to them. The choice of CGA comes from its very handy non-degenerate basis, the fact that it handles spheres, its wide popularity and also from the wide range of geometric transformations that comes with it.

This papers starts with a short introduction to CGA in Sec. 1.1, followed by some notations in Sec. 1.2. The proposal of a construction of CGA from pencils then follows in Sec. 2, allowing some set operator to be introduced in Sec. 2.2, that are extended for the specific case of grade-2 pencils in Sec. 2.3. An applicative case then illustrates our contributions in Sec. 3, before the conclusion in Sec. 4.

1.1. Conformal Geometric Algebra

Conformal Geometric Algebra (CGA) [7] is a well-known family of geometric algebras built on $\mathbb{R}^{d+1,1}$ that allows the representation of any hyperspheres and hyperplanes of dimension $n \leq d$, built by intersection of other hyperspheres or hyperplanes or by the wedge of $n+1$ of their points.

2010 *MSC:* primary 34K06, 34K25; secondary 39.

Keywords: Geometric Algebra, Conformal Mapping, Clifford Algebra, Conformal Geometric Algebra (QCGA), Pencil, Skew Lines.

1.2. Notations

Most of the notations in this paper are inspired from [4] and [6]. In addition, the followings are specified:

A^*	dual [4] of A with $A^* = AI^{-1}$
$\wedge\,,\vee\,,\times\,,\cdot$	outer, anti-outer, commutator and inner products [6]
$A \equiv B$	A equals B up to a non-zero scalar multiplicator

2. CGA view from the perspective of pencils

The previous section introduces CGA in any dimension. This section and the followings focus on CGA of $\mathbb{R}^3$ and its pencils.

2.1. Constructing CGA with pencils

This section aims to show that any objects of CGA is a pencil of spheres.

Definition 2.1. The 4-vectors of CGA are called spheres. Their set is denoted Spheres. They represent any sphere of real, imaginary, or infinite radius. A dual sphere of radius r and center of projective coordinates (x, y, z, w) is of the form:

$$S^* = w^2 e_o + xwe_1 + ywe_2 + zwe_3 + \frac{x^2 + y^2 + z^2 - r^2}{2} e_\infty \tag{2.1}$$

Definition 2.2. A n-pencil is the intersection of n spheres, computed by their anti-outer product $\big(a \vee b = (a^* \wedge b^*)^*\big)$. It is thus a blade of grade $5 - n$.

More precisely, a n-pencil is the vector space generated by the n spheres.

Definition 2.3. The anti-outer-product null space of a n-pencil A, denoted $\mathbb{NA}(A)$ is defined as

$$\mathbb{NA}\,(A) = \{S \in \text{Spheres} \mid A \vee S = 0\} \tag{2.2}$$

It corresponds to the set of all spheres in the object. For now on, the mention of the inclusion of any pencil A in a pencil B refers to the property $\mathbb{NA}(A) \subset \mathbb{NA}(B)$.

Definition 2.4. A point is a 4-pencil representing a location in the space. Denoted p, a point of projective coordinates (x, y, z, w) is of the form:

$$\mathrm{p} = w^2 e_o + xwe_1 + ywe_2 + zwe_3 + \frac{x^2 + y^2 + z^2}{2} e_\infty \tag{2.3}$$

A point also happens to be the dual of a sphere of radius zero.

Definition 2.5. The outer-product null space of a n-pencil A, denoted $\mathbb{NO}\,(A)$ is defined as

$$\mathbb{NO}\,(A) = \{\mathrm{p} \in \text{Points} \mid A \wedge \mathrm{p} = 0\} \tag{2.4}$$

It corresponds to the set of all points (thought as 4-pencils) in which the object is included. For now on, the mention of the inclusion of any point p in a pencil A refers to the property $\mathrm{p} \in \mathbb{NO}(A)$.

Definition 2.6. A pencil is said flat iif. it does not contain any sphere of finite radius. Flat pencils are the pencils containing the point at infinity e_∞. A pencil is said round iif. it contains at least one finite-radius sphere.

All geometric objects of CGA being blades and all blades of CGA being pencils, then all objects of CGA can be defined as pencils.

2.2. Set operator on pencils: the easy cases

Previous section presents pencils as vector spaces of spheres, this section introduces set operators for them. Let's first define an empty pencil and the pencil of all spheres. Relying on Def. 2.3 gives us I as the only blade containing no spheres, and 1 as the only blade containing every spheres. Therefore I is the empty pencil and 1 the pencil of all spheres.

Definition 2.7. Two pencils are said independent iif they share no sphere.

