Local Space Structure by Geometric Algebra Using the Hurwitz Unit Quaternions

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Introduction of Prerequisites

Light photons propagate *retarded* huge distances through physical space, but what do we know about one local fermion beside it exclude the others? We ask:

What is the quality trait of the space locality for one indivisible spin½ fermion?

We start internal in a spatial unit sphere which surface we call S^2, and state:
The center point is that which has no part in in Euclidian Geometry nor in Physics.
Two geometric points support one line segment as one 1-vector $\longmapsto$ ***direction***.
Three 1-vector ***direction*** as spokes from a fictive center support a circumference S^1 like a Mercedes star ✇ circle wheel. Then our intuition of a unit sphere gives that :
To support the spherical surface S^2 from a fictitious center we need four 1-vector ***directions*** as unit spokes, radial pointing-out four points on the spherical surface.
We call this fiction a *tetraon* and see it as a fundamental structure idea for a spatial sphere forming a locality of four ***directions***. We prefer a symmetric regular tetraon.
Defining the locality, we prefer a unit sphere forming an *inside* and an *outside*.

The Geometric Algebra

For the intuition of product operations with the ***directions*** in physical space we use the Geometric Algebra $\mathcal{G}_{3,0}(\mathbb{R})$ that is an extract from Space-Time Algebra STA, $\mathcal{G}_{1,3}(\mathbb{R})$ introduced by David Hestenes [1].[6] The spacetime aspect demand that any oscillating action on the surface S^2 never exceed the *retardation of information*, known as the speed of light. This restriction we store in the idea of angular momentum as spacelike bivector generators,[7] and the timelike bivectors manifest their ontological existence in the 1-vector ***directions***.[8] The spacelike bivectors is angular generators for active spinors in the even subalgebra of quaternions $\mathbb{H} = \mathcal{G}_{0,2}^{\perp}(\mathbb{R}) \sim \mathcal{G}_{3,0}^{+}(\mathbb{R}) \subset \mathcal{G}_{1,3}^{+}(\mathbb{R})$.

The present study

We find sixteen *interconnected* 2-spinors of Hurwitz unit-½-quaternions generated from eight orientated bivectors of four spatial ***directions*** intern relative stable in a regular tetrahedron space symmetry of the locality inside a unit sphere.

To get an intuition of the physical impact we link the bivector generators to the angular momentum of internal oscillations inside one spherical spin½ fermion. We take a superposition of these four active angular 1-spinors to format one indivisible fermion.

►► Main Letter + Appendix ►►

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[5] The book: Research on the a priori of Physic [7], and a book manuscript under preparation.
[6] Hestenes called $\mathcal{G}_{3,0}(\mathbb{R})$ the *Pauli Algebra*, generated by the orthonormal set $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1 \in \mathcal{G}_{3,0}$.
[7] Example: $\boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \gamma_2\gamma_1 = i\gamma_3\gamma_0$, by the pseudoscalar $\boldsymbol{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i = \gamma_0\gamma_1\gamma_2\gamma_3$.
[8] An example from STA is the ***directions***: $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$, that exist over counting times $\gamma_0[\gamma_0^{-1}]=1$.

1. Local Space Structure by Geometric Algebra Using the Hurwitz Unit Quaternions

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Introduction:

This letter use the work of Adolf Hurwitz's (1859-1919) on Number Theory of Quaternions [2] [3] (in German) to find a normal invariant subgroup of sixteen unit-½-quaternions by superposition of the orthonormal bivector basis. These unit 2-spinors interconnect by product relations; and bivector generators of these form a local regular tetrahedron space structure.

1.1. The Unit Quaternion Group and the Linear 2-spinor Algebra $\mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R})$

We define the fundamental units of the multiplicative quaternion group $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$.[9] Recall the strong *interconnected* multiplication structure of the quaternion basis

$$\boxed{\boldsymbol{i}_1{}^2 = \boldsymbol{i}_2{}^2 = \boldsymbol{i}_3{}^2 = -1, \qquad \begin{array}{ccccc} \boldsymbol{i}_1 & = & -\boldsymbol{i}_2\boldsymbol{i}_3 & = & \boldsymbol{i}_3\boldsymbol{i}_2\,, \\ \boldsymbol{i}_2 & = & -\boldsymbol{i}_3\boldsymbol{i}_1 & = & \boldsymbol{i}_1\boldsymbol{i}_3\,, \\ \boldsymbol{i}_3 & = & -\boldsymbol{i}_1\boldsymbol{i}_2 & = & \boldsymbol{i}_2\boldsymbol{i}_1\,, \end{array}}^{10} \quad \begin{array}{c} \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = +1, \\ \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1 = -1, \end{array} \tag{1.1}$$

this presumes the geometric perpendicular plane bivector basis introduced in Appendix:

$$\underbrace{\boldsymbol{i}_3 \perp \boldsymbol{i}_1 \perp \boldsymbol{i}_2 \perp \boldsymbol{i}_3}_{\text{Geometric perpendicular}} \implies \underbrace{\boldsymbol{i}_1\cdot\boldsymbol{i}_2 = \boldsymbol{i}_2\cdot\boldsymbol{i}_3 = \boldsymbol{i}_3\cdot\boldsymbol{i}_1 = 0}_{\text{Algebraic orthogonal}} \implies \underbrace{\boldsymbol{i}_3 \coloneqq \boldsymbol{i}_2\boldsymbol{i}_1}_{\textit{Perturbated}} \tag{1.2}$$

The geometric perpendicular unit quaternion multiplication group have eight elements

$$\mathbb{U}_\perp(\mathbb{H})_8 = \left\{ \begin{array}{cccc} 1 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3, & \boldsymbol{i}_1 = \boldsymbol{i}_3\boldsymbol{i}_2, & \boldsymbol{i}_2 = \boldsymbol{i}_1\boldsymbol{i}_3, & \boldsymbol{i}_3 = \boldsymbol{i}_2\boldsymbol{i}_1, \\ -1 = \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1, & -\boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3, & -\boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1, & -\boldsymbol{i}_3 = \boldsymbol{i}_1\boldsymbol{i}_2 \end{array} \right\}_8, \tag{1.3}$$

that stays closed inside, performing an orthonormal *interconnectivity* structure of its three perpendicular plane directions and one independent unit scalar. All these four units $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ have two opposite orientations in symmetry. The division inverse is

$$\boxed{\boldsymbol{i}_k^{-1} = \frac{\boldsymbol{i}_k}{\boldsymbol{i}_k^2} = \frac{\boldsymbol{i}_k}{-1} = -\boldsymbol{i}_k = \boldsymbol{i}_k^\dagger \iff \boldsymbol{i}_k^{-1}\boldsymbol{i}_k = \frac{\boldsymbol{i}_k}{\boldsymbol{i}_k} = 1} \implies \boldsymbol{i}_k^\dagger\boldsymbol{i}_k = \boldsymbol{i}_k\boldsymbol{i}_k^\dagger = 1. \tag{1.4}$$

We use the conjugate notation ($\psi^\dagger$ to ψ) which is bivector inverse. – Performing the *isometric measure* of one plane unit ***direction*** $\boldsymbol{i}_k$ on the others $\boldsymbol{i}_j$ in $\mathbb{U}_\perp(\mathbb{H})_8$: $\boldsymbol{i}_k\boldsymbol{i}_j\boldsymbol{i}_k^\dagger = \boldsymbol{i}_j^\dagger = \boldsymbol{i}_j^{-1} = -\boldsymbol{i}_j$, for $j \neq k$, simply confirms that (1.3) is a *normal invariant* orthogonal subgroup which generates the ***full linear quaternion algebra*** from two units

$$\mathbb{H} = \mathcal{G}_{0,2}^\perp(\mathbb{R}) \longleftarrow span_{\mathbb{R}}\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} \quad \longleftarrow \mathcal{G}en_{\mathbb{R}}^\wedge\{\boldsymbol{i}_1, \boldsymbol{i}_2\}, \ \text{(etc. perturbated.)} \tag{1.5}$$

Every 2-spinor of the quaternion algebra $\mathbb{H} = \mathcal{G}_{0,2}^\perp(\mathbb{R}) \sim \mathcal{G}_{3,0}^+(\mathbb{R}) \subset \mathcal{G}_{3,0}(\mathbb{R})$, we write

$$S \coloneqq \lambda_0 + \textstyle\sum_{k=1}^3 \lambda_k\boldsymbol{i}_k = \lambda_0 1 + \lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3, \qquad \text{for } \forall\lambda_\kappa \in \mathbb{R}. \tag{1.6}$$

1.2. The Hurwitz Unit Quaternion Subgroup

Adolf Hurwitz (1859-1919) introduced in 1896 [2], and 1919 [3] in his work on the number theory of quaternions a closed subgroup set of 24 unit quaternion elements

$$\mathbb{U}(\mathbb{H}) \coloneqq \left\{ \pm 1, \ \pm\boldsymbol{i}_1, \ \pm\boldsymbol{i}_2, \ \pm\boldsymbol{i}_3, \ \left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16} \right\}_{24}, \tag{1.7}$$

which in a geometric context perform a *spatial structure* group from its subgroup (1.3)

$$\mathbb{U}_\perp(\mathbb{H})_8 \ \overrightarrow{\subset}\ \mathbb{U}(\mathbb{H})_{24} \ \overrightarrow{\subset}\ \mathbb{H} \ \overleftarrow{\sim}\ \mathcal{G}_3^+(\mathbb{R}) \ \overleftarrow{\subset}\ \mathcal{G}_{3,0}(\mathbb{R}) \leftarrow \mathcal{G}en_{\mathbb{R}}^\wedge\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}, \tag{1.8}$$

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[9] Hamilton named the quaternion basis $\boldsymbol{i} \equiv \boldsymbol{i}_3$, $\boldsymbol{j} \equiv \boldsymbol{i}_2$, $\boldsymbol{k} \equiv \boldsymbol{i}_1$, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$.

[10] Remark opposite orientation noted by Hurwitz [2] §1.(1). – A product $\wedge$ algebra $\mathcal{G}en_{\mathbb{R}}^\wedge\{\boldsymbol{i}_j, \boldsymbol{i}_k\}$.

1.3. The Sixteen Hurwitz unit-½-quaternions (½-versors)

This (1.3) orthonormal subgroup $\mathbb{U}_{\perp}(\mathbb{H})_8$ with eight 2^3 elements make by linear superposition the new units to the *Hurwitz Unit Quaternion Group* $\mathbb{U}(\mathbb{H}) \subset \mathbb{H}$, (1.7).

$$\varrho_{\epsilon} \in \mathbb{U}_{½}(\mathbb{H}) = \left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16}. \tag{1.9}$$

These extra sixteen units: $\mathbb{U}_{½}(\mathbb{H})_{16} = \mathbb{U}(\mathbb{H})_{24}/\mathbb{U}_{\perp}(\mathbb{H})_8$, are not mutually orthogonal. The index $\boldsymbol{\epsilon}$ of the unit element ϱ_{ϵ} indicate sixteen 2^4 combination of orientation signs

$$\boxed{\epsilon_{\mu} = \pm 1} \text{ for } \mu = 0,1,2,3\,; \quad \text{giving the index } \boldsymbol{\epsilon} \leftarrow (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \tag{1.10}$$

From this, we write each unit

$$\varrho_{\epsilon} = \tfrac{1}{2}(\epsilon_0 1 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3) = \varrho_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}. \text{ (sixteen different)} \tag{1.11}$$

these sixteen specific 2-rotors (unit 2-spinors) we call *unit-½-quaternions* or ½-versors, all $\varrho_{\epsilon} \in \mathbb{H}$. They have conjugation that just is a reversed 2-rotor (multiplication inverse)

$$\varrho_{\epsilon}^{\dagger} = \varrho_{\epsilon}^{-1} = \tfrac{1}{2}(\epsilon_0 1 - \epsilon_1 \boldsymbol{i}_1 - \epsilon_2 \boldsymbol{i}_2 - \epsilon_3 \boldsymbol{i}_3), \tag{1.12}$$

The *unit* magnitude of these $|\varrho_{\epsilon}| = 1$, we confirm by defining a unit measure

$$\varrho_{\epsilon}\varrho_{\epsilon}^{\dagger} = \varrho_{\epsilon}^{\dagger}\varrho_{\epsilon} = \tfrac{1}{4}(\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) = 1. \tag{1.13}$$

All the 24 unit elements of the *Hurwitz Unit Quaternion Subgroup* are now introduced

$$\mathbb{U}(\mathbb{H}) = \left\{ \pm 1, \ \pm \boldsymbol{i}_1, \ \pm \boldsymbol{i}_2, \ \pm \boldsymbol{i}_3, \ \left\{\tfrac{1}{2}(\epsilon_0 1 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3)\right\}_{16} \right\}_{24} \tag{1.14}$$

To confirm that this is a *closed multiplication group* we make test products:
A unit basis bivector $\boldsymbol{i}_k$ operating on one of its ½-versors give another ½-versor

$$\varrho_{\epsilon'} = \boldsymbol{i}_3 \varrho_{\epsilon} = \tfrac{1}{2}(+\epsilon_0 \boldsymbol{i}_3 + \epsilon_2 \boldsymbol{i}_1 - \epsilon_1 \boldsymbol{i}_2 - \epsilon_3), \qquad \text{etc. for } \boldsymbol{i}_2, \ \boldsymbol{i}_3 \tag{1.15}$$

