Inverse of a Multivector

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Definitions and notations

Let $p,q\in\mathbb{Z}_{\geq 0}, p+q=n$. Non-degenerate real Clifford algebra $\mathcal{C}\ell_{p,q}$ is the associative unital algebra generated by the generators $\{e_1,e_2,\dots,e_n\}$ satisfying:

$$e_i^2=1\ \ \forall\ \ i\in\{1,2,\dots,p\}$$
$$e_i^2=-1\ \ \forall\ \ i\in\{p+1,p+2,\dots,n\}$$
$$e_ie_j=-e_je_i\ \ \forall\ \ i\neq j\in\{1,2,\dots,n\}$$

We denote the product by

$$e_{i_1i_2\dots i_l}:=e_{i_1}\cdot e_{i_2}\cdots e_{i_l}:=e_{\{i_1,i_2,\dots,i_l\}}$$

where we assume that $1\leq i_1<i_2<\dots<i_l\leq n$.

We define $e_{\{\}}:=1$.

A general element of $\mathcal{C}\ell_{p,q}$ is called a multivector:

$$U=u\ +\sum_{1\leq i\leq n}u_ie_i\ +\sum_{1\leq i_1<i_2\leq n}u_{i_1i_2}e_{i_1i_2}\ +\dots$$
$$+\sum_{1\leq i_1<i_2<\dots<i_{n-1}\leq n}u_{i_1i_2\dots i_{n-1}}e_{i_1i_2\dots i_{n-1}}\ +\ u_{12\dots n}e_{12\dots n}$$

where $u,u_i,\dots,u_{12\dots n}\in\mathbb{R}$.

We denote the pseudoscalar by $I=e_{12\dots n}$.

We call linear combination of products of k different generators grade k multivectors. The subspace generated by them is denoted by $\mathcal{C}\ell^k_{p,q}$.

The projection of a multivector onto $\mathcal{C}\ell^k_{p,q}$ is denoted by $\langle U\rangle_k$.

The tensor product $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{C}\ell_{p,q}$ is called the complexification of $\mathcal{C}\ell_{p,q}$.

First examples

$$\mathcal{C}\ell_{0,1}\cong\mathbb{C}\qquad\mathcal{C}\ell_{0,2}\cong\mathbb{H}\qquad\mathcal{C}\ell_{1,0}\cong\mathbb{S}$$

Complex numbers and quaternions are division rings but split complex numbers are not.

In general, not every $\mathcal{C}\ell_{p,q}$ is a division ring. So,

How to find invertible elements and their inverses?

Another example, $\mathcal{C}\ell_{0,3}$ is not a division ring, $e_{12}+e_3$ is not invertible.

Matrix representations, trace, determinant

It is well known that Clifford algebras are isomorphic to matrix algebras. We will first consider the isomorphism of $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{C}\ell_{p,q}$ into matrix algebras:

$\beta:\mathbb{C}\otimes\mathcal{C}\ell_{p,q}\to M_{p,q}$, where

$$M_{p,q}:=\begin{cases}\mathrm{Mat}(2^{\frac{n}{2}},\mathbb{C}), & \text{if } n \text{ is even,}\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \text{if } n \text{ is odd.}\end{cases}$$

This representation is constructed inductively and is faithful.

The dimension of this representation is $N:=2^{\lfloor\frac{n+1}{2}\rfloor}$.

As $\mathcal{C}\ell_{p,q}\hookrightarrow\mathbb{C}\otimes\mathcal{C}\ell_{p,q}$, β gives rise to a representation of $\mathcal{C}\ell_{p,q}$ which we denote by β'.

Using this representation, we can define trace and determinant of multivectors as follows:

$$\mathrm{Tr}(U):=tr(\beta'(U)),\qquad\mathrm{Det}(U):=det(\beta'(U))$$

These are well defined in the sense that any other representation of same dimension would give same trace and determinant.

It follows that a multivector is invertible **iff** $\mathrm{Det}(U)\neq 0$.

One can use the inverse they get from the matrix representation and we are done. But this is boring and computationally challenging. We resort to better ways.

Also, one can check that $\mathrm{Tr}(U)=N\langle U\rangle_0$.

Can determinant of a multivector be expressed in purely multivector terms?

Looking at some inverses

Inverse of a vector: consider a vector

$$a=a_1e_1+a_2e_2+\dots+a_ne_n$$

then $a^2=a_1^2e_1^2+a_2^2e_2^2+\dots+a_n^2e_n^2\in\mathbb{R}$

Thus if $a^2\neq 0$, this gives $a^{-1}=\dfrac{a}{a^2}$.

Let us also look at quaternions:

as $(\mathrm{a}+\mathrm{bi}+\mathrm{cj}+\mathrm{dk})(\mathrm{a}-\mathrm{bi}-\mathrm{cj}-\mathrm{dk})$
$=\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2+\mathrm{d}^2\in\mathbb{R}$

Thus if $q\bar{q}\neq 0$, this gives $q^{-1}=\dfrac{\bar{q}}{q\bar{q}}$.

