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Factorizations of the Conformal Villarceau Motion

Supported by: Austrian Science Fund (FWF): P 33397-N Rotor Polynomials: Algebra and Geometry of Conformal Motions

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The Conformal Villarceau Motion

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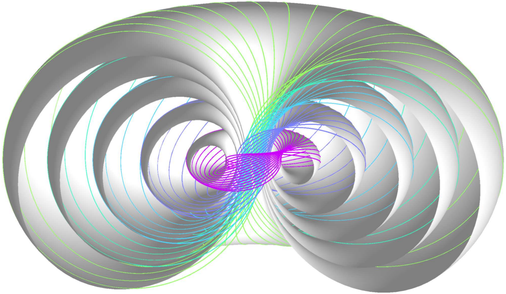

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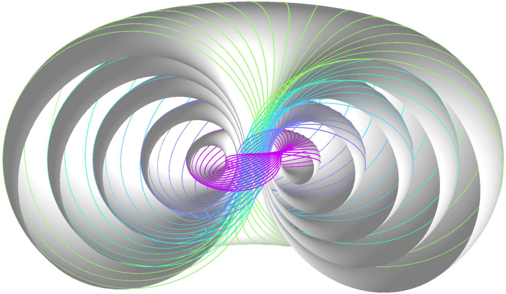

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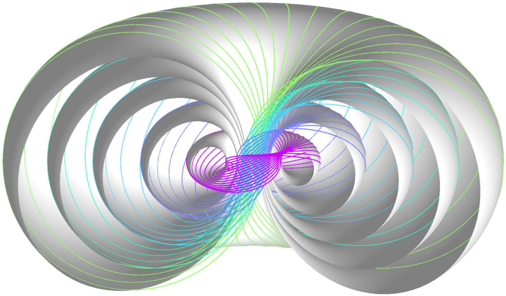

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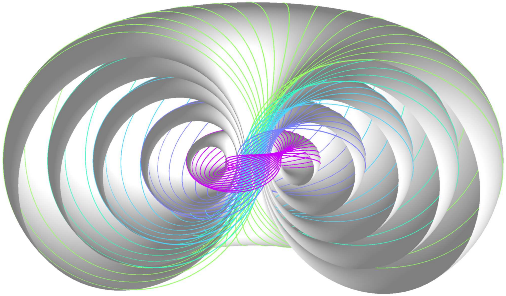

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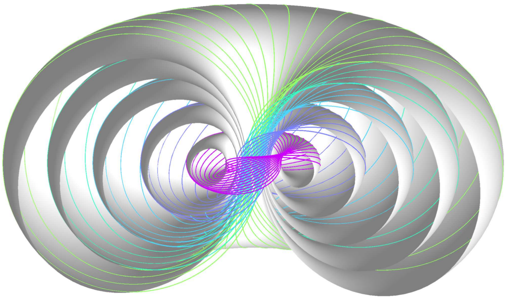

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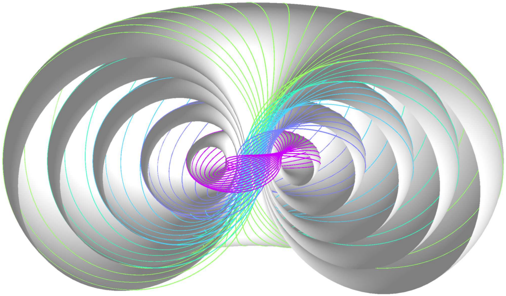

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- The trajectories of all points form a family of nested tori
- The set of unique trajectories forms the Hopf-fibration of $\mathbb{R}^3$

Base Objects in CGA

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We will be using 3D-CGA with the basis $\{e_1, e_2, e_3, e_+, e_-\} \in \mathbb{R}^{4,1}$ and call its even sub-algebra CGA_+.