Having two ½-versors (unit-½-quaternions)

$$\begin{aligned} \varrho_a &= \tfrac{1}{2}(\epsilon_{a0} + \epsilon_{a1}\boldsymbol{i}_1 + \epsilon_{a2}\boldsymbol{i}_2 + \epsilon_{a3}\boldsymbol{i}_3), \quad \text{and} \\ \varrho_b &= \tfrac{1}{2}(\epsilon_{b0} + \epsilon_{b1}\boldsymbol{i}_1 + \epsilon_{b2}\boldsymbol{i}_2 + \epsilon_{b3}\boldsymbol{i}_3), \end{aligned} \tag{1.16}$$

the simple product of these two gives a unit member of $\mathbb{U}(\mathbb{H})$

$$\begin{aligned} \varrho_a \varrho_b = &+¼(\epsilon_{a0}\epsilon_{b0} - \epsilon_{a1}\epsilon_{b1} - \epsilon_{a2}\epsilon_{b2} - \epsilon_{a3}\epsilon_{b3}) \\ &+¼(\epsilon_{a1}\epsilon_{b0} + \epsilon_{a0}\epsilon_{b1} + \epsilon_{a3}\epsilon_{b2} - \epsilon_{a2}\epsilon_{b3})\,\boldsymbol{i}_1 \\ &+¼(\epsilon_{a2}\epsilon_{b0} - \epsilon_{a3}\epsilon_{b1} + \epsilon_{a0}\epsilon_{b2} + \epsilon_{a1}\epsilon_{b3})\,\boldsymbol{i}_2 \\ &+¼(\epsilon_{a3}\epsilon_{b0} + \epsilon_{a2}\epsilon_{b1} - \epsilon_{a1}\epsilon_{b2} + \epsilon_{a0}\epsilon_{b3})\,\boldsymbol{i}_3 = \\ = &\begin{cases} \pm \boldsymbol{i}_k, \text{ if } \epsilon_{ak} = \mp \epsilon_{bk} \text{ for one } k, \text{and } \epsilon_{aj} = \pm \epsilon_{bj} \text{ for } j \neq k, & : \text{bivector}, \\ \pm 1\,, \text{ if } \epsilon_{a0} = \pm \epsilon_{b0} \text{ and } \epsilon_{ak} = \mp \epsilon_{bk}\,, & : \text{scalar}, \\ \text{else}, \ \tfrac{1}{2}(\epsilon_0 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3)\,, \quad \text{in all other cases} & : ½\text{-versor}. \end{cases} \end{aligned} \tag{1.17}$$

The simple multiplication by -1 gives $\varrho_{-\epsilon} = -\varrho_{\epsilon}$, again a ½-versor.
Squaring a ½-versor (1.11) turns the scalar part negative, but still a ½-versor

$$\varrho_{\epsilon}^2 = \epsilon_0 \varrho_{\epsilon} - 1 = -\tfrac{1}{2} + \epsilon_0 \tfrac{1}{2}(+\epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3), \tag{1.18}$$

a positive scalar $\epsilon_0 = 1$ prevents the reversing of the bivector part, then $\varrho^2 = \varrho - 1$, as in [3], simply $\left(\tfrac{1}{2}(1 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3)\right)^2 = \tfrac{1}{2}(-1 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3)$.

1.4. Stability of the Sixteen Hurwitz unit-½-quaternions

From the quaternion basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$, we have sixteen ½-versors (1.9)-(1.11). To simplify the possible perturbation structure between these sixteen elements we introduce extra as in [2] the reduced quaternion 1-spinors operators of the simplest form

$$\xi_{k_\pm} = \alpha(1 \pm \boldsymbol{i}_k) \quad \in \mathbb{H}, \text{ but } \notin \mathbb{U}(\mathbb{H}), \text{ and } \alpha \in \mathbb{R}, \quad k = 1,2,3 \tag{1.19}$$

in three orthogonal ⊥-plane-*directions* of the quaternion basis, with the reciprocal

$$\xi_{k_\pm}^{-1} = \alpha^{-1}(1 \pm \boldsymbol{i}_k)^{-1} = \alpha^{-1}\tfrac{1}{2}(1 \mp \boldsymbol{i}_k), \quad \text{where} \quad \xi_{k_\pm}\xi_{k_\pm}^{-1} = 1, \tag{1.20}$$

and freedom to choose $\alpha = \pm\sqrt{½}$ and we get unit 1-rotor operators of the form

$$U_{\boldsymbol{i}_k} = \sqrt{½}(1 + \boldsymbol{i}_k), \text{ and } U_{\boldsymbol{i}_k}^\dagger = \sqrt{½}(1 - \boldsymbol{i}_k), \text{ then } U_{\boldsymbol{i}_k}U_{\boldsymbol{i}_k}^\dagger = U_{\boldsymbol{i}_k}^\dagger U_{\boldsymbol{i}_k} = 1. \tag{1.21}$$

The trick is to take the product of two different 1-rotors to get a 2-rotor ½-versor, e.g.,

$$\begin{aligned} \varrho_{\mathbf{1}} &= \sqrt{½}(1+\boldsymbol{i}_2)\sqrt{½}(1+\boldsymbol{i}_1) = (1+\boldsymbol{i}_2)½(1+\boldsymbol{i}_1) = ½(1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3), \\ \text{or } = \varrho_{\mathbf{1}} &= (1+\boldsymbol{i}_3)½(1+\boldsymbol{i}_2) = (1+\boldsymbol{i}_1)½(1+\boldsymbol{i}_3) = ½(1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3). \end{aligned} \tag{1.22}$$

We now make a rotation test example; first by sandwich operating with $U_{\boldsymbol{i}_1}$ on (1.11)

$$\begin{aligned} U_{\boldsymbol{i}_1}\varrho_\epsilon U_{\boldsymbol{i}_1}^\dagger &= (1 + \boldsymbol{i}_1)[½(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3)]½(1 - \boldsymbol{i}_1) = \\ &½(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_3\boldsymbol{i}_2 - \epsilon_2\boldsymbol{i}_3), \end{aligned} \tag{1.23}$$

and further operating with $U_{\boldsymbol{i}_2}$ to get a perturbation of signs $\epsilon_k = \pm 1$ in this way

$$\begin{aligned} f(\varrho_\epsilon) = \varrho_{\mathbf{1}}\varrho_\epsilon\varrho_{\mathbf{1}}^\dagger &= U_{\boldsymbol{i}_2}U_{\boldsymbol{i}_1}\varrho_\epsilon U_{\boldsymbol{i}_1}^\dagger U_{\boldsymbol{i}_2}^\dagger = \\ &(1 + \boldsymbol{i}_2)½(1 + \boldsymbol{i}_1)[½(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3)]½(1 - \boldsymbol{i}_1)½(1 - \boldsymbol{i}_2) \\ &= \tfrac{1}{2}(\epsilon_0 + \epsilon_2\boldsymbol{i}_1 + \epsilon_3\boldsymbol{i}_2 + \epsilon_1\boldsymbol{i}_3). \end{aligned} \tag{1.24}$$

The scalar sign $\epsilon_0 = \pm 1$ is not altered by the permutation rotation. A dextral 2-rotor ½-versor permutating operator $\varrho_{\mathbf{1}}$ (1.22) building on the dextral basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ does not change the chirality by the permutation $\varrho_\epsilon \longrightarrow f(\varrho_\epsilon)$. This symmetry is just the same as permutate changing the reference basis sequence of names.

$$\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} \xrightarrow{f} \{\boldsymbol{i}_3, \boldsymbol{i}_1, \boldsymbol{i}_2\} \xrightarrow{f} \{\boldsymbol{i}_2, \boldsymbol{i}_3, \boldsymbol{i}_1\} \tag{1.25}$$

1.5. The Normal Invariant Unit ½-versor Part-group of Quaternions

In general, a sandwich operating on a *Unit-½-quaternion* (½-versor) ϱ_b (1.16) with left acting part ϱ_a and its right operating reversed part $\varrho_a^\dagger$ always gets one of the sixteen *Unit-½-quaternions*, normal invariant ½-versor units inside this subgroup $\mathbb{U}_{½}(\mathbb{H}) \subset \mathbb{H}$

$$\varrho_c = \varrho_a\varrho_b\varrho_a^\dagger = \varrho_a\varrho_b\varrho_a^{-1} = \tfrac{1}{2}(\epsilon_{c0} + \epsilon_{c1}\boldsymbol{i}_1 + \epsilon_{c2}\boldsymbol{i}_2 + \epsilon_{c3}\boldsymbol{i}_3), \tag{1.26}$$

were the sign signature components $\epsilon_{c\mu} = \pm 1$ fulfil

$$\begin{aligned} \epsilon_{c0} &= \epsilon_{b0}, \\ \epsilon_{c1} &= ½(\epsilon_{a1}\epsilon_{a2} + \epsilon_{a0}\epsilon_{a3})\epsilon_{b2} + ½(\epsilon_{a1}\epsilon_{a3} - \epsilon_{a0}\epsilon_{a2})\epsilon_{b3}, \\ \epsilon_{c2} &= ½(\epsilon_{a2}\epsilon_{a3} + \epsilon_{a0}\epsilon_{a1})\epsilon_{b3} + ½(\epsilon_{a2}\epsilon_{a1} - \epsilon_{a0}\epsilon_{a3})\epsilon_{b1}, \\ \epsilon_{c3} &= ½(\epsilon_{a3}\epsilon_{a1} + \epsilon_{a0}\epsilon_{a2})\epsilon_{b1} + ½(\epsilon_{a3}\epsilon_{a2} - \epsilon_{a0}\epsilon_{a1})\epsilon_{b2}. \end{aligned} \tag{1.27}$$

Then the sixteen ½-versor units (1.9) are mutually *interconnected* in a stable structure

$$\varrho_\epsilon \in \mathbb{U}_{½}(\mathbb{H}) = \left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16},$$

by their mutual interaction of its own members as unit rotations operators. All these sixteen possibilities of unit 2-rotor ½-versors perform a normal invariant subgroup, which comparts in a tetrahedron structure of ***directions***. This we designate it *the normal regular tetrahedron subgroup of quaternions* $\mathbb{U}_{½}(\mathbb{H}) \subset \mathbb{U}(\mathbb{H})_{24} \subset \mathbb{H}$. (the half-versors)

1.6. The Sixteen *Unit ½-versor Quaternion* Basis with Tetrahedron Directions

The sixteen distinct ½-versors of $\mathbb{U}_{½}(\mathbb{H})$ (1.9) we write out as

$$\begin{aligned} \varrho_0 &= \tfrac{1}{2}(+1 - \boldsymbol{i}_1 - \boldsymbol{i}_2 - \boldsymbol{i}_3), & \varrho_0^\dagger &= \tfrac{1}{2}(+1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3), \\ \varrho_1 &= \tfrac{1}{2}(+1 - \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3), & \varrho_1^\dagger &= \tfrac{1}{2}(+1 + \boldsymbol{i}_1 - \boldsymbol{i}_2 - \boldsymbol{i}_3), \\ \varrho_2 &= \tfrac{1}{2}(+1 + \boldsymbol{i}_1 - \boldsymbol{i}_2 + \boldsymbol{i}_3), & \varrho_2^\dagger &= \tfrac{1}{2}(+1 - \boldsymbol{i}_1 + \boldsymbol{i}_2 - \boldsymbol{i}_3), \\ \varrho_3 &= \tfrac{1}{2}(+1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 - \boldsymbol{i}_3), & \varrho_3^\dagger &= \tfrac{1}{2}(+1 - \boldsymbol{i}_1 - \boldsymbol{i}_2 + \boldsymbol{i}_3), \end{aligned} \tag{1.28}$$

$$\begin{aligned} \varrho_0^2 &= -\varrho_0^\dagger = \tfrac{1}{2}(-1 - \boldsymbol{i}_1 - \boldsymbol{i}_2 - \boldsymbol{i}_3), & -\varrho_0 &= \tfrac{1}{2}(-1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3) = (\varrho_0^\dagger)^2, \\ \varrho_1^2 &= -\varrho_1^\dagger = \tfrac{1}{2}(-1 - \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3), & -\varrho_1 &= \tfrac{1}{2}(-1 + \boldsymbol{i}_1 - \boldsymbol{i}_2 - \boldsymbol{i}_3) = (\varrho_1^\dagger)^2, \\ \varrho_2^2 &= -\varrho_2^\dagger = \tfrac{1}{2}(-1 + \boldsymbol{i}_1 - \boldsymbol{i}_2 + \boldsymbol{i}_3), & -\varrho_2 &= \tfrac{1}{2}(-1 - \boldsymbol{i}_1 + \boldsymbol{i}_2 - \boldsymbol{i}_3) = (\varrho_2^\dagger)^2, \\ \varrho_3^2 &= -\varrho_3^\dagger = \tfrac{1}{2}(-1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 - \boldsymbol{i}_3), & -\varrho_3 &= \tfrac{1}{2}(-1 - \boldsymbol{i}_1 - \boldsymbol{i}_2 + \boldsymbol{i}_3) = (\varrho_3^\dagger)^2. \end{aligned} \tag{1.29}$$

These unit ½-versors consist of a scalar ±½, plus a spinning[11] plane area bivector $\pm\mathbf{A}_\kappa$

$$\varrho_\kappa = +\frac{1}{2} + \mathbf{A}_\kappa, \quad \text{and} \quad \varrho_\kappa^\dagger = +\frac{1}{2} + \mathbf{A}_\kappa^\dagger, \quad \text{with} \quad \varrho_\kappa + \varrho_\kappa^\dagger = 1 \tag{1.30}$$

