Hamiltonian mechanics in terms of geometric algebras

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Stephen, Leo and Charles

Solution of Rigid Body Dynamics in PGA by S. De Keninck and L. Dorst, based on ideas of C. Gunn: the motion is characterized by a *motor* M; an invertible PGA element that represents a combination of rotation and translation. The kinematics equation reads

$$\dot{M} = -\frac{1}{2} M B_b, \tag{1}$$

where B_b is a bivector that represents a generator of the motion in the body frame. The dynamics equation then reads

$$\dot{B}_b = I_b^{-1}(B_b \times I_b(B_b) + F_b), \tag{2}$$

where $\times$ is the geometric algebra commutator product defined by $a \times b = 1/2(ab - ba)$ and where $F_b = \widetilde{M} F_w M$ is an element in PGA that uniformly represents the total forces and torques in the body frame. For a rigid body consisting of a set of discrete points $m_i X_i$, the total body inertia map I_b is given by

$$I_b(B) = \sum_i m_i X_i \vee (X_i \times B) \tag{3}$$

and the Lagrangian of the system can be described as $L = \frac{1}{2} B \vee I(B)$.

Standard approach

Alternatively, one can find the above equations of motion by taking this Lagrangian as an input and by using the Lagrangian or Hamiltonian formalism. Namely, the configuration space of a rigid body is the Euclidean Lie group

$$SE(3) = SO(3) \rtimes \mathbb{R}^3,$$

and the rigid body motion can be viewed as a solution of the optimal control problem

$$\int L \, dt \to \min,$$

where $L : TSE(3) \to \mathbb{R}$ is the associated Lagrangian function. This approach is known in physics as the principle of least action. By the Pontryagin maximum principle, the solution satisfies Hamilton's equations defined by the Hamiltonian associated with Lagrangian L. Indeed, as we shall see, we get equations equivalent to (1) and (2) in this way.

Lie groups

We start from a general approach where the Euclidean group $SE(3)$ is replaced by an arbitrary Lie group G and where the classical Hamiltonian by an arbitrary function on its cotangent bundle. The form of Hamilton's equations in such cases is well-known, and if G is formed by rotors of a geometric algebra, we can directly translate this general result for dynamics on Lie groups into the GA language. The case of the rigid body corresponds to the choice of geometric algebra PGA and $G = SE(3)$. Of course, we may choose $SE(n)$ to obtain an n-dimensional version of the rigid body motion. But not just that, we may freely choose Hamiltonian to describe various interactions, and we may also change geometric algebra, for example, to CGA, which then leads to Lie group $G = SO(n+1,1)$. In particular, we get a description of the rigid body motion in CGA, and we also get its generalization to an "elastic body motion".

Lagrangian approach

The classical Lagrangian dynamics is formulated in terms of a function on this tangent bundle, the so-called Lagrangian function $L : T\mathbb{R}^n \to \mathbb{R}$, which may contain all physical information concerning the system and the forces acting on it. According to Hamilton's principle, the evolution of a physical system between two specified states $q_0 = q(t_0)$ and $q_1 = q(t_1)$ is then determined by a trajectory $q(t)$ in the configuration space that is a stationary point of the action functional

$$\mathcal{S}[q] = \int_{t_0}^{t_1} L(q, \dot{q}) dt. \tag{4}$$

Computing the first variation, one finds that this requirement is equivalent to well-known Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{5}$$

Hamiltonian approach

The Hamiltonian form of the equations of motion is given in terms of conjugate momentum covectors. The passage from the tangent bundle to the cotangent bundle is realized by the Legendre transformation $T\mathbb{R}^n \to T^*\mathbb{R}^n : (q, \dot{q}) \mapsto (q, p)$, where $p \in T_q^*\mathbb{R}^n$ is the conjugate momentum defined by

$$p = \frac{\partial L}{\partial \dot{q}}. \tag{6}$$

Note that we assume that the Legendre transformation is globally invertible. This property is usually referred to as the hyperregularity of the Lagrangian function in literature. The Hamiltonian dynamics is then equivalent to the Lagrangian dynamics and can be described by introducing the Hamiltonian function $H : T^*\mathbb{R}^n \to \mathbb{R}$:

$$H(q, p) = p \cdot \dot{q} - L(q, \dot{q}), \tag{7}$$

where $\dot{q}$ is viewed as a function of (q, p) by inverting the Legendre transformation.

