On Rank of Multivectors in Geometric Algebra

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3. SVD in GA
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Real Clifford geometric algebra (GA)

Let us consider the real Clifford geometric algebra (GA) $\mathcal{G}_{p,q}$ with the identity element $e \equiv 1$ and the generators e_a, $a = 1, 2, \ldots, n$, where $n = p + q \geq 1$:

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad \eta = (\eta_{ab}) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1).$$

Consider the subspaces $\mathcal{G}^k_{p,q}$ of grades $k = 0, 1, \ldots, n$, which elements are linear combinations of the basis elements $e_A = e_{a_1 a_2 \ldots a_k} = e_{a_1} e_{a_2} \cdots e_{a_k}$ with ordered multi-indices of length k. An arbitrary multivector $M \in \mathcal{G}_{p,q}$ has the form

$$M = \sum_A m_A e_A \in \mathcal{G}_{p,q}, \qquad m_A \in \mathbb{R},$$

where we have a sum over arbitrary multi-index A of length from 0 to n. The projection of M onto the subspace $\mathcal{G}^k_{p,q}$ is denoted by $\langle M \rangle_k$.
The grade involution and reversion of a multivector $M \in \mathcal{G}_{p,q}$ are denoted by

$$\widehat{M} = \sum_{k=0}^{n} (-1)^k \langle M \rangle_k, \qquad \widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k \tag{1}$$

$$\widehat{M_1 M_2} = \widehat{M_1} \widehat{M_2}, \qquad \widetilde{M_1 M_2} = \widetilde{M_2} \widetilde{M_1}, \qquad \forall M_1, M_2 \in \mathcal{G}_{p,q}. \tag{2}$$

Complexified Clifford GA

Let us consider the complexified Clifford geometric algebra $\mathcal{G}^{\mathbb{C}}_{p,q} := \mathbb{C} \otimes \mathcal{G}_{p,q}$. An arbitrary element of $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$ has the form

$$M = \sum_A m_A e_A \in \mathcal{G}^{\mathbb{C}}_{p,q}, \qquad m_A \in \mathbb{C}.$$

Note that $\mathcal{G}^{\mathbb{C}}_{p,q}$ has the following basis of 2^{n+1} elements:

$$e, ie, e_1, ie_1, e_2, ie_2, \ldots, e_{1\ldots n}, ie_{1\ldots n}. \tag{3}$$

In addition to the grade involution and reversion, we use the operation of complex conjugation, which takes complex conjugation only from the coordinates m_A and does not change the basis elements e_A:

$$\overline{M} = \sum_A \overline{m}_A e_A \in \mathcal{G}^{\mathbb{C}}_{p,q}, \qquad m_A \in \mathbb{C}, \qquad M \in \mathcal{G}^{\mathbb{C}}_{p,q}.$$

We have

$$\overline{M_1 M_2} = \overline{M_1}\ \overline{M_2}, \qquad \forall M_1, M_2 \in \mathcal{G}^{\mathbb{C}}_{p,q}.$$

Unitary space on GA

Let us consider an operation of Hermitian conjugation $\dagger$ in $\mathcal{G}^{\mathbb{C}}_{p,q}$:

$$M^\dagger := M|_{e_A \to (e_A)^{-1},\ m_A \to \overline{m}_A} = \sum_A \overline{m}_A (e_A)^{-1}. \qquad (4)$$

We have the following two other equivalent definitions of this operation:

$$M^\dagger = \begin{cases} e_{1\ldots p}\overline{\widetilde{M}}e^{-1}_{1\ldots p}, & \text{if } p \text{ is odd,} \\ e_{1\ldots p}\overline{\widetilde{\widehat{M}}}e^{-1}_{1\ldots p}, & \text{if } p \text{ is even,} \end{cases} = \begin{cases} e_{p+1\ldots n}\overline{\widetilde{M}}e^{-1}_{p+1\ldots n}, & \text{if } q \text{ is even,} \\ e_{p+1\ldots n}\overline{\widetilde{\widehat{M}}}e^{-1}_{p+1\ldots n}, & \text{if } q \text{ is odd.} \end{cases} \qquad (5)$$