$\kappa = 0,1,2,3$. Recall (1.13) for each k, $\not{\Sigma}_{\kappa}$, $\varrho_\kappa \varrho_\kappa^\dagger = \frac{1}{4}(\epsilon_0^2 + \epsilon_1^2 + \epsilon_1^2 + \epsilon_1^2)_\kappa = \frac{1}{4} + \frac{3}{4} = 1$. The spin½ state $j = \frac{1}{2}$, gives $\lambda = j(j+1) = \frac{3}{4}$, [6]. The plane area generating bivectors in (1.30) for the spin½ intrinsic angular momentum for each κ are achieved by

$$\mathbf{A}_\kappa = \tfrac{1}{2}(\varrho_\kappa - \varrho_\kappa^\dagger), \quad \text{and} \quad \mathbf{A}_\kappa^\dagger = \tfrac{1}{2}(\varrho_\kappa^\dagger - \varrho_\kappa), \quad \mathbf{A}_\kappa \mathbf{A}_\kappa^\dagger = \tfrac{3}{4}, \not{\Sigma}_{\kappa}. \tag{1.31}$$

These eight bivectors in four pairs of reversed orientations $\mathbf{A}_\kappa^\dagger = -\mathbf{A}_\kappa$ we just write

$$\begin{aligned} \mathbf{A}_0 &= \tfrac{1}{2}(-\boldsymbol{i}_1 - \boldsymbol{i}_2 - \boldsymbol{i}_3), & \mathbf{A}_0^\dagger &= \tfrac{1}{2}(+\boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3), \\ \mathbf{A}_1 &= \tfrac{1}{2}(-\boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3), & \mathbf{A}_1^\dagger &= \tfrac{1}{2}(+\boldsymbol{i}_1 - \boldsymbol{i}_2 - \boldsymbol{i}_3), \\ \mathbf{A}_2 &= \tfrac{1}{2}(+\boldsymbol{i}_1 - \boldsymbol{i}_2 + \boldsymbol{i}_3), & \mathbf{A}_2^\dagger &= \tfrac{1}{2}(-\boldsymbol{i}_1 + \boldsymbol{i}_2 - \boldsymbol{i}_3), \\ \mathbf{A}_3 &= \tfrac{1}{2}(+\boldsymbol{i}_1 + \boldsymbol{i}_2 - \boldsymbol{i}_3), & \mathbf{A}_3^\dagger &= \tfrac{1}{2}(-\boldsymbol{i}_1 - \boldsymbol{i}_2 + \boldsymbol{i}_3), \end{aligned} \tag{1.32}$$

representing four plane ***direction*** of face areas of a regular tetrahedron internal in a ³space locality sphere **Fig. 1**. Note the algebraic rule of addition balance

$$\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = 0 \tag{1.33}$$

In the geometric mother algebra $\mathcal{G}_{3,0}(\mathbb{R})$ with the pseudoscalar $\boldsymbol{i} \coloneqq \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ for the quaternions $\mathbb{H} = \mathcal{G}_{0,2}^{\perp}(\mathbb{R}) \sim \mathcal{G}_{3,0}^{+} \subset \mathcal{G}_{3,0}(\mathbb{R}) \leftarrow \mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ from which the idea of bivectors coms, that

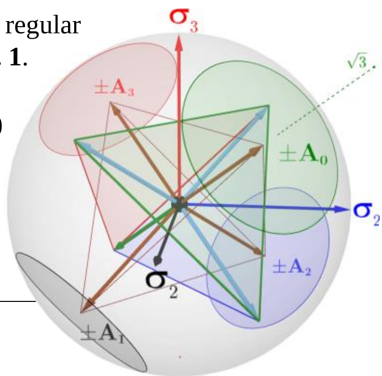


Fig. 1 The local sphere *tetrahedron* structure.
Four bivector ***directions*** of spinning plane angular momenta, illustrated circular rotating areas $|\mathbf{A}_\kappa| = \sqrt{3/4}$.
Eight relative *invariant* possibilities $\pm\mathbf{A}_\kappa$ and outwards *directions*.

[11] Remember that a bivector is rotating *invariant* in its own plane, so its angular spin is free.

Gives the quaternion basis (1.1): $\boldsymbol{i} \implies \quad \boldsymbol{i}_1 = \boldsymbol{i}\boldsymbol{\sigma}_1, \quad \boldsymbol{i}_2 = \boldsymbol{i}\boldsymbol{\sigma}_2, \quad \boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3,$ (1.34)
and the bivector areas (1.32) $\mathbf{A}_\kappa = \boldsymbol{i}\mathbf{k}_\kappa$, have the dual ***direction*** 1-vectors

$$\left\{\begin{aligned} \mathbf{k}_0 &= \tfrac{1}{2}(-\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3), \\ \mathbf{k}_1 &= \tfrac{1}{2}(-\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3), \\ \mathbf{k}_2 &= \tfrac{1}{2}(+\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3), \\ \mathbf{k}_3 &= \tfrac{1}{2}(+\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3) \end{aligned}\right\}_4, \qquad (1.35)$$

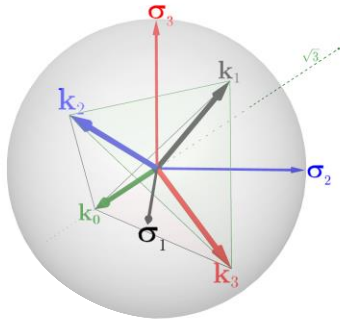


Fig. 2 A *tetraon* basis set $\{\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$, where the dextral structure $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ is preserved from Cartesian basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$, armed from $\mathbf{k}_0$. Note $|\mathbf{k}_\kappa| = \frac{1}{2}\sqrt{3}$.

that are displayed outwards in **Fig. 2**. This set $\mathbf{k}_\kappa$ and its parity inverted $\overline{\mathbf{k}_\kappa}$ is given dual from (1.31)

$$\begin{aligned} \mathbf{k}_\kappa &= \boldsymbol{i}\tfrac{1}{2}(\varrho_\kappa^\dagger - \varrho_\kappa), & \mathbf{k}_\kappa^2 = \tfrac{3}{4}, \text{ and} \\ \overline{\mathbf{k}_\kappa} &= -\mathbf{k}_\kappa = \boldsymbol{i}\tfrac{1}{2}(\varrho_\kappa - \varrho_\kappa^\dagger), & \text{see Fig. 1.} \end{aligned} \qquad (1.36)$$

These eight 1-vectors we ***direct*** outward from a locality center perpendicular normal to the faces (1.32) of a regular tetrahedron. The set $\mathbf{k}_\kappa$ we call a *tetraon*, and the parity inverted is also a *tetraon* combines as an *octaon*. These ***directions*** are dual to the plane area invariant spin rotating bivectors $\pm\mathbf{A}_\kappa = \pm\boldsymbol{i}\mathbf{k}_\kappa$, displayed **Fig. 1**

The minor disadvantage with this set $\{\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$ is the magnitudes $\sqrt{3/4}$, for each element, $\mathbf{k}_\kappa^2 = \frac{3}{4}$. The advantage is the balance $\mathbf{k}_0 + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$, of a regular *tetraon* symmetry which entails $\mathbf{k}_0 = -(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$, that is the antagonist 1-vector ***direction*** to the dextral structure 1,2,3, displayed in **Fig. 2**.

The locality spatial structure of this we elaborate in the Appendix: § 2.5-2.6

We normalise the regular *tetraon* ***directions*** $\mathbf{u}_\kappa = \frac{2}{\sqrt{3}}\mathbf{k}_\kappa$, to a structure basis set $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ supporting the local unit sphere, so $\mathbf{u}_0^2 = \mathbf{u}_1^2 = \mathbf{u}_2^2 = \mathbf{u}_3^2 = 1$,

$$\mathbf{u}_\kappa \mathbf{u}_\kappa = 4, \quad \text{and the algebraic rule} \quad \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0 \qquad (1.37)$$

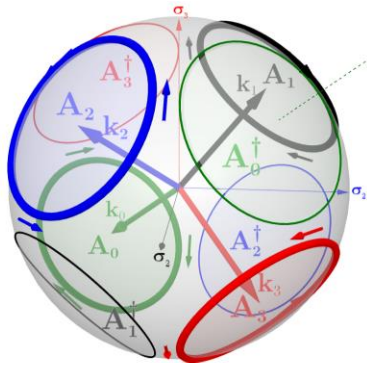
The spin½ 1-vector ***directions*** is then $\mathbf{s}_\kappa = \tfrac{1}{2}\mathbf{u}_\kappa = \sqrt{1/3}\,\mathbf{k}_\kappa$, and the angular momentum generating bivector is $\mathbf{S}_\kappa = \tfrac{1}{2}\boldsymbol{i}\mathbf{u}_\kappa = \sqrt{1/3}\,\mathbf{A}_\kappa$ in the regular tetrahedron structure where

$$\sum \mathbf{S}_\kappa \mathbf{S}_\kappa = -1, \quad \mathbf{S}_\kappa \mathbf{S}_\kappa^\dagger = 1, \quad \text{with the rule} \quad \mathbf{S}_0 + \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 = 0\,. \qquad (1.38)$$

Taking these four ***directions*** spin½ bivectors as generators of 1-spinor cyclic oscillators

$$\psi_{\kappa\pm}^{1/2} \sim \rho_\kappa U_{\phi_\kappa \pm} = \rho_\kappa e^{\pm\frac{1}{2}\boldsymbol{i}\mathbf{u}_\kappa \phi_\kappa} = \rho_\kappa(\cos \tfrac{1}{2}\phi_\kappa \pm \boldsymbol{i}\mathbf{u}_\kappa \sin \tfrac{1}{2}\phi_\kappa), \qquad (1.39)$$

Fig. 3 Display idea of spin½ Fermion cyclic 1-spinor oscillations of the tetrahedron ***direction*** symmetry in a local sphere. The four circle angular momenta bivectors $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ outwards dextral orientated, and their reversed outwards sinistral $\mathbf{A}_\kappa^\dagger = -\mathbf{A}_\kappa$ for fluctuating oscillations, that on the spherical surface does not exceed the *retarding speed of information*. Besides the eight circular oscillating bivector areas, there are six shapes ✧ in the three perpendicular *directions* of the orthogonal basis idea, they are not cyclic closed, hence no angular momentum.



that oscillate in the interconnected relative invariant plane ***directions*** of $\mathbf{A}_\kappa$ in a locality sphere displayed in **Fig. 3**. This result in angular momentum from the time *direction* $\phi_\kappa\gamma_0[\gamma_0^{-1}]$ of the phase development $\phi_\kappa = t\omega$. The bivector ***directions*** $\mathbf{S}_\kappa = \sqrt{1/3}\mathbf{A}_\kappa$ is the same as the spatial *directions* of the ½-versors $\varrho_{\kappa\pm} = \pm\frac{1}{2} + \mathbf{A}_\kappa$ (1.28)-(1.30) that is mutual interconnected by the measure operation (1.26).

1.7. The Hurwitz Integer Quaternions

Adolf Hurwitz introduced integer quaternions [2],§3,eq.(3) and [3],Vor.4,eq.(14),

$$g = \frac{1}{2}(g_0 + g_1\boldsymbol{i}_1 + g_2\boldsymbol{i}_2 + g_3\boldsymbol{i}_3), \tag{1.40}$$

where all the $g_\mu \in \mathbb{Z}$ are exclusive *odd* or *even* integers for different excitation combinations of physical localised quaternions, which should be studied more.

We have above studied the indivisible case with $g_\mu = \epsilon_\mu = \pm 1$, for $\mu = 0,1,2,3$; giving the geometric tetrahedron structure of a physical space locality.

Alternative choosing some $g_\nu = 0$, force the others even, $g_\mu \geq 2$ or 0, for $\mu \neq \nu$. The simple case is counting in the ***directions*** of the quaternion basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$, e.g., one direction $\boldsymbol{i}_3$ gives the 1-rotor oscillation $e^{\boldsymbol{i}_3\phi}$ transversal wavefront propagating external with the light speed of information. The next often used case is $g_0 = 2$ and one $g_k = \pm 2$ and two $g_j = 0$ for $j \neq k$, $j, k = 1,2,3$, as $(1 \pm \boldsymbol{i}_k)$ with the modulo 2 norm

$$(1 + \boldsymbol{i}_k)(1 - \boldsymbol{i}_k) = 2 \quad \leftrightarrow \quad (1 + \boldsymbol{i}_k)(1 + \boldsymbol{i}_k)^{-1} = 1, \tag{1.41}$$

that also preserve as (1.23) the *interconnectivity* tetrahedron structure of (1.28)→(1.32).

1.8. Concluding Idea of Unit Quaternions in Physics

This brief introduction the Unit Quaternion Group based on the over hundred years old work by Adolf Hurwitz on integer quaternions inspired an algebraic view of the geometric tetrahedron structure of local Space in physics, where the *non-directional* scalar parts of quaternions mix their impact into normal invariant stable symmetry structure of *interconnected* bivector *directions* of plane rotation invariant angular retarded activity inside a spherical *locality structure* considered as one indivisible ***entity***.