First of all, if two pencils are not independents, then their anti-outer product is 0. The independence criterion brings the following theorems:

Theorem 2.8. *The union $A \cup B$ of two independent pencils A and B is $A \vee B$. The set complement of any pencil A is its dual A^*. The intersection of two pencils A and B whose complement are independent is $A \wedge B$.*

Proof. Consider A and B independents n and m-pencils. $A = S_1 \vee \cdots \vee S_n$, $B = S_{n+1} \vee \cdots \vee S_{m+n}$, then $A \vee B = S_1 \vee \cdots \vee S_{m+n}$, hence the union. Also $A \vee A^* \equiv 1$ is the whole space, hence A^* is the complement of A. The union and complement operators are enough to create the intersection operator, hence taking any A and B so that A^* and B^* are independent, $(A^* \vee B^*)^* = A \wedge B$ is the intersection of A and B. □

The operators used ensure closure of the resulting sets. Commutativity is also given by the anti-commutativity of the products, as the sign of a pencil is irrelevant.

Theorem 2.9. *Any round n-pencil A can be decomposed into a flat $(n-1)$-pencil $\mathrm{Flat}(A) = A \wedge e_\infty$ and its smallest sphere $\mathrm{Small}(A) = A \wedge \mathrm{Flat}(A)$.*

Proof. This theorem is proven for every grade. For 2-pencils, the fact that a circle C is the dual of a point pair $\mathrm{p}_1 \wedge \mathrm{p}_2$ is used. For clarity, two functions Center and Radius are used, giving the center and the radius of an object.

$$\mathrm{Small}(1) = e_\infty^* \qquad \mathrm{Small}(S) = S \qquad \text{with } S \text{ a sphere} \tag{2.5}$$

$$\mathrm{Small}(I) = 0 \qquad \mathrm{Small}(\mathrm{p}) \equiv \mathrm{p}^* \qquad \text{with p a point} \tag{2.6}$$

$$\mathrm{Small}(\mathrm{p}_1 \wedge \mathrm{p}_2) \equiv \mathrm{Center}(\mathrm{p}_1 \wedge \mathrm{p}_2) - \frac{\mathrm{Radius}^2(\mathrm{p}_1 \wedge \mathrm{p}_2)}{2} e_\infty \tag{2.7}$$

$$\mathrm{Small}\big((\mathrm{p}_1 \wedge \mathrm{p}_2)^*\big) \equiv \mathrm{Center}(\mathrm{p}_1 \wedge \mathrm{p}_2) + \frac{\mathrm{Radius}^2(\mathrm{p}_1 \wedge \mathrm{p}_2)}{2} e_\infty \tag{2.8}$$

$$\equiv \mathrm{Center}(C) - \frac{\mathrm{Radius}^2(C)}{2} e_\infty \tag{2.9}$$

In every case, the resulting sphere is the smallest possible, with all the points of the pencil on its equator. The case of the 0-pencil differs as it contains no spheres, therefore the result is 0. For a 5-pencil, the result is the dual of the point at infinity, which can be interpreted as a sphere of radius 0 centered on it. □

This brings meaning to 1-vectors that are not points (duals of spheres of non-null radius), which can now be interpreted as the intersection of a flat point Fp (their center) and a small sphere S of same center, hence of the form $\mathrm{Fp} \vee S$.

2.3. Set operator on 2-pencils

The previous section introduces set operators for independent and equal pencils, while this section introduces set operators for 2-pencils that are neither independent nor equal. 2-pencils consist of circles with real or imaginary radii. When two circles are not independent, they intersect at a common sphere. Therefore this section examines two circles $C_1 = S_a \vee S_b$ and $C_2 = S_a \vee S_c$. The initial step involves finding the symmetric difference $C_1 \triangle C_2 = (C_1 \cup C_2) \setminus (C_1 \cap C_2)$.

Theorem 2.10. *Consider S_a, S_b and S_c three independent spheres with $S_a^2 \neq 0$.*

$$(S_a \vee S_b) \times (S_a \vee S_c) \equiv (S_b \vee S_c)^* \tag{2.10}$$

Proof. Consider $C_1^* = S_a \wedge S_b$ and $C_2^* = S_a \wedge S_c$ with S_a of non-zero radius and potentially a plane. S_a, S_b and S_c can be chosen orthogonal, meaning that their inner product two by two is 0.

$$C_1 \times C_2 = -C_1^* \times C_2^* = (S_a \wedge S_b)(S_a \wedge S_c) = -S_a^2 (S_b \wedge S_c) \equiv S_c \tag{2.11}$$

Hence the property is true for a shared non-punctual sphere. □

This permits a general formulation for the symmetric difference of two circles.