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Natural Objects in CGA

The vectors in CGA naturally can be viewed as spheres:

$$\underset{([m_1,m_2,m_3],r)}{\text{Sphere}} \hookrightarrow m_1e_1 + m_2e_2 + m_3e_3 + \frac{m_1^2 + m_2^2 + m_3^2 - r^2}{2}e_\infty + e_0 \quad =: S$$

$$\underset{([m_1,m_2,m_3])}{\text{Point}} \hookrightarrow m_1e_1 + m_2e_2 + m_3e_3 + \frac{m_1^2 + m_2^2 + m_3^2 - 0^2}{2}e_\infty + e_0 \quad =: P$$

$$\underset{(n=[n_1,n_2,n_3],d)}{\text{Plane}} \hookrightarrow n_1e_1 + n_2e_2 + n_3e_3 + de_\infty + 0e_0 \quad =: E$$

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A transformation y can be applied to a general element $x \in \mathrm{CGA}$ by sandwiching.

$$x \mapsto yx\tilde{y} = x'$$

Simple Motions

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Simple conformal motions R are characterized by a 2-blade B and given as

$$R(t) = e^{tB} = \begin{cases} \cos(t\beta) + B\frac{\sin(t\beta)}{\beta} & \text{if } B^2 = -\beta^2 \\ 1 + Bt & \text{if } B^2 = 0 \\ \cosh(t\beta) + B\frac{\sinh(t\beta)}{\beta} & \text{if } B^2 = \beta^2 \end{cases} .$$

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Through reparametrization, we can write $[e^{tB}] = [s + B]$ for appropriate $s(t)$.

Simple Motions

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The condition $y\tilde{y} = \tilde{y}y \in \mathbb{R}$ defines the *Study variety* $\mathcal{S}$ of conformal Kinematics.
For $y \in \mathcal{S}$,

$$\langle y\tilde{y}\rangle_0 = \langle \tilde{y}y\rangle_0 = 0$$

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Taking the reparametrization of simple motions we see

$$(s+B)\widetilde{(s+B)} = s^2 + (B+\tilde{B})s + B\tilde{B} \in \mathbb{R}$$

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Simple conformal motions are described by lines on $\mathcal{S}$ and elements in $\mathcal{S} \cap \mathcal{N}$ describe singular transformations.

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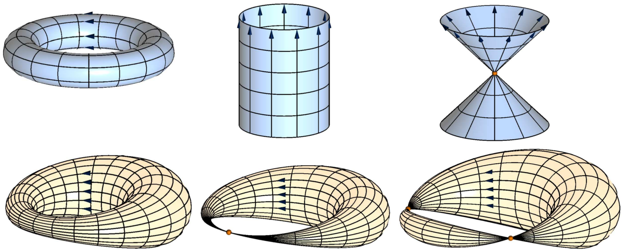

Figure: Simple conformal motions and euclidean counterpart

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We define the action of C on $x \in \mathrm{CGA}$ as

$$x \mapsto C x \tilde{C}$$

and call it a *rational conformal motion.*

Spinor Polynomials

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Definition (Spinor Polynomial)

A polynomial $C(t) := \sum_{i=0}^{n} c_i t^i$ with $c_i \in \mathrm{CGA}_+$ fulfilling

- $C\tilde{C} = \tilde{C}C \neq 0 \in \mathbb{R}[t]$
- $n > 0$

is called a spinor polynomial and describes a conformal motion.

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- There exist polynomials with infinite factorizations
- The linear polynomial $t - h$ is a right factor of C if and only if h is a right zero.

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Now split off this factor and start again.

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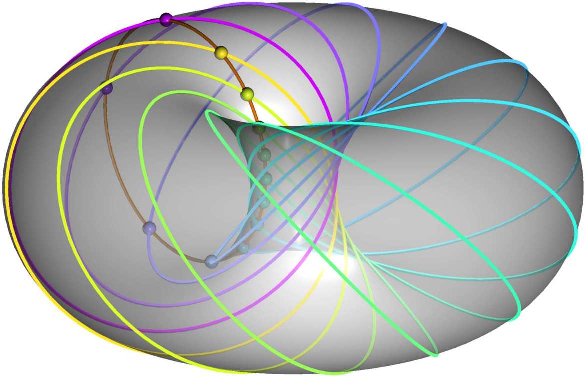