The operation

$$(M_1, M_2) := \langle M_1^\dagger M_2 \rangle_0 \geq 0$$

is a (positive definite) scalar product. Using this scalar product we introduce inner product space over the field of complex numbers (unitary space) in $\mathcal{G}^{\mathbb{C}}_{p,q}$. We have a norm

$$||M|| := \sqrt{(M, M)} = \sqrt{\langle M^\dagger M \rangle_0} \qquad (6)$$

Matrix representation of $\mathcal{G}^{\mathbb{C}}_{p,q}$

Let us consider the following faithful representation (isomorphism) of the complexified geometric algebra

$$\beta : \mathcal{G}^{\mathbb{C}}_{p,q} \quad \to \quad \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{C}), & \text{if } n \text{ is even,} \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

Let us denote the size of the corresponding matrices by

$$N := 2^{[\frac{n+1}{2}]},$$

where square brackets mean taking the integer part. Note that we use block-diagonal matrices in the cases $n = 1 \mod 2$.

Let us present an explicit form of one β' of these representations of $\mathcal{G}^{\mathbb{C}}_{p,q}$.
Let us consider the case $p=n$, $q=0$. To obtain the matrix representation for another signature with $q\neq 0$, we should multiply matrices $\beta'(e_a)$, $a=p+1,\ldots,n$ by imaginary unit i.
We have $\beta'(e)=I_N$ and $\beta'(e_{a_1a_2\ldots a_k})=\beta'(e_{a_1})\beta'(e_{a_2})\cdots\beta'(e_{a_k})$.
In the case $n=1$, we take $\beta'(e_1)=\mathrm{diag}(1,-1)$.
Suppose we know $\beta'_a:=\beta'(e_a)$, $a=1,\ldots,n$ for some fixed odd $n=2k+1$. Then for $n=2k+2$, we take the same $\beta'(e_a)$, $a=1,\ldots,2k+1$, and

$$\beta'(e_{2k+2})=\begin{pmatrix} 0 & I_{\frac{N}{2}} \\ I_{\frac{N}{2}} & 0 \end{pmatrix}.$$

For $n=2k+3$, we take

$$\beta'(e_a)=\begin{pmatrix} \beta'_a & 0 \\ 0 & -\beta'_a \end{pmatrix}, \qquad a=1,\ldots,2k+2,$$

and

$$\beta'(e_{2k+3})=\begin{pmatrix} i^{k+1}\beta'_1\cdots\beta'_{2k+2} & 0 \\ 0 & -i^{k+1}\beta'_1\cdots\beta'_{2k+2} \end{pmatrix}.$$

Note that for this matrix representation we have

$$(\beta'(e_a))^\dagger = \eta_{aa}\beta'(e_a), \qquad a = 1, \ldots, n,$$

where † is the Hermitian transpose of a matrix. Using the linearity, we get that Hermitian conjugation of matrix is consistent with Hermitian conjugation of corresponding multivector:

$$\beta'(M^\dagger) = (\beta'(M))^\dagger, \qquad M \in \mathcal{G}^{\mathbb{C}}_{p,q}. \tag{8}$$

Note that the same is not true for an arbitrary matrix representations β of the form (7). It is true the matrix representations $\gamma = T^{-1}\beta' T$ obtained from β' using the matrix T such that $T^\dagger T = I$.

Unitary Lie groups in GA

Let us consider the group

$$\mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q} = \{M \in \mathcal{G}^{\mathbb{C}}_{p,q} : M^\dagger M = e\}, \tag{9}$$

which we call a unitary group in $\mathcal{G}^{\mathbb{C}}_{p,q}$. Note that all the basis elements e_A of $\mathcal{G}_{p,q}$ belong to this group by the definition.
Using (7) and (8), we get the following isomorphisms to the classical matrix unitary groups:

$$\mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n}{2}}), & \text{if } n \text{ is even}, \\ \mathrm{U}(2^{\frac{n-1}{2}}) \times \mathrm{U}(2^{\frac{n-1}{2}}), & \text{if } n \text{ is odd}, \end{cases} \tag{10}$$

where

$$\mathrm{U}(k) = \{A \in \mathrm{Mat}(k, \mathbb{C}), \quad A^\dagger A = I\}. \tag{11}$$

Singular value decomposition (SVD)

Theorem

For an arbitrary $A \in \mathbb{C}^{n\times m}$*, there exist matrices* $U \in \mathrm{U}(n)$ *and* $V \in \mathrm{U}(m)$ *such that*

$$A = U\Sigma V^\dagger, \qquad (12)$$

where

$$\Sigma = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad k = \min(n, m), \qquad \mathbb{R} \ni \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0.$$

Note that choosing matrices $U \in \mathrm{U}(n)$ *and* $V \in \mathrm{U}(m)$*, we can always arrange diagonal elements of the matrix* Σ *in decreasing order* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$.