Hamiltonian approach

Namely, the Euler-Lagrange equations (5) are equivalent to the well-known Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{8}$$

$$\dot{p} = -\frac{\partial H}{\partial q}. \tag{9}$$

Let $\mathfrak{g}$ be the associated Lie algebra. The tangent map (differential) at the identity to left translation map $\ell_g : G \to G$, defined as $\ell_g(h) = gh$, defines a map $(\ell_g)_* : \ T_e G = \mathfrak{g} \to T_g G$ from the Lie algebra to a tangent space to G. It is easy to see that this map has the inverse $(\ell_{g^{-1}})_* : \ T_g G \to \mathfrak{g}$ and thus is a linear isomorphism

$$(\ell_{g^{-1}})_* : \ TG \cong G \times \mathfrak{g}. \tag{10}$$

In other words, by left trivialization, we can identify the tangent bundle TG of a Lie group G with $G \times \mathfrak{g}$. Similarly, we can use the cotangent map to left translation $(\ell_g)^* : T_g^* G \to T_e^* G = \mathfrak{g}^*$, defined by $(\ell_g)^*(\mu)(\xi) = \mu((\ell_g)_*(\xi))$ for each $\xi \in \mathfrak{g}$ and $\mu \in T_g^* G$, to get a global trivialization of the cotangent bundle $T^* G$:

$$(\ell_g)^* : \ T^* G \cong G \times \mathfrak{g}^*. \tag{11}$$

Dynamics on Lie groups

The next ingredient we need to formulate the Hamilton's equations on a Lie group G is the coadjoint operator, for each $\mu \in \mathfrak{g}^*$ and $\xi, \zeta \in \mathfrak{g}$. defined by

$$\mathrm{ad}^*_\xi(\mu)(\zeta) = \mu(\mathrm{ad}_\xi(\zeta)) = \mu([\xi, \zeta]), \tag{12}$$

where $[,]$ denotes the Lie bracket in $\mathfrak{g}$. Now we are ready to formulate a coordinate-free Hamiltonian dynamics on Lie group G. Using the identification (11), the Hamiltonian function may be viewed as

$$H : G \times \mathfrak{g}^* \to \mathbb{R}.$$

Its derivative with respect to the second argument, which we write as $\partial H/\partial\mu$, is a linear map $\mathfrak{g}^* \to \mathbb{R}$, therefore it can be naturally considered as an element of $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. The derivative of the Hamiltonian function with respect to the first (group) argument, which we write as $\partial H/\partial g$, may be seen as a directional derivative and thus as a vector in the cotangent space $T^*_g G$.

Proposition

(Hamilton's equations on a Lie group) Let G be a Lie group, $\mathfrak{g}$ its associated Lie algebra, and let $\xi = \partial H/\partial\mu \in \mathfrak{g}$. The Hamilton's equations on G are a system on $G \times \mathfrak{g}^*$ given by

$$\dot{g} = (\ell_g)_*\xi, \tag{13}$$

$$\dot{\mu} = \mathrm{ad}^*_\xi\,\mu - (\ell_g)^*\frac{\partial H}{\partial g}. \tag{14}$$

GA formulation

The general form of Hamilton's equations on a Lie group given above allows a direct translation into the GA language if $G = \mathbb{G}(V)$ is the Lie group of rotors in a geometric algebra induced on a vector space V equipped with a quadratic form. In the case of a non-degenerate form of signature (p, q), the Lie group G is a covering of the orthogonal Lie group $SO(p, q)$. If we allow a degenerate quadratic form with a kernel of dimension r, the resulting geometric algebra can be embedded into the geometric algebra induced by a non-degenerate quadratic form of signature $(p + r, q + r)$. Hence G is a covering of a Lie subgroup of $SO(p + r, q + r)$. In any case, the associated Lie algebra is the algebra of bivectors,