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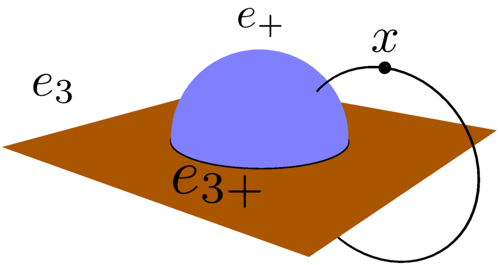

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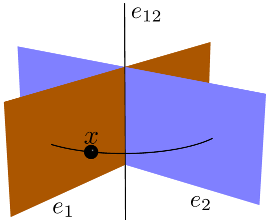

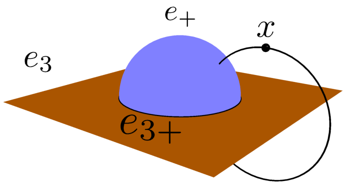

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$$C = (\cos(\tfrac{\varphi}{2}) - B_- \sin(\tfrac{\varphi}{2}))(\cos(\tfrac{\varphi}{2}) - B_+ \sin(\tfrac{\varphi}{2})).$$

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Using $t = \cot(\frac{\varphi}{2})$ we can rewrite C as a spinor polynomial.

$$\begin{aligned} C &= (t - B_-)(t - B_+) \\ &= t^2 - t(e_{12} + e_{3+}) + e_{123+}. \end{aligned}$$

Factorization of the Villarceau Motion

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$$C\tilde{C} = t^4 + 1 = (t^2 + 1)^2 = M^2$$

for $M := (t^2 + 1)$

Factorization of the Villarceau Motion

- Find factors of $C\tilde{C}$
$$C\tilde{C} = t^4 + 1 = (t^2 + 1)^2 = M^2$$
for $M := (t^2 + 1)$
- Divide C by one monic, quadratic factor and extract the remainder.

$$\begin{aligned} C &= t^2 - t(e_{12} + e_{3+}) + e_{123+} \\ &= 1 \cdot (t^2 + 1) + R \\ R &:= r_1 t + r_0 \\ r_1 &:= (-e_{12} - e_{3+}) \\ r_0 &:= e_{123+} - 1 \end{aligned}$$

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- Find h_2 so that the following holds:
  - $t - h_2$ is a right zero of R
    $$r_1 h_2 + r_2 = 0$$
  - h_2 is a zero of M
    $$h_2^2 + 1 = 0$$
  - $t - h$ is a spinor polynomial
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- We get the result
  $$h_2 = e_{12} + s_x x + s_y y + s_z z,$$
  where
  $$s_x = 2(e_{1+} - e_{23}), \quad s_y = 2(e_{2+} + e_{13}), \quad s_z = 2(e_{3+} - e_{12}),$$
  $$\text{and} \quad x^2 + y^2 + (z - \tfrac{1}{4})^2 - \tfrac{1}{16} = 0.$$

Linear Factors of the Villarceau Motion

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- Because s_x, s_y, s_x are pairwise perpendicular, h_2 lies on a sphere.

$$
\begin{aligned}
h_2(u,v) &= m + \frac{1}{4}S(u,v) \\
m &= \frac{1}{2}(e_{12} + e_{3+}) \\
S(u,v) &= \sin(u)\cos(v)s_x + \sin(u)\sin(v)s_y + \cos(u)s_z
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- By polynomial division of C by $t - h_2$ we get the second linear factor

$$h_1(u,v) = m - \frac{1}{4}S(u,v)$$

- The two factors of C lie antipodal on a sphere centered at m.

Analysis of Linear Factors

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The norm of h_1 and h_2 is always $1 \Rightarrow h_1$ and h_2 are conformal rotations.