Diagonal elements of the matrix Σ are called singular values, they are square roots of eigenvalues of the matrices $AA^\dagger$ or $A^\dagger A$. Columns of the matrices U and V are eigenvectors of the matrices $AA^\dagger$ and $A^\dagger A$ respectively.

Theorem (SVD in GA)

For an arbitrary multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$*, there exist multivectors* $U, V \in \mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q}$*, where*

$$\mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q} = \{U \in \mathcal{G}^{\mathbb{C}}_{p,q} : U^{\dagger} U = e\}, \qquad U^{\dagger} := \sum_{A} \overline{u}_A (e_A)^{-1},$$

such that

$$M = U \Sigma V^{\dagger}, \tag{13}$$

where multivector Σ *belongs to the subspace* $K \in \mathcal{G}^{\mathbb{C}}_{p,q}$*, which is a real span of a set of* $N = 2^{[\frac{n+1}{2}]}$ *fixed basis elements (3) of* $\mathcal{G}^{\mathbb{C}}_{p,q}$ *including the identity element* e*.*

Example

In the case $\mathcal{G}^{\mathbb{C}}_{2,0} \cong \mathrm{Mat}(2,\mathbb{C})$, we have

$$\beta'(e)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta'(e_1)=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \beta'(e_2)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \beta'(e_{12})=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrices $\beta'(e)$ and $\beta'(e_1)$ are real and diagonal, we get the subspace

$$K = \mathrm{span}(e, e_1).$$

Consider the multivector
$M = (1+i)e + (1-i)e_1 + (1+i)e_2 + (-1+i)e_{12} \in \mathcal{G}^{\mathbb{C}}_{2,0}$. We can choose

$$U = \frac{1+i}{2\sqrt{2}}e + \frac{-1+i}{2\sqrt{2}}e_1 + \frac{-1+i}{2\sqrt{2}}e_2 + \frac{-1-i}{2\sqrt{2}}e_{12} \in \mathrm{G}\mathcal{G}^{\mathbb{C}}_{2,0}, \qquad (14)$$

$$V = \frac{1+i}{2\sqrt{2}}e + \frac{-1+i}{2\sqrt{2}}e_1 + \frac{1-i}{2\sqrt{2}}e_2 + \frac{-1-i}{2\sqrt{2}}e_{12} \in \mathrm{G}\mathcal{G}^{\mathbb{C}}_{2,0}, \qquad (15)$$

$$U^\dagger U = V^\dagger V = e, \quad M = U\Sigma V^\dagger, \quad \Sigma = 2(e+e_1) \in K = \mathrm{span}(e, e_1). \qquad (16)$$

Determinant and characteristic polynomial in GA

We can introduce the notion of determinant

$$\mathrm{Det}(M) := \det(\beta(M)) \in \mathbb{R}, \qquad M \in \mathcal{G}^{\mathbb{C}}_{p,q},$$

where β is (7), and the notion of characteristic polynomial

$$\varphi_M(\lambda) := \mathrm{Det}(\lambda e - M) = \lambda^N - C_{(1)}\lambda^{N-1} - \cdots - C_{(N-1)}\lambda - C_{(N)} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R},$$
$$M \in \mathcal{G}^{\mathbb{C}}_{p,q}, \quad N = 2^{[\frac{n+1}{2}]}, \quad C_{(k)} = C_{(k)}(M) \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}, \quad k = 1, \ldots, N. \qquad (17)$$

The following method based on the Faddeev–LeVerrier algorithm allows us to recursively obtain basis-free formulas for all the characteristic coefficients $C_{(k)}$, $k = 1, \ldots, N$ (17):

$$M_{(1)} := M, \qquad M_{(k+1)} = M(M_{(k)} - C_{(k)}), \qquad (18)$$
$$C_{(k)} := \frac{N}{k}\langle M_{(k)}\rangle_0, \qquad k = 1, \ldots, N. \qquad (19)$$

In this method, we obtain high coefficients from the lowest ones.