$$\mathfrak{g} \cong \Lambda^2 V.$$

GA formulation

The dual Lie algebra $\mathfrak{g}^* \cong \Lambda^2 V^*$ can be identified with $\mathfrak{g}$ in the non-degenerate case. Namely, the defining bilinear form extends then extends to a non-degenerate bilinear form on bivectors via $\langle A, B\rangle = \langle A\tilde{B}\rangle_0$, where $\langle\rangle_0$ denotes the projection to the scalar part and $\tilde{B}$ is the reversion of B. Another possibility of viewing the dual algebra $\mathfrak{g}^*$ in GA is via a pseudoscalar I (or volume form). Namely, if we associate a $\mu^\sharp \in \Lambda^{n-2} V$ to each $\mu \in \mathfrak{g}^* \cong \Lambda^2 V^*$ such that $\mu^\sharp \wedge B = \mu(B) I$ for each bivector $B \in \Lambda^2 V$, we get an isomorphism of Lie algebras

$$\mathfrak{g}^* \cong \Lambda^{n-2} V,$$

where the Lie bracket on the right-hand side is given by the commutator with respect to the geometric product in $\mathbb{G}(V)$. It is easy to show that, in this identification, the coadjoint action corresponds to a commutator in GA of a bivector and an element of grade $n-2$,

$$(\mathrm{ad}^*_\xi \mu)^\sharp = [\mu^\sharp, \xi].$$

GA formulation

Then the Hamiltonian is a function

$$H : G \times \Lambda^{n-2} V \to \mathbb{R},$$

the inverse Legendre transformation gives a bivector

$$B = \frac{\partial H}{\partial \mu} \in \Lambda^2 V \tag{15}$$

while the derivative of the Hamiltonian function with respect to the group variable, expressed in the right trivialization is an element

$$F_s = \frac{\partial H}{\partial M} \tilde{M} \in \Lambda^{n-2} V.$$

This term corresponds in the Newtonian picture to forces expressed in the world frame.

Hamiltonian on Lie groups

Then, the Hamiltonian dynamics on the Lie group G in GA language look as follows.

Proposition

Let $\mathbb{G}(V)$ be a geometric algebra over a quadratic space V of dimension n and of any signature, and let G be the Lie group of its invertible elements. The Hamilton's equations on G read

$$\dot{M} = MB$$
$$\dot{\mu} = [\mu, B] - \tilde{M}F_s M$$

where $M \in G$, $\mu \in \Lambda^{n-2}V$, $B = \partial H/\partial\mu$ is the bivector obtained by the inverse Legendre transformation, and where $[\,,]$ is the commutator with respect to the geometric product in $\mathbb{G}(V)$.

PGA

PGA is generated by vector space V and quadratic form of degenerate signature $(3, 0, 1)$, i.e. with basis (e_0, e_1, e_2, e_3) such that

$$e_1^2 = e_2^2 = e_3^2 = 1 \text{ and } e_0^2 = 0.$$

Due to the one-dimensional kernel generated by e_0, the bivectors form Lie algebra $\mathfrak{g} = \mathfrak{se}(3) = \operatorname{span}\{e_0 \wedge e_i, e_i \wedge e_j\}$. Hence any bivector is of the form

$$B = \sum v_i e_0 \wedge e_i + \sum \omega_k e_i \wedge e_j,$$

where we assume that k is the complementary index to indices i, j.

PGA

For the dual Lie algebra we have $\mathfrak{g}^* = \mathrm{span}\{e_i \wedge e_j, e_0 \wedge e_i\}$. Actually, it has the same structure as $\mathfrak{g}$, and a dual bivector can be written as

$$\mu = \sum p_i (e_0 \wedge e_i)^* + \sum \ell_k (e_i \wedge e_j)^*,$$

where the star denotes the usual PGA duality. The dependence of rigid body Lagrangian $L(M, B)$ on the fiber variable is given by

$$L(\, , B) = \frac{1}{2} m v^2 + \frac{1}{2} \sum J_k \omega_k^2$$

and thus the Legendre transformation $\mathfrak{g} \to \mathfrak{g}^*$ is given by linear function

$$I(B) = \frac{\partial L}{\partial B} = m v_i (e_0 \wedge e_i)^* + J_k \omega_k (e_i \wedge e_j)^*.$$