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2. Appendix:

2.1. Angular of Space Structure by Geometric Algebra

The fundament for Geometric Algebra is the idea of multiplication product of geometric objects, called *vectors*, *which brings something forth*. The simplest geometric object of ***grade***-1 is the 1-vector, which ***direction*** connect two geometric points[12]. The simplest actual product of two angular different 1-vectors **a** and **b** is separable into two parts, the *inner symmetric* of ***grade***-0, and the *outer antisymmetric* of ***grade***-2

$$\underbrace{\mathbf{ab}}_{product} = \underbrace{\tfrac{1}{2}(\mathbf{ab}+\mathbf{ba})}_{inner\ symmetric} + \underbrace{\tfrac{1}{2}(\mathbf{ab}-\mathbf{ba})}_{outer\ antisymmetric} = \underbrace{\mathbf{a}\cdot\mathbf{b}}_{scalar\ grade\text{-}0} + \underbrace{\mathbf{a}\wedge\mathbf{b}}_{bivector\ area\ grade\text{-}2}\ . \quad [1], [4] \qquad (2.1)$$

The classical Aristotelian three-dimensional space has: *length*$_1$, *breadth*$_2$, *depth*$_3$, ***directions*** we interpreted from a unit perpendicular basis displayed **Fig. 4**. Any 1-vector

$$\mathbf{r} = x_0\mathbf{u}_0 = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3, \quad \text{where} \quad x_\mu \in \mathbb{R}, \qquad (2.2)$$

is formed in ***direction*** $\mathbf{u}_0$ orientated spherical outwards. The additive linear combination (2.2) spans the 1-vector space $V_3(\mathbb{R}) \leftarrow span_{\mathbb{R}}\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ from the unit 1-vector basis set $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ constructed isometric $|\boldsymbol{\sigma}_1| = |\boldsymbol{\sigma}_2| = |\boldsymbol{\sigma}_3| = 1$, and perpendicular

$$\underbrace{\boldsymbol{\sigma}_3\perp\boldsymbol{\sigma}_1\perp\boldsymbol{\sigma}_2\perp\boldsymbol{\sigma}_3}_{\text{Geometric perpendicular}} \Longrightarrow \underbrace{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_1 = 0}_{\text{Algebraic orthogonal}}\ , \ \underbrace{\boldsymbol{\sigma}_k^2 = \epsilon_{AM} = 1}_{\text{Physical isometric}}\ . \qquad (2.3)$$

defined orthonormal and positive definite quadratic form by ***grade***-0 scalars of (2.1). This artefact orthonormal 1-vector basis $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ generate a Geometric Algebra $\mathcal{G}_{3,0}(\mathbb{R}) \leftarrow \mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ by making all products of these three unit basis 1-vectors: First the ***grade***-0 scalar units $\boldsymbol{\sigma}_k^2 = 1$. Then the three ***grade***-2 unit bivector plane areas

$$\boxed{\begin{array}{lllll} \boldsymbol{i}_1 := \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 & = & -\boldsymbol{i}_2\boldsymbol{i}_3 & = & \boldsymbol{i}_3\boldsymbol{i}_2 \\ \boldsymbol{i}_2 := \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 & = & -\boldsymbol{i}_3\boldsymbol{i}_1 & = & \boldsymbol{i}_1\boldsymbol{i}_3 \\ \boldsymbol{i}_3 := \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 & = & -\boldsymbol{i}_1\boldsymbol{i}_2 & = & \boldsymbol{i}_2\boldsymbol{i}_1 \end{array}} \Longrightarrow \begin{cases} \boldsymbol{i}_1{}^2 = \boldsymbol{i}_2{}^2 = \boldsymbol{i}_3{}^2 = -1, \\ \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1 = -1, \\ \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = +1. \end{cases} \qquad (2.4)$$

And the top ***grade***-3 of $\mathcal{G}_{3,0}(\mathbb{R})$ is the unit trivector commuting *chiral pseudoscalar*

$$\boldsymbol{i} \equiv \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 \Longrightarrow \boldsymbol{i}_1 = \boldsymbol{i}\boldsymbol{\sigma}_1,\ \boldsymbol{i}_2 = \boldsymbol{i}\boldsymbol{\sigma}_2,\ \boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3, \ \text{as the duals } \textbf{Fig. 5} \qquad (2.5)$$

By this, we created an orthogonal set of eight units which makes the linear span [5]

$$\mathcal{G}_{3,0}(\mathbb{R}) \longleftarrow span_{\mathbb{R}}\Big\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3,\ \{1,\ \boldsymbol{i}_1,\ \boldsymbol{i}_2,\ \boldsymbol{i}_3\},\ \boldsymbol{i}\Big\}_8 \leftarrow \mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}. \qquad (2.6)$$

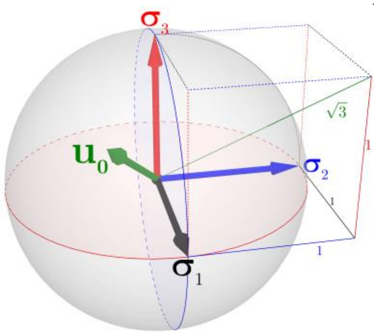
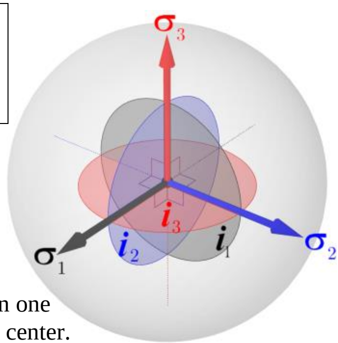


Fig. 4 Cartesian unit basis $\boldsymbol{\sigma}_3\perp\boldsymbol{\sigma}_1\perp\boldsymbol{\sigma}_2\perp\boldsymbol{\sigma}_3$ for a Bloch sphere ***direction*** $\mathbf{u}_0$.

Fig. 5 Perpendicular dual planes bivector basis $\boldsymbol{i}_3\perp\boldsymbol{i}_1\perp\boldsymbol{i}_2\perp\boldsymbol{i}_3$, that intersect in one local spherical center.



[12] A point is that which has no part (E.I.De.1.) see [7] §4.3, and Euclid's Elements, Book I .

Every multivector A in $\mathcal{G}_{3,0}(\mathbb{R}) = \mathcal{G}(V_3,\mathbb{R})$ can be resolved in eight mixed dimensions

$$\begin{aligned} A &= \alpha + \underbrace{x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3}_{} + \underbrace{\beta_1\boldsymbol{i}_1 + \beta_2\boldsymbol{i}_2 + \beta_3\boldsymbol{i}_3}_{} + \nu\boldsymbol{i}\ , \\ A &= \langle A\rangle_0 + \quad \langle A\rangle_1 \quad + \quad \langle A\rangle_2 \quad + \langle A\rangle_3 \end{aligned} \tag{2.7}$$

These separate into even and odd parts $\mathcal{G}_{3,0}(\mathbb{R}) = \mathcal{G}_3 = \mathcal{G}_3^+ + \mathcal{G}_3^-$,

$$A = \langle A\rangle^+ + \langle A\rangle^- = \langle A\rangle^+_{0,2} + \langle A\rangle^-_{1,3} = \underbrace{\alpha + \mathbf{b}\boldsymbol{i}}_{+} + \underbrace{\mathbf{a} + \nu\boldsymbol{i}}_{-} \tag{2.8}$$

Any arbitrary bivector plane *direction* $\boldsymbol{i}_0$ we span from the basis $\{\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$ as

$$\mathbf{B} = \langle\mathbf{B}\rangle_2 = \beta_0\boldsymbol{i}_0 \coloneqq \beta_1\boldsymbol{i}_1 + \beta_2\boldsymbol{i}_2 + \beta_3\boldsymbol{i}_3 = \mathbf{b}\boldsymbol{i} = (\beta_1\boldsymbol{\sigma}_1 + \beta_2\boldsymbol{\sigma}_2 + \beta_3\boldsymbol{\sigma}_3)\boldsymbol{i}. \tag{2.9}$$

These ***grade***-2 bivector blades $\langle A\rangle_2$ in $\mathcal{G}_{3,0}(\mathbb{R})$ are generators for plane angular change. The even part of (2.7)-(2.8) we get from the even basis: $\mathcal{G}^+_{3,0}(\mathbb{R}) \leftarrow span_{\mathbb{R}}\{1,\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$.

2.2. The Quaternions ℍ in the Even Algebra are 2-spinors

Every spinor of the quaternion group $\mathbb{H} = \mathcal{G}^{\perp}_{0,2}(\mathbb{R}) \sim \mathcal{G}^+_{3,0}(\mathbb{R}) \subset \mathcal{G}_{3,0}(\mathbb{R})$ we write

$$S \coloneqq \lambda_0 + \textstyle\sum_{k=1}^{3}\lambda_k\boldsymbol{i}_k = \lambda_0 1 + \lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3, \quad \text{for } \forall\lambda_\kappa \in \mathbb{R}. \ (1.6)\leftarrow(2.10)$$

The auto product square is $S^2 = SS = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \in \mathbb{R}$. But the *conjugated* reversed orientated of a *quaternion* in ℍ we define as $S^\dagger = \widetilde{S} = \lambda_0 - \lambda_k\boldsymbol{i}_k \in \mathbb{H}$.

$$|S|^2 = S\widetilde{S} = SS^\dagger = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0\,, \tag{2.11}$$

is the quadratic measure of a quaternion with magnitude $|S| = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2} \in \mathbb{R}$.

The uniqueness of the quaternion group $\mathbb{H} \sim \mathcal{G}^{\perp}_{0,2}(\mathbb{R})$ is that its unit basis $(\boldsymbol{i}_k^{\,2} = -1)$ is geometric perpendicular, which imply the strong *interconnectivity* (2.4), giving (1.2)

$$\underbrace{\boldsymbol{i}_3 \perp \boldsymbol{i}_1 \perp \boldsymbol{i}_2 \perp \boldsymbol{i}_3}_{\text{Geometric perpendicular}} \implies \underbrace{\boldsymbol{i}_1\cdot\boldsymbol{i}_2 = \boldsymbol{i}_2\cdot\boldsymbol{i}_3 = \boldsymbol{i}_3\cdot\boldsymbol{i}_1 = 0}_{\text{Algebraic orthogonal}} \implies \underbrace{\boldsymbol{i}_3 \coloneqq \boldsymbol{i}_2\boldsymbol{i}_1}_{\text{Perturbated}} \tag{2.12}$$

This *interconnection* $\boldsymbol{i}_3 = \boldsymbol{i}_2\boldsymbol{i}_1$, make the two basis elements generate the full group

$$\mathbb{H} = \mathcal{G}^{\perp}_{0,2}(\mathbb{R}) \longleftarrow span_{\mathbb{R}}\{1,\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\} \longleftarrow \mathcal{G}en^{\wedge}_{\mathbb{R}}\{\boldsymbol{i}_1,\boldsymbol{i}_2\}, \text{ etc. perturbated. } (1.5) \leftarrow(2.13)$$

2.3. Active Angular Spinors

If more than one plane is in play elements $S \in \mathbb{H}$ in ←(2.10) is called 2-spinors, else if only one plane *direction* is active, having a 1-spinor $S \coloneqq \lambda_0 1 + \beta_0\boldsymbol{i}_0$ from (2.9), or e.g.,

$$S_3 = (\lambda_0 + \lambda_3\boldsymbol{i}_3) = \varrho(\cos\tfrac{1}{2}\phi + \boldsymbol{i}_3\sin\tfrac{1}{2}\phi) = \varrho e^{+\boldsymbol{i}_3\frac{1}{2}\phi} = \varrho U_3, \tag{2.14}$$

The 2-spinor ←(2.10) we often want normalised as a 2-rotor $\Psi_{\mathbb{H}} = U = \widehat{S} = S/|S|$

$$\Psi_{\mathbb{H}} = \lambda_0 1 + \lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3, \quad\text{and}\quad \Psi^\dagger_{\mathbb{H}} = \lambda_0 1 - \lambda_1\boldsymbol{i}_1 - \lambda_2\boldsymbol{i}_2 - \lambda_3\boldsymbol{i}_3. \tag{2.15}$$

This *unit-quaternion* was by Hamilton (1805-65) called a *versor*, we call it a *2-rotor* demanding $\Psi_{\mathbb{H}}\Psi^\dagger_{\mathbb{H}} = \lambda_\mu\lambda_\mu = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. – We separate in two 1-spinors

$$\Psi_{\mathbb{H}} = 1(\lambda_0 + \lambda_3\boldsymbol{i}_3) + \boldsymbol{i}_1(\lambda_1 + \lambda_2\boldsymbol{i}_3) = 1S_3 + \boldsymbol{i}_1 S_2\,, \tag{2.16}$$

where the first is S_3 (2.14) and the different second 1-spinor is

$$S_2 = (\lambda_1 + \lambda_2\boldsymbol{i}_3) = \rho(\cos\tfrac{1}{2}\psi + \boldsymbol{i}_3\sin\tfrac{1}{2}\psi) = \rho e^{+\boldsymbol{i}_3\frac{1}{2}\psi} = \rho U_2. \tag{2.17}$$

Then these two 1-spinors (2.14) and (2.17) form 2-rotor (2.16) normalised $\varrho^2 + \rho^2 = 1$. A third 1-spinor S_1 is possible in composition with S_3 from (2.15) giving the separation

$$\Psi_{\mathbb{H}} = 1(\lambda_0 + \lambda_3\boldsymbol{i}_3) + \boldsymbol{i}_2(\lambda_2 - \lambda_1\boldsymbol{i}_3) = 1S_3 + \boldsymbol{i}_2 S_1\,, \tag{2.18}$$

but the oscillating 1-spinors S_1 is synchronised to S_2. The idea is that the three 1-spinors S_1, S_2, S_3 in their oscillations possesses angular momenta in the ***directions*** $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$.