The determinant is minus the last coefficient

$$\mathrm{Det}(M) = -C_{(N)} = -M_{(N)} = U(C_{(N-1)} - M_{(N-1)}) \tag{20}$$

and has the property

$$\mathrm{Det}(M_1 M_2) = \mathrm{Det}(M_1)\mathrm{Det}(M_2), \qquad M_1, M_2 \in \mathcal{G}^{\mathbb{C}}_{p,q}. \tag{21}$$

The inverse of a multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$ can be computed as

$$M^{-1} = \frac{\mathrm{Adj}(M)}{\mathrm{Det}(M)} = \frac{C_{(N-1)} - M_{(N-1)}}{\mathrm{Det}(M)}, \qquad \mathrm{Det}(M) \neq 0. \tag{22}$$

Rank in GA

Let us introduce the notion of rank of a multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$:

$$\mathrm{rank}(M) := \mathrm{rank}(\beta(M)) \in \{0, 1, \dots, N\}, \tag{23}$$

where β is (7).

Lemma

The rank of multivector $\mathrm{rank}(M)$ (23) is well-defined, i.e. it does not depend on the representation β (7).

Below we present another equivalent definition, which does not depend on the matrix representation β. We use the fact that rank is the number of nonzero singular values in the SVD and Vieta formulas.

Theorem

We have the following properties of the rank of arbitrary multivectors $M_1, M_2, M_3 \in \mathcal{G}^{\mathbb{C}}_{p,q}$:

$$\mathrm{rank}(M_1 U) = \mathrm{rank}(U M_1) = \mathrm{rank}(M_1), \qquad \forall \text{ invertible } U \in \mathcal{G}^{\mathbb{C}}_{p,q}, \tag{24}$$

$$\mathrm{rank}(M_1 M_2) \leq \min(\mathrm{rank}(M_1), \mathrm{rank}(M_2)), \tag{25}$$

$$\mathrm{rank}(M_1 M_2) + \mathrm{rank}(M_2 M_3) \leq \mathrm{rank}(M_1 M_2 M_3) + \mathrm{rank}(M_2), \tag{26}$$

$$\mathrm{rank}(M_1) + \mathrm{rank}(M_3) \leq \mathrm{rank}(M_1 M_3) + N. \tag{27}$$

Theorem

We have

$$\mathrm{rank}(M) = \mathrm{rank}(\widehat{M}) = \mathrm{rank}(\widetilde{M}) = \mathrm{rank}(\overline{M}) \tag{28}$$

$$= \mathrm{rank}(M^\dagger) = \mathrm{rank}(M^\dagger M) = \mathrm{rank}(M M^\dagger), \qquad \forall M \in \mathcal{G}^{\mathbb{C}}_{p,q}. \tag{29}$$

Lemma

Suppose that a square matrix $A \in \mathbb{C}^{N\times N}$ *is diagonalizable. Then*

$$\mathrm{rank}(A) = N \quad \Leftrightarrow \quad C_{(N)} \neq 0; \tag{30}$$

$$\mathrm{rank}(A) = k \in \{1, \dots, N-1\} \Leftrightarrow C_{(k)} \neq 0,\ C_{(j)} = 0, j = k+1, \dots, N; \tag{31}$$

$$\mathrm{rank}(A) = 0 \quad \Leftrightarrow \quad A = 0. \tag{32}$$

Lemma

For an arbitrary multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, *we have*

$$C_{(N)}(M^\dagger M) = 0 \quad \Longleftrightarrow \quad C_{(N)}(M) = 0, \tag{33}$$

$$C_{(1)}(M^\dagger M) = 0 \quad \Longleftrightarrow \quad M = 0. \tag{34}$$

Theorem (Rank in GA)

Let us consider an arbitrary multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$ *and* $T := M^\dagger M$. *We have*

$$\operatorname{rank}(M) = \begin{cases} N, & \text{if } C_{(N)}(M) \neq 0, \\ N-1, & \text{if } C_{(N)}(M) = 0 \text{ and } C_{(N-1)}(T) \neq 0, \\ N-2 & \text{if } C_{(N)}(M) = C_{(N-1)}(T) = 0 \text{ and } C_{(N-2)}(T) \neq 0 \\ \cdots & \\ 2, & \text{if } C_{(N)}(M) = C_{(N-1)}(T) = \cdots = C_{(3)}(T) = 0 \text{ and} \\ & C_{(2)}(T) \neq 0, \\ 1, & \text{if } C_{(N)}(M) = C_{(N-1)}(T) = \cdots = C_{(2)}(T) = 0 \text{ and} \\ & M \neq 0, \\ 0, & \text{if } M = 0. \end{cases}$$