Similar idea extends to a paravector:

as $(a+\sum_{1\leq i\leq n}a_ie_i)(a-\sum_{1\leq i\leq n}a_ie_i)=a^2-\sum_{1\leq i\leq n}a_i^2e_i^2\in\mathbb{R}$

Thus if $A\widehat{A}\neq 0$, $A^{-1}=\dfrac{\widehat{A}}{A\widehat{A}}$.

Operations of conjugation

Classical operations of conjugation:

Grade involution ^: $U|_{e_i\mapsto -e_i}$

Grade reversion ~: $U|_{e_{i_1}e_{i_2}\cdots e_{i_k}\mapsto e_{i_k}e_{i_{k-1}}\cdots e_{i_1}}$

Clifford conjugation: superposition of ^ and ~.

Writing explicitly, $U=\sum_{k=0}^{n}\langle U\rangle_k$,

$$\widehat{U}=\sum_{k=0}^{n}(-1)^k\langle U\rangle_k=\sum_{k=0}^{n}(-1)^{\binom{k}{1}}\langle U\rangle_k$$

$$\widetilde{U}=\sum_{k=0}^{n}(-1)^{\frac{k(k-1)}{2}}\langle U\rangle_k=\sum_{k=0}^{n}(-1)^{\binom{k}{2}}\langle U\rangle_k$$

Aiming to continue the pattern above, we define more operations of conjugations as follows:

$$U^{\triangle_j}=\sum_{k=0}^{n}(-1)^{\binom{k}{2^{j-1}}}\langle U\rangle_k\ \text{for}\ \ 1\leq j\leq m:=[\log_2 n]+1$$

grade k	0	1	2	3	4	5	6	7
id	+	+	+	+	+	+	+	+
$\triangle_1=\widehat{\ }$	+	−	+	−	+	−	+	−
$\triangle_2=\widetilde{\ }$	+	+	−	−	+	+	−	−
$\triangle_1\triangle_2$	+	−	−	+	+	−	−	+
$\triangle_3$	+	+	+	+	−	−	−	−
$\triangle_1\triangle_3$	+	−	+	−	−	+	−	+
$\triangle_2\triangle_3$	+	+	−	−	−	−	+	+
$\triangle_1\triangle_2\triangle_3$	+	−	−	+	−	+	+	−

Note that: $\widehat{UV}=\widehat{U}\widehat{V}$ and $\widetilde{UV}=\widetilde{V}\widetilde{U}$ but in general $(UV)^{\triangle_j}\neq U^{\triangle_j}V^{\triangle_j}$ or $V^{\triangle_j}U^{\triangle_j}$.

Trace can be expressed in terms of the above operations of conjugations as:

$$\langle U\rangle_0=\frac{1}{2^m}(U+U^{\triangle_1}+U^{\triangle_2}+\dots+U^{\triangle_1\cdots\triangle_m})$$

A closer look at multiplication

1) The # swaps to get from $(a_1a_2\dots a_k)b$ to $b(a_1a_2\dots a_k)$ is k.

2) If $e_a\notin\{e_{i_1},e_{i_2},\dots,e_{i_k}\}$ then
$$(e_{i_1}e_{i_2}\dots e_{i_k})e_a=(-1)^k\ e_a(e_{i_1}e_{i_2}\dots e_{i_k})$$

3) If $e_a\in\{e_{i_1},e_{i_2},\dots,e_{i_k}\}$ then
$$(e_{i_1}e_{i_2}\dots e_{i_k})e_a=(-1)^{k-1}\ e_a(e_{i_1}e_{i_2}\dots e_{i_k})$$

4) In general, if d generators are common,
$$(e_{i_1}e_{i_2}\dots e_{i_k})(e_{j_1}e_{j_2}\dots e_{j_l})$$
$$=(-1)^{k(l-d)+(k-1)d}\ (e_{j_1}e_{j_2}\dots e_{j_l})(e_{i_1}e_{i_2}\dots e_{i_k})$$
$$=(-1)^{kl-d}(e_{j_1}e_{j_2}\dots e_{j_l})(e_{i_1}e_{i_2}\dots e_{i_k})$$

An e.g., $I(e_{i_1}e_{i_2}\dots e_{i_l})=(-1)^{(n-1)k}\ (e_{i_1}e_{i_2}\dots e_{i_l})I$

Thus,
$$\mathrm{cen}(\mathcal{C}\ell_{p,q})=\begin{cases}\mathcal{C}\ell^0_{p,q}, & \text{if } n \text{ is even,}\\ \mathcal{C}\ell^0_{p,q}\oplus\mathcal{C}\ell^n_{p,q}, & \text{if } n \text{ is odd,}\end{cases}$$