2.4. The Quantum Quality of spin½ Angular Momentum

Coming to angular momentum we drop the oscillating 1-spinor content S_1, S_2, S_3, and concentrate on the perpendicular angular momenta 1-vector ***directions*** $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$, and their dual transversal angular area bivector ***directions*** $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$.

In (2.6) we introduced the full geometric algebra $\mathcal{G}_{3,0}(\mathbb{R}) \leftarrow \mathcal{G}en^{\wedge}_{\mathbb{R}}\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$. From this odd basis, we give the chiral pseudoscalar duality map to the even part $\{\boldsymbol{i}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\} \longrightarrow \{-\boldsymbol{i}^2, \boldsymbol{i}\boldsymbol{\sigma}_1, \boldsymbol{i}\boldsymbol{\sigma}_2, \boldsymbol{i}\boldsymbol{\sigma}_3\} = \{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} = \{\boldsymbol{\sigma}_k^2, \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\}$. **Fig. 5** displays perpendicular plans $\boldsymbol{i}_1 \perp \boldsymbol{i}_2 \perp \boldsymbol{i}_3$ intersecting in one local center point representing one possible indivisible spin½ fermion. In the traditional Quantum Mechanics ^{3}space spherical angular momentum state $|\lambda, m\rangle$ see [6], [7] has the lowest exited state of quantum number $j = \frac{1}{2}$, $\lambda = j(j+1) = \frac{3}{4}$, $m = \pm\frac{1}{2} \;\Rightarrow\; |\frac{3}{4}, \pm\frac{1}{2}\rangle$, two eigenvalue state

$$\mathbf{s}_3 \left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle \doteq \pm\tfrac{1}{2}\hbar\boldsymbol{\sigma}_3 \left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle, \quad \text{and} \quad J^2 \left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle \doteq \hbar^2 \tfrac{3}{4} \left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle, \; (\hbar = 1) \tag{2.19}$$

Where $J^2 = \mathbf{k}^2 = \mathbf{s}_1^2 + \mathbf{s}_2^2 + \mathbf{s}_3^2 = \mathbf{s}_k\mathbf{s}_k \rightarrow \frac{3}{4}$, (Casimir invariant) for the first excitation. From this we presume three perpendicular ***directions*** spin½ 1-vector operators

$$\mathbf{s}_1 = \tfrac{1}{2}\hbar\boldsymbol{\sigma}_1, \quad \mathbf{s}_2 = \tfrac{1}{2}\hbar\boldsymbol{\sigma}_2, \quad \mathbf{s}_3 = \tfrac{1}{2}\hbar\boldsymbol{\sigma}_3, \quad \in \mathcal{G}_{3,0}^- \;\longleftarrow\; \mathbf{s}_k = -\hbar\boldsymbol{i}\mathbf{S}_k, \tag{2.20}$$

where we dual introduce cyclic oscillation planes with angular momenta bivectors

$$\mathbf{S}_1 = \tfrac{1}{2}\hbar\boldsymbol{i}_1, \quad \mathbf{S}_2 = \tfrac{1}{2}\hbar\boldsymbol{i}_2, \quad \mathbf{S}_3 = \tfrac{1}{2}\hbar\boldsymbol{i}_3 \quad \in \mathcal{G}_{3,0}^+(\mathbb{R}), \qquad \mathbf{S}_k^\dagger = -\mathbf{S}_k \tag{2.21}$$

The *commutator product* of the angular momenta bivector operators *interconnect*

$$[\mathbf{S}_2, \mathbf{S}_1] = 2[\mathbf{S}_2 \times \mathbf{S}_1] = \mathbf{S}_2\mathbf{S}_1 - \mathbf{S}_1\mathbf{S}_2 = \hbar\mathbf{S}_3 \;\Rightarrow\; \underbrace{\mathbf{S}_2\mathbf{S}_1 = \tfrac{1}{2}\hbar\mathbf{S}_3}_{\textit{orthogonal product}}, \tag{2.22}$$

where we use the *interconnectivity* (2.4) of the quaternion basis together with (2.30).

$$\mathbf{S}_2\mathbf{S}_1 = \tfrac{1}{2}\mathbf{S}_3, \qquad \mathbf{S}_3\mathbf{S}_2 = \tfrac{1}{2}\mathbf{S}_1, \qquad \mathbf{S}_1\mathbf{S}_3 = \tfrac{1}{2}\mathbf{S}_2, \qquad (\hbar = 1) \tag{2.23}$$

The conjugation is area reversion $\mathbf{S}_k^\dagger = -\mathbf{S}_k$, with the dual parity inversion $\overline{\mathbf{s}_k} = -\mathbf{s}_k$, In the traditional QM the 1-vector operator commutator *interconnectivity* is

$$[\mathbf{s}_1, \mathbf{s}_2] = \mathbf{s}_1\mathbf{s}_2 - \mathbf{s}_2\mathbf{s}_1 = \hbar\boldsymbol{i}\mathbf{s}_3 = \hbar\mathbf{S}_3 = \tfrac{1}{2}\hbar^2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \tfrac{1}{2}\hbar^2\boldsymbol{i}_3, \tag{2.24}$$

Three basic orthogonal angular momenta operator *directions* have *interconnectivity*

Commutators		Orthogonal Geometric Products		
grade-2, ***grade***-2 → ***grade***-2	***grade***-1, ***grade***-1 → ***grade***-2	***grade***-2⊗***grade***-2 → ***grade***-2	***grade***-1⊗***grade***-1 → ***grade***-2	
$[\mathbf{S}_2, \mathbf{S}_1] = \hbar\mathbf{S}_3$	$= [\mathbf{s}_1, \mathbf{s}_2] = \hbar\boldsymbol{i}\,\mathbf{s}_3$	$\mathbf{S}_2\mathbf{S}_1 = \hbar\frac{1}{2}\mathbf{S}_3$	$= \mathbf{s}_1\mathbf{s}_2 = \hbar\boldsymbol{i}\frac{1}{2}\mathbf{s}_3$	(2.25)
$[\mathbf{S}_3, \mathbf{S}_2] = \hbar\mathbf{S}_1$	$= [\mathbf{s}_2, \mathbf{s}_3] = \hbar\boldsymbol{i}\,\mathbf{s}_1$	$\mathbf{S}_3\mathbf{S}_2 = \hbar\frac{1}{2}\mathbf{S}_1$	$= \mathbf{s}_2\mathbf{s}_3 = \hbar\boldsymbol{i}\frac{1}{2}\mathbf{s}_1$	(2.26)
$[\mathbf{S}_1, \mathbf{S}_3] = \hbar\mathbf{S}_2$	$= [\mathbf{s}_3, \mathbf{s}_1] = \hbar\boldsymbol{i}\,\mathbf{s}_2$	$\mathbf{S}_1\mathbf{S}_3 = \hbar\frac{1}{2}\mathbf{S}_2$	$= \mathbf{s}_3\mathbf{s}_1 = \hbar\boldsymbol{i}\frac{1}{2}\mathbf{s}_2$	(2.27)

This is an a priori concept idea for ***quantum*** operations of angular momenta which possesses ***qualities of directions***, with the orthogonal geometric algebraic reason (2.4). Two orthogonal angular momenta ***directions*** do not commute, but the interesting thing is, that all these orthogonal operator components anticommute

$$\mathbf{s}_j\mathbf{s}_k + \mathbf{s}_k\mathbf{s}_j = 0 \quad \text{and dual} \quad \mathbf{S}_k\mathbf{S}_j + \mathbf{S}_j\mathbf{S}_k = 0, \qquad k \neq j, \text{ for } j, k = 1,2,3. \tag{2.28}$$

The dangerous advantage of the algebraic orthogonal structure for the geometric perpendicular planes of angular momenta makes the scalar ***quality grade***-0 immaterial.

2.5. Eight Possibilities for Angular Momentum Superposition at One Locality

In (2.20)-(2.21) we realise that each of the three components has two orientation possibilities of opposite signs $\boxed{\epsilon_k = \pm 1}$ for $k = 1,2,3$ giving index $\boldsymbol{\epsilon} \leftarrow (\epsilon_1, \epsilon_2, \epsilon_3)$. We form the full angular momentum as the superposition of the bivector components

$$\mathbf{A}_\epsilon = \epsilon_1 \mathbf{S}_1 + \epsilon_2 \mathbf{S}_2 + \epsilon_3 \mathbf{S}_3, \quad \text{and dual} \qquad (2.29)$$

$$\mathbf{k}_\epsilon = \epsilon_1 \mathbf{s}_1 + \epsilon_2 \mathbf{s}_2 + \epsilon_3 \mathbf{s}_3 \quad = -\boldsymbol{i}\mathbf{A}_\epsilon . \qquad (2.30)$$

There are eight possibilities $\mathbf{S}_\epsilon \in \{ \epsilon_1 \mathbf{S}_1 + \epsilon_2 \mathbf{S}_2 + \epsilon_3 \mathbf{S}_3 \}_8$

The squared of these orthogonal components gives in Superposition the Casimir invariant

$$\mathbf{A}_\epsilon^2 = (\epsilon_1 \mathbf{S}_1)^2 + (\epsilon_2 \mathbf{S}_2)^2 + (\epsilon_3 \mathbf{S}_3)^2 = -\tfrac{3}{4}, \qquad (2.31)$$

Dual this is $\mathbf{k}_\epsilon^2 = \frac{3}{4}$, and squared magnitude $\mathbf{A}_\epsilon \mathbf{A}_\epsilon^\dagger = \frac{3}{4}$.

The eight 1-vector ***directions*** for the rotation axis

$$\mathbf{k}_\epsilon \in \left\{\tfrac{1}{2}(\pm\boldsymbol{\sigma}_1 \pm\boldsymbol{\sigma}_2 \pm\boldsymbol{\sigma}_3)\right\}_8 \subset \mathcal{G}_{3,0}^{-}(\mathbb{R}). \qquad (2.32)$$

are displayed in **Fig. 6** as the qualitative superposition of perpendicular components of one local indivisible particle.

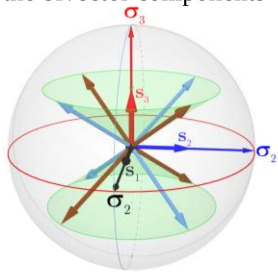


Fig. 6 Eight 1-vectors compositions $\pm\frac{1}{2}\boldsymbol{\sigma}_1 \pm \frac{1}{2}\boldsymbol{\sigma}_2 \pm \frac{1}{2}\boldsymbol{\sigma}_3$, four arms of two orientations a *tetraon* and a *tetraon*, combines as an *octaon*.

→ **Fig. 1**.

The possibilities scheme:

The bivector angular momentum } and { **its 1-vector axis outwards from one centre**.

Reverse,	Full,	Components,	*Dual*,	Full,	Components,	*Inverse*.

$$\left.\begin{array}{rl} & \mathbf{A}_{0---} = -\mathbf{S}_1 - \mathbf{S}_2 - \mathbf{S}_3, \\ & \mathbf{A}_{1-++} = -\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3, \\ & \mathbf{A}_{2+-+} = +\mathbf{S}_1 - \mathbf{S}_2 + \mathbf{S}_3, \\ & \mathbf{A}_{3++-} = +\mathbf{S}_1 + \mathbf{S}_2 - \mathbf{S}_3, \\ \mathbf{A}_{3++-}^\dagger = & \mathbf{A}_{3--+} = -\mathbf{S}_1 - \mathbf{S}_2 + \mathbf{S}_3, \\ \mathbf{A}_{2+-+}^\dagger = & \mathbf{A}_{2-+-} = -\mathbf{S}_1 + \mathbf{S}_2 - \mathbf{S}_3, \\ \mathbf{A}_{1-++}^\dagger = & \mathbf{A}_{1+--} = +\mathbf{S}_1 - \mathbf{S}_2 - \mathbf{S}_3, \\ \mathbf{A}_{0---}^\dagger = & \mathbf{A}_{0+++} = +\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3, \end{array}\right\} = \boldsymbol{i} \left\{\begin{array}{ll} \mathbf{k}_{0---} = -\mathbf{s}_1 - \mathbf{s}_2 - \mathbf{s}_3 & \\ \mathbf{k}_{1-++} = -\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 & \\ \mathbf{k}_{2+-+} = +\mathbf{s}_1 - \mathbf{s}_2 + \mathbf{s}_3 & \\ \mathbf{k}_{3++-} = +\mathbf{s}_1 + \mathbf{s}_2 - \mathbf{s}_3 & \\ \mathbf{k}_{3--+} = -\mathbf{s}_1 - \mathbf{s}_2 + \mathbf{s}_3 & = \overline{\mathbf{k}_{3++-}} \\ \mathbf{k}_{2-+-} = -\mathbf{s}_1 + \mathbf{s}_2 - \mathbf{s}_3 & = \overline{\mathbf{k}_{2+-+}} \\ \mathbf{k}_{1+--} = +\mathbf{s}_1 - \mathbf{s}_2 - \mathbf{s}_3 & = \overline{\mathbf{k}_{1-++}} \\ \mathbf{k}_{0+++} = +\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 & = \overline{\mathbf{k}_{0---}} \end{array}\right. \qquad (2.33)$$

These different combinations of bivectors result in the following impact:

1. The shift of signs in one *direction* changes the chirality with *dextral* or *sinistral* orientation.
2. The shift of the sign in two of three *directions* changes the *spin up* + or *down* − orientation.
3. The shift of sign in three *directions* is a *parity inversion* of 1-vectors, or just a *bivector reversion* that consists of the full bivector *reversion* shift of both 1. and 2., $\mathbf{A}_\kappa \rightarrow \mathbf{A}_\kappa^\dagger$.