Example

For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $n = p + q = 1$, we have

$$\operatorname{rank}(M) = \begin{cases} 2, & \text{if } M\widehat{M} \neq 0, \\ 1, & \text{if } M\widehat{M} = 0 \text{ and } M \neq 0, \\ 0, & \text{if } M = 0. \end{cases} \tag{35}$$

Example

For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $n = p + q = 2$, we have

$$\operatorname{rank}(M) = \begin{cases} 2, & \text{if } M\widetilde{\widehat{M}} \neq 0, \\ 1, & \text{if } M\widetilde{\widehat{M}} = 0 \text{ and } M \neq 0, \\ 0, & \text{if } M = 0. \end{cases} \tag{36}$$

Example

For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $n=p+q=3$, we have

$$\mathrm{rank}(M) = \begin{cases} 4, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} \neq 0, \\ 3, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = 0 \text{ and } T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T\widehat{T}\widetilde{T} + \widetilde{\widehat{T}}\widehat{T}\widetilde{T} \neq 0, \\ 2, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T\widehat{T}\widetilde{T} + \widetilde{\widehat{T}}\widehat{T}\widetilde{T} = 0 \text{ and} \\ & T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + \widehat{T}\widetilde{T} \neq 0, \\ 1, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T\widehat{T}\widetilde{T} + \widetilde{\widehat{T}}\widehat{T}\widetilde{T} = \\ & = T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + \widehat{T}\widetilde{T} = 0 \text{ and } M \neq 0, \\ 0, & \text{if } M = 0, \end{cases}$$

where $T := M^\dagger M$.

Example

Let us consider the $\triangle$-operation $((M_1 M_2)^{\triangle} \neq M_1^{\triangle} M_2^{\triangle};\ (M_1 M_2)^{\triangle} \neq M_2^{\triangle} M_1^{\triangle}\ )$

$$M^{\triangle} := \sum_{k=0}^{n} (-1)^{\frac{k(k-\mathbf{1})(k-\mathbf{2})(k-\mathbf{3})}{\mathbf{24}}} \langle M \rangle_k = \sum_{k=0,1,2,3 \bmod 8} \langle M \rangle_k - \sum_{k=4,5,6,7 \bmod 8} \langle M \rangle_k. \qquad (37)$$

For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $n = p+q = 4$, we have $(T := M^{\dagger} M)$

$$\mathrm{rank}(M) = \begin{cases} 4, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} \neq 0, \\ 3, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = 0 \text{ and} \\ & T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T(\widehat{T}\widetilde{T})^{\triangle} + \widetilde{\widehat{T}}(\widehat{T}\widetilde{T})^{\triangle} \neq 0, \\ 2, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T(\widehat{T}\widetilde{T})^{\triangle} + \widetilde{\widehat{T}}(\widehat{T}\widetilde{T})^{\triangle} = 0 \\ & \text{and } T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + (\widehat{T}\widetilde{T})^{\triangle} \neq 0, \\ 1, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T(\widehat{T}\widetilde{T})^{\triangle} + \widetilde{\widehat{T}}(\widehat{T}\widetilde{T})^{\triangle} = \\ & = T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + (\widehat{T}\widetilde{T})^{\triangle} = 0 \text{ and } M \neq 0, \\ 0, & \text{if } M = 0. \end{cases}$$

The case of normal multivectors

We call *a normal multivector* $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$ a multivector with the property $M^\dagger M = MM^\dagger$. *Hermitian multivectors* $M^\dagger = M$, *anti-Hermitian multivectors* $M^\dagger = -M$, *unitary multivectors* $M^\dagger M = e$ are the particular cases of normal multivectors. For example, the basis elements e_A are unitary by the definition. Note that all unitary multivectors have rank equal to N.