Theorem 2.11. *Consider C_1 and C_2 two circles.*

$$C_1 \triangle C_2 = (C_1 \times C_2)^* + C_1 \vee C_2 + C_1 \wedge C_2^* \tag{2.12}$$

Proof. The three terms yield $C_1 \triangle C_2$ across three distinct scenarios: independence, equality, and neither. Each term evaluates to 0 in the absence of the respective case, thereby complementing one another. □

Theorem 2.12. *If C_1 and C_2 are round and share exactly one common round sphere, then*

$$C_1 \cup C_2 = C_1 \vee \mathrm{Flat}(C_1 \triangle C_2) \tag{2.13}$$

Proof.

$$\begin{aligned}
C_1 &= S_a \vee S_b && (2.14)\\
F &= \mathrm{Flat}(C_1 \times C_2) = \lambda S_b + \mu S_c && (2.15)\\
F \vee C_1 &= S_a \vee S_b \vee (\lambda S_b + \mu S_c) \qquad \text{with } \lambda, \mu \in \mathbb{R} && (2.16)\\
&\equiv S_a \vee S_b \vee S_c && (2.17)
\end{aligned}$$

□

Theorem 2.13. *If l_1 and l_2 are two lines lying on a common plane P, then:*

$$l_1 \cup l_2 = l_1 \wedge \big((l_1 \triangle l_2) \vee e_{123}\big) \tag{2.18}$$

Proof.

$$l_1 = P \vee P_\perp \quad \text{with} \quad P_\perp = l_1 \wedge P^* \tag{2.19}$$

$$l_1 \triangle l_2 = (p_1 \wedge u \wedge e_\infty) \triangle (p_2 \wedge v \wedge e_\infty) = p_3 \wedge w \wedge e_\infty \tag{2.20}$$

$$\text{with } u, v \text{ and } w \text{ Euclidean vectors of CGA}$$

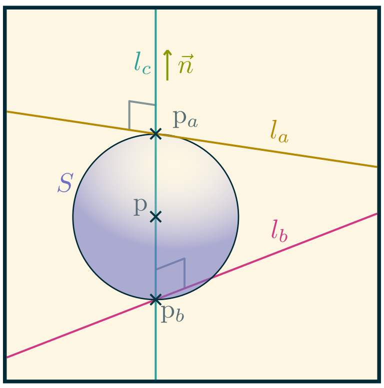
$$l_3 \vee e_{123} = l_3 \cdot e_{o\infty} \equiv w \quad \text{and} \quad P = l_1 \wedge w \tag{2.21}$$

$$l_1 \cup l_2 = P_\perp \vee l_2 \tag{2.22}$$

□

3. Applicative case: smallest sphere tangent to two skew lines

Two skew lines l_a and l_b are by definition not co-planar. There is exactly one point p that minimizes the distance to both lines, passing through a line l_c perpendicular to l_a and l_b and intersecting them in p_a and p_b. The sphere S of center p and passing through p_a and p_b is the smallest sphere tangent to l_a and l_b, and its diameter is their distance. Finding p is a problem often encountered in computer vision, which is why Dorst et al. proposed a PGA-based algorithm to find p_a and p_b from a point of each line and their directional vector. As an alternative, Alg 1 produces the sphere S. The line l_c is found using Th. 2.11. Subtracting the direction of l_c to both pencils l_a and l_b using Th. 2.8 gives two parallel planes $P_{\|,a}$ and $P_{\|,b}$. Summing them results in the plane in-between them, passing through p. This plane is then intersected with l_c, giving the flat point $\mathrm{Fp} = \mathrm{p} \wedge e_\infty$, whose dual is the pencil of all spheres centered on p, then constrained by p_a to get S.



Algorithm 1: Find smallest tangent sphere of two skew lines

Function `skew_lines_sphere`
- **Input:** l_a, l_b
- **Output:** $\mathrm{p} \wedge e_\infty$
- $l_c \leftarrow (l_a \times l_b)^*$
- $n \leftarrow l_c \vee e_{123}$ `// Euclidean vector`
- $P_{\|,a} \leftarrow l_a \wedge (l_a \wedge n)^*$
- $P_{\|,b} \leftarrow l_b \wedge (l_b \wedge n)^*$
- $\mathrm{Fp} \leftarrow (P_{\|,a} + P_{\|,b}) \vee l_c$
- $\mathrm{Fp}_a \leftarrow P_{\|,a} \vee l_c$
- $x_a \leftarrow -(e_{o\infty} \cdot (\mathrm{Fp}_a \wedge e_o))/(e_{o\infty} \cdot \mathrm{Fp}_a)$
- $\mathrm{p}_a \leftarrow e_o + x_a + \frac{1}{2}x_a^2 e_\infty$
- **return** $\mathrm{Fp}^* \wedge \mathrm{p}_a$

Figure 1. The algorithm and an illustration of the use case.

4. Conclusion

This paper introduced an interpretation of CGA objects purely based on pencils of spheres, as well as some set operators taking advantage of this new framework. This contribution is illustrated by an application that finds a sphere of interest of a pair of skew lines. That specific applicative case could be generalized to circles since lines are circles. In a broader picture, this paper is a first step toward a set theory re-interpretation of CGA powered by pencils of curves. As a future work, we intend to extend the set operators to the yet unsupported cases, as well as proposing more applicative cases to demonstrate their potency.

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