The equal magnitudes of these eight 2^3 superposition angular momenta are

$$|\mathbf{A}_\kappa| = |\mathbf{A}_\kappa^\dagger| = |\mathbf{k}_\kappa| = |\overline{\mathbf{k}}_\kappa| = \sqrt{3/4} = \sqrt{\mathbf{k}_\epsilon^2}. \qquad (2.34)$$

All eight possibilities have the projection impact $\pm\frac{1}{2}$ in one ***direction***, e.g., $\pm\boldsymbol{\sigma}_3$.

We recall that the idea of angular momentum as preserved bivector areas has its origin in Kepler's 2nd Law found before Newton introduced the artefact idea of mass, which is not necessary for the local internal structure of one indivisible spin½ particle.

We instead have eight *qualities* $(2^3\pm)$ of the full resulting angular momentum bivector

$$\mathbf{A}_\epsilon \to \ \mathbf{A}_{(2^{3\pm})} \ = \ \tfrac{1}{2}(\pm\boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3) \quad \in \mathcal{G}^+_{3,0}(\mathbb{R}). \tag{2.35}$$

This is an alternative structure for the internal oscillations of a fermion, not in the perpendicular planes displayed in **Fig. 5**, but still inside the unit quaternion structure of one 2-rotor $\Psi_{\mathbb{H}} = \lambda_0 1 + \lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3$, for one single indivisible spin½ fermion.

This is the ontological reason to invent a geometric interpretation of the quaternion idea displayed in **Fig. 5** with six spatial ***directions*** from the intersection locality center together with one scalar giving the orthonormal set $\{\pm 1, \pm\boldsymbol{i}_1, \pm\boldsymbol{i}_2, \pm\boldsymbol{i}_3\}$ of three planes. Each four basic units appear when $\lambda_\mu^2 = 1$, for $\lambda_{\mu\neq\nu}^2 = 0$, $\nu, \mu = 0{,}1{,}2{,}3$. The symmetric structure of (2.35) appears when $\lambda_\mu^2 = \frac{1}{4} \Leftrightarrow \lambda_\mu = \pm\frac{1}{2}$, resulting in sixteen other units

$$\left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16} \tag{2.36}$$

We have already treated the bivector ***directions*** of this in (2.35), dual display **Fig. 6**. These (2.36) are the extra new units to the *Hurwitz Unit Quaternion Group*, [2], [3].

2.6. The Central Directions of a Tetrahedron

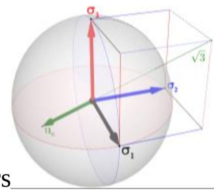
The 1-vector ***directions*** in (2.33) is the same as in (1.35) displayed **Fig. 2**

$$\begin{Bmatrix} \mathbf{k}_0 &= \frac{1}{2}(-\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3), \\ \mathbf{k}_1 &= \frac{1}{2}(-\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3), \\ \mathbf{k}_2 &= \frac{1}{2}(+\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3), \\ \mathbf{k}_3 &= \frac{1}{2}(+\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3) \end{Bmatrix}_4 \to \begin{Bmatrix} \mathbf{u}_0 &= \frac{1}{\sqrt{3}}(-\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3), \\ \mathbf{u}_1 &= \frac{1}{\sqrt{3}}(-\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3), \\ \mathbf{u}_2 &= \frac{1}{\sqrt{3}}(+\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3), \\ \mathbf{u}_3 &= \frac{1}{\sqrt{3}}(+\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3) \end{Bmatrix}_4 . \tag{2.37}$$

It is urgent to note that a four permutation $\{0{,}1{,}2{,}3\} \to \{1{,}2{,}3{,}0\}$ change the chirality, while $\{0, 1{,}2{,}3\} \to \{0, 2{,}3{,}1\}$ is dextral preserve, $\mathbf{k}_0 = -(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$, is antagonistic.[13]

Conversely from a presumed set $\{\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$, we find the orthonormal set

$$\boldsymbol{\sigma}_1 = \mathbf{k}_2 + \mathbf{k}_3, \quad \boldsymbol{\sigma}_2 = \mathbf{k}_3 + \mathbf{k}_1, \quad \boldsymbol{\sigma}_3 = \mathbf{k}_1 + \mathbf{k}_2, \tag{2.38}$$



From this orthonormal set, we need a fourth 1-vector to support a local unit sphere $\mathbf{u}_0 = -\frac{1}{\sqrt{3}}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3)$. This locality center requirement is that four points support a sphere surface; three points support a circle circumference like a three-1-vectors Mercedes star ✇; two points support a line segment as one 1-vector ⟼ ***direction***; and one single point supports nothing nor any locality *direction* structure of physics.

This set $\{\mathbf{u}_0, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ is not symmetric in its support of locality sphere in ^{3}space. The regular *tetraon* ***directions*** set $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, (2.38) is just the normalised units $\mathbf{u}_\kappa = \sqrt{2/3}\,\mathbf{k}_\kappa$, so $\mathbf{u}_0^2 = \mathbf{u}_1^2 = \mathbf{u}_2^2 = \mathbf{u}_3^2 = 1$, that structure support the local unit sphere,

$$\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0, \qquad \text{the algebraic rule for the symmetry.} \tag{2.39}$$

Note $\mathbf{u}_\kappa \mathbf{u}_\kappa = 4$, and further the linear dependency of one of the three others, e.g., $\mathbf{u}_0 = -(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3)$, like $\mathbf{u}_0 = -\sqrt{1/3}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3)$ above. This regular tetrahedron symmetry in space obeys six 1-rotor relations with a scalar for covariant projection

[13] Looking at the chirality of your hand depends on your arm as the fixed₀ rotation reference.

$$\begin{aligned} \mathbf{u}_1\mathbf{u}_0 &= \mathbf{u}_1\wedge\mathbf{u}_0 - \tfrac{1}{3}, & \mathbf{u}_3\mathbf{u}_0 &= \mathbf{u}_3\wedge\mathbf{u}_0 - \tfrac{1}{3}, \\ \mathbf{u}_2\mathbf{u}_1 &= \mathbf{u}_2\wedge\mathbf{u}_1 - \tfrac{1}{3}, & \mathbf{u}_3\mathbf{u}_1 &= \mathbf{u}_3\wedge\mathbf{u}_1 - \tfrac{1}{3}, \\ \mathbf{u}_0\mathbf{u}_2 &= \mathbf{u}_0\wedge\mathbf{u}_2 - \tfrac{1}{3}, & \mathbf{u}_3\mathbf{u}_2 &= \mathbf{u}_3\wedge\mathbf{u}_2 - \tfrac{1}{3}, \end{aligned} \qquad (2.40)$$

where $\mathbf{u}_\kappa\cdot\mathbf{u}_\kappa = 1$, $\mathbf{u}_\kappa\cdot\mathbf{u}_\nu = -\frac{1}{3}$, for $\kappa \neq \nu$, and $\cos\beta = -\frac{1}{3} \Rightarrow \beta \sim 109.47°$.

In our artefact humanistic mathematical approach to physics, we have the freedom to model in myriad ways. The known example is the Cartesian $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ and now the transformation $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\} \rightarrow \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to a non-orthogonal unit basis of a regular tetraon inside unit sphere. By (2.37) the symmetry condition (2.39)-(2.40) is fulfilled.

2.7. An Autonomous Four-Angular-Momenta Component Bivector

The ideas of Kepler's 2nd Law and the Laplace discovery 1799 (including Newton's 1st Law) both the *area magnitude* and *direction* of angular development are preserved. This fundamental idea of physics is: *The Law of Angular Momentum Conservation*. The angular momentum is best represented by constant bivectors. This idea we will a priori use internally for one local autonomous indivisible spin½ fermion $\Psi_{½}$ structure of space. We make an artefact local internal model for this[14] of four cyclic oscillating components representing angular momenta *direction* bivectors $\{\boldsymbol{i}\mathbf{u}_0, \boldsymbol{i}\mathbf{u}_1, \boldsymbol{i}\mathbf{u}_2, \boldsymbol{i}\mathbf{u}_3\}$ Each four-component possesses then conserved spin½ angular momenta bivector components, each with two reversed orientation possibilities of the rotation oscillations

$$\begin{aligned} S_0 &= +\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_0, & S_1 &= +\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_1, & S_2 &= +\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_2, & S_3 &= +\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_3, \\ S_0^\dagger &= -\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_0, & S_1^\dagger &= -\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_1, & S_2^\dagger &= -\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_2, & S_3^\dagger &= -\tfrac{1}{2}\boldsymbol{i}\mathbf{u}_3, \end{aligned} \qquad (2.41)$$

where we omit $\hbar=1$ the internal autonomous measure norm for the unit spherical structure circumscribing the regular tetrahedron with these four face *directions* **Fig. 7**

2.8. Superposition of Angular 1-spinor Wave Functions

These four plane rotation invariant bivector for angular momenta components represent four 1-spinors of active cyclic oscillating wavefunctions

$$\psi_{\kappa\pm}^{½} \sim \rho_\kappa U_{\phi_\kappa} = \rho_\kappa e^{\pm½\boldsymbol{i}\mathbf{u}_\kappa\phi_\kappa} = \rho_\kappa(\cos ½\phi_\kappa \pm \boldsymbol{i}\mathbf{u}_\kappa \sin ½\phi_\kappa), \qquad (2.42)$$

each with one of two orientation possibilities, $(+)$ or $(-)$, [not both $(\pm)_\kappa$] for all each $\kappa = 0,1,2,3$ internal in one particle locality.

For one local indivisible fermion $\Psi_{½}$, we get the full wavefunction by superposition

$$\psi_{\wedge} = \sum_{\kappa=0}^{3} \rho_\kappa e^{\pm½\boldsymbol{i}\mathbf{u}_\kappa\phi_\kappa} = \sum_{\kappa=0}^{3} \rho_\kappa(\cos ½\phi_\kappa \pm \boldsymbol{i}\mathbf{u}_\kappa \sin ½\phi_\kappa) \qquad (2.43)$$

These multiple fluctuations can be simplified by demanding the scalar sum to vanish

$$\rho_0 \cos ½\phi_0 + \rho_1 \cos ½\phi_1 + \rho_2 \cos ½\phi_2 + \rho_3 \cos ½\phi_3 = 0. \qquad (2.44)$$

The unit ellipsoidal demand $\sum_{\kappa=0}^{3} \rho_\kappa^2 = 1$ can by spherical symmetry be simplified presuming $\rho_\kappa = \pm\frac{1}{2}$ for the four 1-spinors, absorbing the options $(\pm)_\kappa$ in (2.43).
Two pairs $(\phi_{\kappa_1}\phi_{\kappa_2}), (\phi_{\kappa_3}\phi_{\kappa_4})$ of phase synchronising of four phase angles $\phi_0, \phi_1, \phi_2, \phi_3$,

[14] This we considered as an a priori transcendental substance unknown to us, thus not measurable.

$$\left.\begin{array}{ll}\text{if}: \ \rho_{\kappa_1} = +\rho_{\kappa_2} \ \Rightarrow & \phi_{\kappa_2} = \phi_{\kappa_1} + 4n\pi + 2\pi \\ \text{if}: \ \rho_{\kappa_1} = -\rho_{\kappa_2} \ \Rightarrow & \phi_{\kappa_2} = \phi_{\kappa_1} + 4n\pi \end{array}\right\} \ \kappa_2 \neq \kappa_1 \quad \left.\begin{array}{ll}\text{if}: \ \rho_{\kappa_3} = +\rho_{\kappa_4} \ \Rightarrow & \phi_{\kappa_4} = \phi_{\kappa_3} + 4n\pi + 2\pi \\ \text{if}: \ \rho_{\kappa_3} = -\rho_{\kappa_4} \ \Rightarrow & \phi_{\kappa_4} = \phi_{\kappa_3} + 4n\pi \end{array}\right\} \ \kappa_4 \neq \kappa_3 \quad \begin{array}{c}\text{All different} \\ \kappa_l = 0,1,2,3; \\ l = 1,2,3,4.\end{array} \tag{2.45}$$

In this way we make the scalar (cosine) part disappear from our artefact model for the substance. We cannot experience these ϕ_{κ_l} direct (the idea is a priori transcendental). The four bivector oscillating components (2.43) will not disappear in their four different *directions*, therefor we can reduce the wavefunction to

$$\psi_{\wedge} = \pm i\mathbf{u}_0\rho_0 \sin \tfrac{1}{2}\phi_0 \ \pm i\mathbf{u}_1\rho_1 \sin \tfrac{1}{2}\phi_1 \ \pm i\mathbf{u}_2\rho_2 \sin \tfrac{1}{2}\phi_2 \ \pm i\mathbf{u}_3\rho_3 \sin \tfrac{1}{2}\phi_3, \tag{2.46}$$

synchronised after the phase angle pattern (2.45), where there are only *two* independent development *parameters* ϕ_a, ϕ_b of the double cyclic fluctuation of a wavefunction. These two can be parametrised by t through their reciprocity measure frequencies: $\phi_a \leftarrow \omega_a t$, $\phi_b \leftarrow \omega_b t$, probably with jitter $|\omega_a| - |\omega_b|$, but constant energy $|\omega_a|+|\omega_b|$. The idea, *two* angular development *parameters* agree with the orthogonal case where two 1-spinors agree with the two degrees freedom for a 2-spinor in $\mathbb{H}\sim\mathcal{G}^{\perp}_{0,2}(\mathbb{R})$, (2.16).