Theorem

Let us consider a normal ($M^\dagger M = MM^\dagger$) multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$. We have

$$
\mathrm{rank}(M) = \begin{cases} N, & \text{if } C_{(N)}(M) \neq 0, \\ N-1, & \text{if } C_{(N)}(M) = 0 \text{ and } C_{(N-1)}(M) \neq 0, \\ N-2 & \text{if } C_{(N)}(M) = C_{(N-1)}(M) = 0 \text{ and } C_{(N-2)}(M) \neq 0 \\ \cdots & \\ 2, & \text{if } C_{(N)}(M) = C_{(N-1)}(M) = \cdots = C_{(3)}(M) = 0 \text{ and} \\ & C_{(2)}(M) \neq 0, \\ 1, & \text{if } C_{(N)}(M) = C_{(N-1)}(M) = \cdots = C_{(2)}(M) = 0,\ M \neq 0, \\ 0, & \text{if } M = 0. \end{cases}
$$

Example

For an arbitrary normal ($M^\dagger M = MM^\dagger$) multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $n = p + q = 3$, we have

$$\mathrm{rank}(M) = \begin{cases} 4, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} \neq 0, \\ 3, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = 0 \text{ and } M\widetilde{\widehat{M}}\widehat{M} + M\widetilde{\widehat{M}}\widetilde{M} + M\widehat{M}\widetilde{M} + \widetilde{\widehat{M}}\widehat{M}\widetilde{M} \neq 0, \\ 2, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = M\widetilde{\widehat{M}}\widehat{M} + M\widetilde{\widehat{M}}\widetilde{M} + M\widehat{M}\widetilde{M} + \widetilde{\widehat{M}}\widehat{M}\widetilde{M} = 0 \text{ and} \\ & M\widetilde{\widehat{M}} + M\widehat{M} + M\widetilde{M} + \widetilde{\widehat{M}}\widehat{M} + \widetilde{\widehat{M}}\widetilde{M} + \widehat{M}\widetilde{M} \neq 0, \\ 1, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = M\widetilde{\widehat{M}}\widehat{M} + M\widetilde{\widehat{M}}\widetilde{M} + M\widehat{M}\widetilde{M} + \widetilde{\widehat{M}}\widehat{M}\widetilde{M} = \\ & = M\widetilde{\widehat{M}} + M\widehat{M} + M\widetilde{M} + \widetilde{\widehat{M}}\widehat{M} + \widetilde{\widehat{M}}\widetilde{M} + \widehat{M}\widetilde{M} = 0 \text{ and } M \neq 0, \\ 0, & \text{if } M = 0. \end{cases}$$

Example

For an arbitrary normal ($M^\dagger M = MM^\dagger$) multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $n = p + q = 4$, we have

$$\operatorname{rank}(M) = \begin{cases} 4, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} \neq 0, \\ 3, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = 0 \text{ and} \\ & M\widetilde{\widehat{M}}\widehat{M} + M\widetilde{\widehat{M}}\widetilde{M} + M(\widehat{M}\widetilde{M})^{\triangle} + \widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} \neq 0, \\ 2, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = M\widetilde{\widehat{M}}\widehat{M} + M\widetilde{\widehat{M}}\widetilde{M} + M(\widehat{M}\widetilde{M})^{\triangle} + \widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = 0 \\ & \text{and } M\widetilde{\widehat{M}} + M\widehat{M} + M\widetilde{M} + \widetilde{\widehat{M}}\widehat{M} + \widetilde{\widehat{M}}\widetilde{M} + (\widehat{M}\widetilde{M})^{\triangle} \neq 0, \\ 1, & \text{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = M\widetilde{\widehat{M}}\widehat{M} + M\widetilde{\widehat{M}}\widetilde{M} + M(\widehat{M}\widetilde{M})^{\triangle} + \widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = \\ & = M\widetilde{\widehat{M}} + M\widehat{M} + M\widetilde{M} + \widetilde{\widehat{M}}\widehat{M} + \widetilde{\widehat{M}}\widetilde{M} + (\widehat{M}\widetilde{M})^{\triangle} = 0 \text{ and } M \neq 0, \\ 0, & \text{if } M = 0. \end{cases}$$

Conclusions

- We implement the notion of rank of multivector in complexified Clifford geometric algebras without using the corresponding matrix representations. Theorem involves only operations in geometric algebras.
- We use natural implementations of SVD, determinant, and characteristic polynomial in GA without using the corresponding matrix representations.
- Note that the results of this work are valid not only for complexified Clifford geometric algebras, but also for real Clifford geometric algebras, since we can use the same matrix representations in the real case (but these matrix representations will have non-minimal dimension in this case).
- New explicit formulas for the rank in the cases of dimensions 3 and 4 can be used in various applications of geometric algebras in physics, engineering, and computer science.

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Thank you for your attention!