PGA

The corresponding Hamiltonian reads

$$H(\, , \mu) = p_i \frac{p_i}{m} - \frac{1}{2} m (\frac{p_i}{m})^2 - \frac{1}{2} \sum J_k (\frac{\ell_k}{J_k})^2 = \frac{p^2}{2m} - \sum \frac{\ell_k^2}{2J_k}$$

and the inverse Legendre transformation is given by the (also linear) function

$$I^{-1}(\mu) = \frac{\partial H}{\partial \mu} = \frac{p_i}{m} e_0 \wedge e_i + \frac{\ell_k}{J_k} e_i \wedge e_j.$$

Due to the linearity, Hamilton's equations can be written as

$$\dot{M} = MB$$

$$I(\dot{B}) = [I(B), B] + \tilde{M} \frac{\partial H}{\partial M} M$$

which is equivalent to the S-L-C equations up to conventions concerning sign and factor 1/2.

CGA

If we view the Euclidean group $SE(3)$ as a subgroup of $SO(4, 1)$, and we take the same Hamiltonian, we get a description of rigid body motion in terms of geometric algebra CGA. Such conformal description also allows a generalization to the motion of an "elastic body." Due to the existence of two null vectors $e_0^2 = e_\infty^2 = 0$, the lie algebra has extra elements. In particular, it is $e_0 \wedge e_\infty$ that generates scaling.

Matrix representation

Recap: G a Lie group, $\mathfrak{g}$ its Lie algebra, H Hamiltonian function $G \times \mathfrak{g} \to \mathbb{R}$, $\xi = \partial H / \partial \mu \in \mathfrak{g}$.
Hamiltonian equations:

$$\text{on G:} \quad \dot{g} = (\ell_g)_* \xi,$$
$$\text{on } \mathfrak{g}^*: \quad \dot{\mu} = \mathrm{ad}^*_\xi \mu - (\ell_g)^* \frac{\partial H}{\partial g}.$$

$G = SE(3)$, then $\mathfrak{g} = se(3) = so(3) \oplus \mathbb{R}^3 \ni (\xi, u)$ and $(\Pi, p) \in se^*(3)$ and under the usual identification $so(3) \cong \mathbb{R}^3$

$$\mathrm{ad}^*_{(\xi,u)}(\Pi, p) = (\Pi \times \xi + p \times u, p \times \xi) \rightsquigarrow$$
$$\frac{d}{dt}\begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} r \times \partial H / \partial \Pi \\ R \partial H / \partial p \end{pmatrix}$$
$$\frac{d}{dt}\begin{pmatrix} \Pi \\ p \end{pmatrix} = \begin{pmatrix} Pi \times \partial H / \partial \Pi + r \times \partial H / \partial r + p \times \partial H / \partial p \\ -R^T \partial H / \partial x + p \times \partial H / \partial \Pi \end{pmatrix}$$

GA representation

In GA we can model the dual Lie algebra $\mathfrak{g}^*$ by the GA duality such that the coadjoint operator becomes the commutator

$$\mathrm{ad}^*_\xi \mu = (\xi \times \mu^*)^*$$

works for G-invariant duality:

- $\mu(\zeta)I = \mu^* \wedge \zeta$
- $\mu(\zeta) = \langle \mu^*, \zeta \rangle$ for nondegenerate GA's

$\Rightarrow$ troubles with the CGA duality in implementations

Conformal model

View $SE(3)$ as a subgroup of $SO(4,1)$.

- $G = SO(4,1)$, Lie algebra $\mathfrak{g} = \mathfrak{so}(4,1)$ modeled on bivectors in CGA $= Cl(4,1)$.
  $V = \mathbb{R}^{4,1}$ with basis $(e_0, e_1, e_2, e_3, e_\infty)$ such that $e_0^2 = e_\infty^2 = 0$

$$\mathfrak{g} = \mathfrak{so}(4,1) = \text{span}\{e_0 \wedge e_i, e_i \wedge e_j, e_i \wedge e_\infty, e_0 \wedge e_\infty\}$$

- Lagrangian $l(B) = \frac{1}{2}mv^2 + \frac{1}{2}\sum J_i\omega_i^2 + \frac{1}{2}J_0\omega_0^2 \rightsquigarrow$ an extra term in the Legendre transformation $J_0(e_0 \wedge e_\infty)^*$
- initial shrinking (expanding) possible

An (easy) implementation in Ganja.js (A versatile and multiplatform Algebra generator with a focus on education and visualization available at BiVector.net)

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