2.9. The Full Angular Momentum of an Autonomous Tetrahedron Structure

From such a non-orthogonal unitary basis $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ (2.39)-(2.40) of *regular tetraon structure* for one local indivisible spin½ fermion $\Psi_{½}$, we formulate an imagined bivector function superposition of the four cyclic plane *directions* angular momenta $\mathbf{S}_\kappa$ of the tetrahedron oscillations giving sixteen possibilities

$$\mathbf{S}_{\wedge} = \ \pm\tfrac{1}{2}i\mathbf{u}_0 \ \pm\tfrac{1}{2}i\mathbf{u}_1 \ \pm\tfrac{1}{2}i\mathbf{u}_2 \pm\tfrac{1}{2}i\mathbf{u}_3 \qquad \leftrightarrow \qquad \{\textstyle\sum_\kappa \mathbf{S}^{\pm}_{\kappa}\}_{16}, \tag{2.47}$$

Square this to a (Casimir) invariant we get a negative value -1, with the expectation

$$\mathbf{S}^{\dagger}_{\wedge}\mathbf{S}_{\wedge} \ = \ \mathbf{S}_{\wedge}\mathbf{S}^{\dagger}_{\wedge} \ = 1. \tag{2.48}$$

This is the criterium to achieve one amount of *one local indivisible* spin½ fermion $\Psi_{½}$.

2.10. The Impact of the Tetraon Directions

We made the imagined idea of the circle oscillating frequency energy components for the one local $\Psi_{½}$, of the form (2.43) $e^{\pm\frac{1}{2}i\mathbf{u}_\kappa\phi_\kappa}$ in four regular symmetric bivector *directions* angular momenta $\mathbf{S}^{\pm}_{\kappa} = \pm\frac{1}{2}i\mathbf{u}_\kappa$ in (2.41) which claim possesses the *tetrahedron* structure. We defined these from the four *outwards* orientated *directions* of the unit basis $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of a regular tetraon inside supporting a symmetric unit sphere.

These four positive *outwards* orientated 1-vector ***directions*** $\mathbf{u}_\kappa$ generates each the ***directed*** angular momentum rotation axial spin½ 1-vectors dual to (2.41)

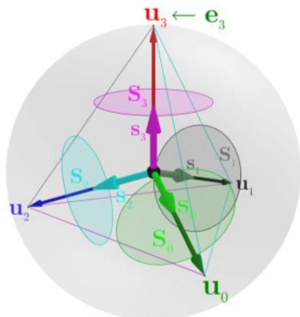


Fig. 7 The *regular tetrahedron idea* for spin½ angular momenta $\mathbf{S}_\kappa$ symmetry of each one *local indivisible* fermion $\Psi_{½}$. The unit *tetraon* basis is $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, which halflings fulfil $\mathbf{s}_0^2 + \mathbf{s}_1^2 + \mathbf{s}_1^2 + \mathbf{s}_1^2 = 1$, and $\mathbf{s}_0 + \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 = 0$ for *outwards* regularity from one local center, internal in a *unit spherical symmetry of locality*.

$$\begin{aligned} \mathbf{s}_0 &= +\tfrac{1}{2}\mathbf{u}_0, & \overline{\mathbf{s}_0} &= -\tfrac{1}{2}\mathbf{u}_0, \\ \mathbf{s}_1 &= +\tfrac{1}{2}\mathbf{u}_1, & \overline{\mathbf{s}_1} &= -\tfrac{1}{2}\mathbf{u}_1, \\ \mathbf{s}_2 &= +\tfrac{1}{2}\mathbf{u}_2, & \overline{\mathbf{s}_2} &= -\tfrac{1}{2}\mathbf{u}_2, \\ \mathbf{s}_3 &= +\tfrac{1}{2}\mathbf{u}_3, & \overline{\mathbf{s}_3} &= -\tfrac{1}{2}\mathbf{u}_3. \end{aligned} \qquad (2.49)$$

The first column is displayed in **Fig. 7**. The intuition problem here is that the 1-vectors is its own reversed $\mathbf{s}_\kappa^\dagger = \mathbf{s}_\kappa = ¼\mathbf{s}_\kappa^{-1}$ but the 1-vectors have parity-inverted opposites $\overline{\mathbf{s}_\kappa} = -\mathbf{s}_\kappa$ which is chiral dual to reversion.
From this, we immediately get $\mathbf{s}_0^2 + \mathbf{s}_1^2 + \mathbf{s}_1^2 + \mathbf{s}_1^2 = 1$ for the unit spherical symmetry with the extra demand of positive *outwards* orientated regular tetraon symmetry
$\mathbf{s}_0 + \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 = 0 \quad \Longrightarrow \mathbf{s}_0 = -(\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3)$ or $\mathbf{s}_3 = (-\mathbf{s}_0 - \mathbf{s}_1 - \mathbf{s}_2)$, etc.
Then the double orientations of the angular momenta of these four 1-vector *directions* of one local autonomous $\Psi_{½}$ form sixteen possibilities of 1-vector superpositions

$$\mathbf{s}_\wedge = (\pm\mathbf{s}_0 \pm\mathbf{s}_1 \pm\mathbf{s}_2 \pm\mathbf{s}_3), \qquad \text{and their duals} \qquad (2.50)$$

$$\mathbf{S}_\wedge = \boldsymbol{i}(\pm\mathbf{s}_0 \pm\mathbf{s}_1 \pm\mathbf{s}_2 \pm\mathbf{s}_3) = (\pm\mathbf{S}_0 \pm\mathbf{S}_1 \pm\mathbf{S}_2 \pm\mathbf{S}_3) = \boldsymbol{i}\mathbf{s}_\wedge \qquad (2.51)$$

To understand the impact of these possibilities we look at projections of three of these spin components into the fourth *direction*. We use Euclidean geometric 1-vector *projection* into one of these *direction* components $\mathbf{s}_\kappa = \frac{1}{2}\mathbf{u}_\kappa$, for $\kappa = 0{,}1{,}2{,}3$:

$$P_{\mathbf{u}_\kappa}\mathbf{x} = P_{\mathbf{s}_\kappa}\mathbf{x} \equiv \mathbf{s}_\kappa^{-1}\mathbf{s}_\kappa{\cdot}\mathbf{x} = (\mathbf{x}{\cdot}\mathbf{s}_\kappa)\mathbf{s}_\kappa^{-1} = (\mathbf{x}{\cdot}\mathbf{u}_\kappa)\mathbf{u}_\kappa^{-1} = \mathbf{x}{\cdot}\mathbf{u}_\kappa\mathbf{u}_\kappa, \quad \not{\Sigma}_{\kappa} \qquad (2.52)$$

because $\mathbf{u}_\kappa^2 = 1 \Rightarrow \mathbf{u}_\kappa^{-1} = \mathbf{u}_\kappa$ we remark $\mathbf{s}_\kappa^{-1}\mathbf{s}_\kappa = \mathbf{u}_\kappa^{-1}\mathbf{u}_\kappa = 1$, and $\boxed{\mathbf{s}_\kappa^{-1} = 4\mathbf{s}_\kappa = 2\mathbf{u}_\kappa}$.

2.11. Information from one Local Tetraon Symmetric Fermion $\Psi_{½}$

When information is transmitted from one local $\Psi_{½}$ at A to another at B as a receiver, it is propagating along one 1-vector *direction*. We chose to align to one of the units $\mathbf{u}_\kappa \longleftarrow \overrightarrow{AB} \parallel \mathbf{u}_\kappa$. We project, the content of the information into that *direction*.

First, the 1-vector component $\mathbf{s}_\kappa$ itself

$$\begin{aligned} P_{\mathbf{s}_\kappa}\mathbf{s}_\kappa &= (\mathbf{s}_\kappa{\cdot}\mathbf{s}_\kappa)\mathbf{s}_\kappa^{-1} = \tfrac{1}{4}\mathbf{s}_\kappa^{-1} = 1\mathbf{s}_\kappa = (\mathbf{s}_\kappa{\cdot}\mathbf{u}_\kappa)\mathbf{u}_\kappa^{-1} = \tfrac{1}{2}\mathbf{u}_\kappa, \quad \text{or} \\ P_{\mathbf{u}_\kappa}\mathbf{u}_\kappa &= (\mathbf{u}_\kappa{\cdot}\mathbf{u}_\kappa)\mathbf{u}_\kappa^{-1} = 1\mathbf{u}_\kappa^{-1} = 1\mathbf{u}_\kappa, \qquad \text{(no sum, } \not{\Sigma}_{\kappa}\text{)}. \end{aligned} \qquad (2.53)$$

Second, the projections of the other 1-vector directions $\mathbf{s}_\nu,\ \nu \neq \kappa,\ \nu,\kappa = 0{,}1{,}2{,}3.$

$$\begin{aligned} P_{\mathbf{s}_\kappa}\mathbf{s}_\nu &= (\mathbf{s}_\nu{\cdot}\mathbf{s}_\kappa)\mathbf{s}_\kappa^{-1} = -\tfrac{1}{4\cdot 3}\mathbf{s}_\kappa^{-1} = -\tfrac{1}{3}\mathbf{s}_\kappa = -\tfrac{1}{6}\mathbf{u}_\kappa, \quad \text{or} \\ P_{\mathbf{u}_\kappa}\mathbf{u}_\nu &= (\mathbf{u}_\nu{\cdot}\mathbf{u}_\kappa)\mathbf{u}_\kappa^{-1} = -\tfrac{1}{3}\mathbf{u}_\kappa. \end{aligned} \qquad (2.54)$$

Specifically, in *direction* $\mathbf{s}_0$, where $\mathbf{s}_0^2 = ¼$, we have the three projections components

$$P_{\mathbf{s}_0}\mathbf{s}_1 = (\mathbf{s}_1{\cdot}\mathbf{s}_0)\mathbf{s}_0^{-1} = P_{\mathbf{s}_0}\mathbf{s}_2 = (\mathbf{s}_2{\cdot}\mathbf{s}_0)\mathbf{s}_0^{-1} = P_{\mathbf{s}_0}\mathbf{s}_3 = (\mathbf{s}_3{\cdot}\,\mathbf{s}_0)\mathbf{s}_0^{-1} = -\tfrac{1}{3\cdot 4}\mathbf{s}_0^{-1} = -\tfrac{1}{3}\mathbf{s}_0, \qquad (2.55)$$

or specifically in the *direction* $\mathbf{u}_3$, where $\mathbf{u}_3^2 = 1$ we have three projections components

$$P_{\mathbf{u}_3}\mathbf{u}_0 = (\mathbf{u}_0{\cdot}\mathbf{u}_3)\mathbf{u}_3^{-1} = P_{\mathbf{u}_3}\mathbf{u}_1 = (\mathbf{u}_1{\cdot}\mathbf{u}_3)\mathbf{u}_3^{-1} = P_{\mathbf{u}_3}\mathbf{u}_2 = (\mathbf{u}_2{\cdot}\mathbf{u}_3)\mathbf{u}_3^{-1} = -\tfrac{1}{3}\mathbf{u}_3^{-1} = -\tfrac{1}{3}\mathbf{u}_3. \qquad (2.56)$$

We see that these three symmetric projections result in just the same contribution.
Depending on the sign combinations we get different impacts on the full projection into one *direction* of the idea in (2.50) $\{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\} \longrightarrow \mathbf{s}_\wedge = (\pm\mathbf{s}_0 \pm\mathbf{s}_1 \pm\mathbf{s}_2 \pm\mathbf{s}_3)$.

This expressed in units of the four *directions* is

$$\mathbf{s}_\wedge = \tfrac{1}{2}(\epsilon_0\mathbf{u}_0 + \epsilon_1\mathbf{u}_1 + \epsilon_2\mathbf{u}_2 + \epsilon_3\mathbf{u}_3), \tag{2.57}$$

where four-angular-momenta components have two orientations $\epsilon_\nu = \pm 1,\ \nu = 0,1,2,3$.