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Applying H_1 and H_2 with independent indeterminates to a point x gives us the trajectory surface

$$D_x = H_1 H_2 x \tilde{H}_2 \tilde{H}_1.$$

fulfilling the conditions:

- All parameter lines are circles.
- The second fundamental form of D_x is diagonal.

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Hence, D_x is a Dupin cyclide.

Dupin Cyclides D_x

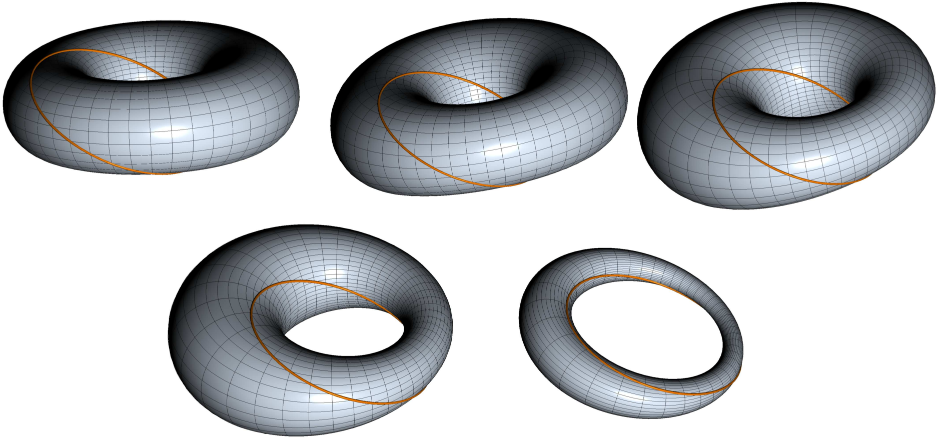

Figure: Different Dupin cyclides with the same Villarceau circle

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Computation shows that none of the elements of this connecting line are invertible.

This helps explain the strange factorization properties.

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Factorizing Spinor Polynomials

Factorization of non-commutative polynomials into linear factors

Input Polynomial C with $deg(C) \geq 1$ monic, no real factors of positive degree

Output$H = (h_1, h_2, \ldots, h_n)$ with $C = (t - h_1)(t - h_2) \ldots (t - h_n)$

$H \leftarrow ()$

while $deg(C) > 0$ **do**

  $M \leftarrow$ Monic quadratic factor of $C\tilde{C}$

  Compute $Q, R \in \mathrm{CGA}_+[t]$, so that $C = QM + R$ ▷ Polynomial Division

  $h \leftarrow$ right zero of R

  $H \leftarrow h + H$ ▷ Concatenation of Tuples

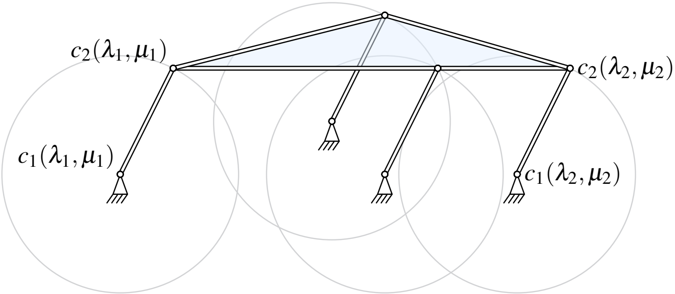
  Compute $C\prime \in \mathrm{CGA}_+[t]$ so that $C = C\prime(t - h)$

  $C \leftarrow C\prime$

end while

return H

The Circular Translation



$$\mathbf{i} = -e_{23}, \quad \mathbf{j} = e_{13}, \quad \mathbf{k} = -e_{12}, \quad \varepsilon = e_{123+} + e_{123-}$$

$$C = t^2 + 1 - \varepsilon(\mathbf{j}t + \mathbf{i})$$

$$C = (t - \mathbf{k} - \varepsilon((1-\mu)\mathbf{j} - \lambda\mathbf{i}))(t + \mathbf{k} - \varepsilon(\lambda\mathbf{i} + \mu\mathbf{j})), \quad (\lambda, \mu) \in \mathbb{R}^2.$$