Table 1 Sixteen combinations resulting in eight different cases of quantitative values:

Projection *direction* $\mathbf{s}_0 \leftarrow \mathbf{u}_0 \leftarrow \overrightarrow{\mathrm{AB}}$

$(\pm\mathbf{s}_0\pm\mathrm{s}_1\pm\mathbf{s}_2\pm\mathbf{s}_3)$	$\rightarrow$	$\mathbf{s}_0 = \frac{1}{2}\mathbf{u}_0$,	$\mathbf{s}_0\mathbf{s}_0^{-1} = 1,$	?
$\mathbf{s}_\wedge \downarrow$	$\mid$	$P_{\mathbf{s}_0}\mathbf{s}_\wedge$,	$(P_{\mathbf{s}_0}\mathbf{s}_\wedge)\mathbf{s}_0^{-1}$,	$\downarrow$
$\{\mathbf{s}_0, \mathrm{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ $(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)$	 $\rightarrow$	$\downarrow$ $q\,\mathbf{s}_0$,	$\downarrow$ q ,	
$(+,+,+,+) \rightarrow$	$\rightarrow$	$+0$,	0 ,	n
$(+,-,+,+)$ $(+,+,-,+)$ $(+,+,+,-)$	$\rightarrow$	$+\frac{1}{3}\mathbf{s}_0$,	$\frac{1}{3}$,	$\overline{\mathrm{d}}$
$(+,-,-,+)$ $(+,-,+,-)$ $(+,+,-,-)$	$\rightarrow$	$+\frac{2}{3}\mathbf{s}_0$,	$\frac{2}{3}$,	u
$(+,-,-,-) \rightarrow$	$\rightarrow$	$+\mathbf{s}_0$,	1 ,	p
$(-,+,+,+) \rightarrow$	$\rightarrow$	$-\mathbf{s}_0$,	-1 ,	e
$(-,+,+,-)$ $(-,+,-,+)$ $(-,-,+,+)$	$\rightarrow$	$-\frac{2}{3}\mathbf{s}_0$,	$-\frac{2}{3}$,	$\overline{\mathrm{u}}$
$(-,-,-,+)$ $(-,-,+,-)$ $(-,+,-,-)$	$\rightarrow$	$-\frac{1}{3}\mathbf{s}_0$,	$-\frac{1}{3}$,	d
$(-,-,-,-) \rightarrow$	$\rightarrow$	-0 ,	0 ,	n

Projection *direction* $\mathbf{u}_3 \leftarrow \overrightarrow{\mathrm{AB}}$

$(\pm\mathbf{u}_0\pm\mathbf{u}_1\pm\mathbf{u}_2\pm\mathbf{u}_3)$	$\mathbf{s}_3 = \frac{1}{2}\mathbf{u}_3$
$2\mathbf{s}_\wedge \downarrow$	$P_{\mathbf{u}_3}\mathbf{s}_\wedge$
$\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ $(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)\frac{1}{2} \rightarrow$	$\downarrow$ $q\mathbf{u}_3^{-1}$
$(+,+,+,+) \rightarrow$	$0\mathbf{u}_3^{-1}$
$(-,+,+,+)$ $(+,-,+,+)$ $(+,+,-,+)$	$\frac{1}{3}\mathbf{u}_3^{-1}$
$(-,-,+,+)$ $(-,+,-,+)$ $(+,-,-,+)$	$\frac{2}{3}\mathbf{u}_3^{-1}$
$(-,-,-,+) \rightarrow$	$1\mathbf{u}_3^{-1}$
$(+,+,+,-) \rightarrow$	$-1\mathbf{u}_3^{-1}$
$(+,+,-,-)$ $(+,-,+,-)$ $(-,+,+,-)$	$-\frac{2}{3}\mathbf{u}_3^{-1}$
$(-,-,+,-)$ $(-,+,-,-)$ $(+,-,-,-)$	$-\frac{1}{3}\mathbf{u}_3^{-1}$
$(-,-,-,-) \rightarrow$	$-0\mathbf{u}_3^{-1}$

(2.58)

Looking for the impact of the autonomous tetraon symmetric four-angular-momentum, we combine the projections for the four components of angular momenta bivectors S_κ, to this we use their dual 1-vector directions $\mathrm{s}_\nu = -\boldsymbol{i}\mathrm{S}_\nu$ forming the superposition $\mathbf{s}_\wedge = \mathbf{s}_{\wedge(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)} = \epsilon_0\mathbf{s}_0 + \epsilon_1\mathrm{s}_1 + \epsilon_2\mathbf{s}_2 + \epsilon_3\mathbf{s}_3$. These four parts with two orientations give in combination $2^4 = 16$ cases of $\mathbf{s}_{\wedge(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}$. In all cases, we project $\mathbf{s}_\wedge$ into one specific information propagation *direction* $\mathbf{s}_\kappa \leftarrow \mathbf{u}_\kappa \leftarrow \overrightarrow{\mathrm{AB}}$

$$P_{\mathbf{s}_\kappa}\mathbf{s}_\wedge = P_{\mathbf{s}_\kappa}(\epsilon_\kappa\mathbf{s}_\kappa) + \textstyle\sum_{\nu\neq\kappa} P_{\mathbf{s}_\kappa}(\epsilon_\nu\mathbf{s}_\nu) = q_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}\,\mathbf{s}_\kappa. \tag{2.59}$$

Writing out for e.g., the *direction* $\mathbf{s}_0 = +\frac{1}{2}\mathbf{u}_0$ using (2.53) and (2.56)

$$P_{\mathbf{s}_0}\mathbf{s}_\wedge = P_{\mathbf{s}_0}(\epsilon_0\mathbf{s}_0) + P_{\mathbf{s}_0}(\epsilon_1\mathrm{s}_1) + P_{\mathbf{s}_0}(\epsilon_2\mathbf{s}_2) + P_{\mathbf{s}_0}(\epsilon_3\mathbf{s}_3) = \left(\epsilon_0 - \tfrac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)\right)\mathbf{s}_0, \tag{2.60}$$

giving the sixteen combinations in **Table 1** resulting in eight different *quantities*.

The opportunities in left side **Table 1** is constructed for the idea of one indivisible spin½ fermion $\Psi_{1/2}$, containing four regular spin½ component quanta $(\epsilon_\nu\mathbf{S}_\nu) = \boldsymbol{i}(\epsilon_\nu\mathbf{s}_\nu)$ as *directional* angular momenta, which for one local $\Psi_{1/2}$ fulfil the condition from (2.48)

$$\textstyle\sum_{\kappa=0}^{3}\mathbf{S}_\kappa^\dagger\mathbf{S}_\kappa = \mathbf{S}_\kappa\mathbf{S}_\kappa^\dagger = 1 \Leftrightarrow \sum_{\kappa=0}^{3}\mathbf{S}_\kappa^2 = -1 \overset{\text{dual}}{\Longleftrightarrow} \mathbf{s}_0^2 + \mathrm{s}_1^2 + \mathbf{s}_1^2 + \mathbf{s}_1^2 = 1, \tag{2.61}$$

is giving the total impact of information to an external B from one $\Psi_{½}$ at A, $\overrightarrow{\mathrm{AB}} \longrightarrow \mathbf{s}_\kappa$:

$$P_{\mathbf{s}_\kappa}\mathbf{s}_\wedge = q_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}\mathbf{s}_\kappa, \quad \text{giving the real numbers} \\ \left(P_{\mathbf{s}_\kappa}\mathbf{s}_\wedge\right)\mathbf{s}_\kappa^{-1} = q_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)} = 1,\ \tfrac{2}{3},\ \tfrac{1}{3},\ 0,\ -0,\ -\tfrac{1}{3},\ -\tfrac{2}{3},\ -1. \tag{2.62}$$

In the measurement interaction, we presume that $\Psi_{½}$ local at A automatically align one angular spinning component $\mathbf{s}_\kappa$ into the interaction *direction* $\overrightarrow{\mathrm{AB}}$, where the norm invariant measure is $\mathbf{s}_\kappa^{-1}\mathbf{s}_\kappa = \mathbf{s}_\kappa\mathbf{s}_\kappa^{-1} = 1$, is local isometric in all possible *directions*.

The right side table of (2.58) considers the unit tetraon (2.39)-(2.40) , where $\mathbf{u}_0^2 + \mathbf{u}_1^2 + \mathbf{u}_2^2 + \mathbf{u}_3^2 = 4$. For the one *direction* e.g., $\mathbf{u}_3$, we get for the sixteen different combinations the resulting projection $P_{\mathbf{u}_3}(2\mathbf{s}_\wedge) = q_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}\mathbf{u}_3^{-1}$. $\boxed{\mathbf{s}_\kappa^{-1} = 4\mathbf{s}_\kappa = 2\mathbf{u}_\kappa}$ Special here we use the spherical isometric unit radius of the local symmetric regular tetraon structure at A as measure norm by $\mathbf{u}_3^{-1}\mathbf{u}_3 = \mathbf{u}_3\mathbf{u}_3^{-1} = 1 \coloneqq |\mathbf{u}_\kappa| \Rightarrow \boxed{\mathbf{u}_\kappa^2 = 1}$

What we have found is, that a fundamental indivisible spin½ locality $\Psi_{½}$ in physical space by its free spherical symmetry autonomously *interconnects* its angular movements in a local tetrahedron structure which carries a *quantitative charge* relative to the surroundings expressed as scalars of the ratio $-1, -⅔, -⅓, -0, +0, +⅓, +⅔, +1$.

2.12. The Tetrahedron Symmetry for Fermi particles

The ½-versors (1.9) have tetrahedron bivector *direction* structure in the local sphere. The scalar part of the ½-versor represents a *quantity* without any *direction* in Space. [15]

The Single Exited *Direction* Structure: The spin½ tetrahedron structure with the unit quantity $|q| = 1$ is simple. One spin½ component $\pm½$ together with the three opposite resulting in $\mp½$ with the sign structure giving the quantity value:

$$(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)\begin{cases} \rightarrow\ (-,-,-,+) \rightarrow q = 1, & \text{for the positron or} \\ \rightarrow\ (+,+,+,-) \rightarrow q = -1, & \text{for the electron} \end{cases} \tag{2.63}$$

The resulting external angular momentum is one unit $\mathbf{L}_3 = \pm 2\mathbf{S}_3 = \pm\frac{2}{2}\mathbf{u}_3 = \pm\mathbf{u}_3$, which directly interacts with the electromagnetic field in which photons have spin ± 1, interacting with one local charge ± 1.

The Local Addition of Exited *Directions*

For the mesons, we try the tetrahedron additive combined structure example

$$(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) \rightarrow \begin{cases} +(-,-,+,+) \rightarrow & q = +⅔, & \mathrm{u}, & \text{up quark}, \\ +(+,+,-,+) \rightarrow & q = +⅓, & \bar{\mathrm{d}}, & \text{down quark}, \\ = \pi^+, \text{spin } +1, \rightarrow & \Sigma q = +1, & \mathrm{u}\bar{\mathrm{d}}, & \text{pion},\ \pi^+. \end{cases} \tag{2.64}$$

There are three 'colour' possibilities for $\pi^+ = \mathrm{u}\bar{\mathrm{d}}$, and $\pi^- = \bar{\mathrm{u}}\mathrm{d}$, etc.

For the baryons, a proton example

$$(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) \rightarrow \begin{cases} +(-,-,+,+) & \rightarrow q = +⅔, & \mathrm{u}, & \text{up quark}, \\ +(-,+,-,+) & \rightarrow q = +⅔, & \mathrm{u}, & \text{up quark}, \\ +(+,-,-,-) & \rightarrow q = -⅓, & \mathrm{d}, & \text{down quark}, \\ \rightarrow \mathrm{p}^+, \text{spin } +½, & \rightarrow \Sigma q = +1, & \mathrm{uud}, & \text{proton},\ \mathrm{p}^+, \end{cases} \tag{2.65}$$

and a neutron example

[15] A candidate for this local *scalar quantity* is the duality product of Energy-Time, $\Delta E{\cdot}\Delta t \sim \frac{1}{2}\hbar$.

$$(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) \to \begin{cases} +(-,-,+,-) & \to\ q = -⅔, & \text{d}, & \text{down quark}, \\ +(-,+,-,+) & \to\ q = +⅔, & \text{u}, & \text{up quark}, \\ +(+,-,-,-) & \to\ q = -⅓, & \text{d}, & \text{down quark}, \\ \to \text{n}^0, \text{spin } -½, & \to\ \Sigma q = 0, & \text{dud}, & \text{neutron, n}^0, \end{cases} \tag{2.66}$$

The *interaction direction* is marked read, with two possible orientations for a spin. It is up to the reader to make a 'colour' permutation interpretation of these ideas.

2.13. Concluding the Bivector Angular Momentum Idea of Quaternions

In this Appendix, we have observed §2.2 that the Geometric Algebra idea of bivectors as the three-dimensional orthogonal basis for the Hamilton quaternions $\mathbb{H}\sim\mathcal{G}_{0,2}^{\perp}(\mathbb{R})\sim\mathcal{G}_{3,0}^{+}(\mathbb{R})$ is geometric defining the physical locality of three perpendicular bivector planes $\{\boldsymbol{i}_1,\boldsymbol{i}_2,\boldsymbol{i}_3\}$ intersecting in one point of spherical locality, shown **Fig. 5**. These planes is rotation invariant conceptualising the most fundamental idea of physics: The preserved Angular Momentum, intrinsics in every active locality.[16]

The intrinsics active angular momentum bivectors in $\mathcal{G}_{3,0}^{+}(\mathbb{R})$ are angular generators for 1-spinors. These combines to 2-spinor quaternions if orthogonal (2.16),(2.18) §2.3.

Three orthogonal spin½ bivectors of angular momentum (2.21)-(2.22) §2.4 combines to a non-orthogonal tetrahedron structure of *directions* for angular momentum bivectors. These are generators for four active oscillating 1-spinors $\in\mathcal{G}_{3,0}^{+}(\mathbb{R})$ associated as 2-spinor $\mathbf{S}_{\wedge}\in\mathbb{H}\sim\mathcal{G}_{3,0}^{+}(\mathbb{R})$ (2.47) with a spatial structure displayed in **Fig. 7**.

The ½-versor idea inspired from the over hundred years old work by Adolf Hurwitz on ½-integer quaternions gives the tetrahedron plane bivector *directions* encapsulating a locality of a spherical center. This is dealt with in the Main Letter, Chapter 1. The four tetrahedron directions each of two orientations are strongly interconnected, so that there in $\mathbb{H}\sim\mathcal{G}_{0,2}^{\perp}(\mathbb{R})$ is only two resulting degrees of freedom. The *non-directional* scalar parts of quaternions mix their impact into normal invariant stable symmetry structure of *interconnected* bivector *directions* of angular activity inside a spherical locality structure considered as one indivisible particle interacting with its surroundings, **Fig. 3**.

Interests: The author declares no conflicts of interest, and no use of external data.

[16] First conceptual expressed in Kepler's 2. Law, the preservation of the angular moving area.