5) Another viewpoint for multiplication: let $A,B\subseteq\{1,2,\dots,n\}$, then e_A,e_B are basis elements of $\mathcal{C}\ell_{p,q}$ and $e_Ae_B=\pm e_{A\Delta B}$. Recasting 4) into this notation, we get
$$e_Ae_B=(-1)^{\binom{|A|}{2}+\binom{|B|}{2}+\binom{|A\Delta B|}{2}}e_Be_A$$

Gradings in Clifford algebra

$\mathcal{C}\ell_{p,q}$ has different gradings:

1) Superalgebra ($\mathbb{Z}_2$ grading): Let
$$\mathcal{C}\ell^+_{p,q}:=\mathcal{C}\ell^0_{p,q}\oplus\mathcal{C}\ell^2_{p,q}\oplus\mathcal{C}\ell^4_{p,q}\oplus\dots:=R_0$$
$$\mathcal{C}\ell^-_{p,q}:=\mathcal{C}\ell^1_{p,q}\oplus\mathcal{C}\ell^3_{p,q}\oplus\mathcal{C}\ell^5_{p,q}\oplus\dots:=R_1$$
then $\mathcal{C}\ell_{p,q}=\mathcal{C}\ell^+_{p,q}\oplus\mathcal{C}\ell^-_{p,q}$ and $R_iR_j\subseteq R_{i+j}\ \forall i,j\in\mathbb{Z}_2$.

2) Quaternion typification ($\mathbb{Z}_2\times\mathbb{Z}_2$ grading): Let
$$\bar{0}:=\mathcal{C}\ell^0_{p,q}\oplus\mathcal{C}\ell^4_{p,q}\oplus\mathcal{C}\ell^8_{p,q}\oplus\dots:=S_{(0,0)}:=T_{(0,1)}$$
$$\bar{1}:=\mathcal{C}\ell^1_{p,q}\oplus\mathcal{C}\ell^5_{p,q}\oplus\mathcal{C}\ell^9_{p,q}\oplus\dots:=S_{(0,1)}:=T_{(1,1)}$$
$$\bar{2}:=\mathcal{C}\ell^2_{p,q}\oplus\mathcal{C}\ell^6_{p,q}\oplus\mathcal{C}\ell^{10}_{p,q}\oplus\dots:=S_{(1,0)}:=T_{(0,0)}$$
$$\bar{3}:=\mathcal{C}\ell^3_{p,q}\oplus\mathcal{C}\ell^7_{p,q}\oplus\mathcal{C}\ell^{11}_{p,q}\oplus\dots:=S_{(1,1)}:=T_{(1,0)}$$
then $\mathcal{C}\ell_{p,q}=\bar{0}\oplus\bar{1}\oplus\bar{2}\oplus\bar{3}$ and
$$\{S_i,S_j\}\subseteq S_{i+j},\ [T_i,T_j]\subseteq T_{i+j}\quad\forall i,j\in\mathbb{Z}_2\times\mathbb{Z}_2$$

One can think of $\bar{0},\bar{1},\bar{2},\bar{3}$ as span of 1, i, j and k in the case of quaternions. Once that is done, then it becomes clear why the name 'quaternion typification' has been chosen because quaternions also have $\mathbb{Z}_2\times\mathbb{Z}_2$ grading.

$$\mathbb{E}\mathbb{A}\subseteq\mathbb{A}\qquad\forall\mathbb{A}\in\{\mathbb{I},\mathbb{J},\mathbb{K}\}$$
$$\mathbb{J}\mathbb{K},\mathbb{K}\mathbb{J}\subseteq\mathbb{I}\ ,\quad\mathbb{K}\mathbb{I},\mathbb{I}\mathbb{K}\subseteq\mathbb{J}\ ,\quad\mathbb{I}\mathbb{J},\mathbb{J}\mathbb{I}\subseteq\mathbb{K}$$

Norm functions

Express how operations of conjugation and quaternion typification mingle with each other.

We call function $N:\mathcal{C}\ell_{p,q}\to\mathbb{R}$ satisfying
$N(U)=Uf(U)$ for some function $f:\mathcal{C}\ell_{p,q}\to\mathcal{C}\ell_{p,q}$

It is then clear that if for some U, $N(U)\neq 0$ then we immediately have an inverse: $U^{-1}=\dfrac{f(U)}{N(U)}$.

We have the following norm functions:

$N(U)=U\widehat{U},\quad n=1;$

$N(U)=U\widehat{\widetilde{U}},\quad n=2;$

$N(U)=U\widetilde{U}\widehat{U}\widehat{\widetilde{U}},\quad n=3;$

$N(U)=U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle},\quad n=4;$ Note: $(\triangle_3=\triangle)$

$N(U)=U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle})^{\triangle},\quad n=5$

$$N(U)=U\Big(\frac{1}{3}\widetilde{U}\widehat{U}\widehat{\widetilde{U}}(\widehat{U}\widehat{\widetilde{U}}U\widetilde{U})^{\triangle}$$
$$+\frac{2}{3}\widetilde{U}((\widehat{U}\widehat{\widetilde{U}})^{\triangle}((\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U})^{\triangle})^{\triangle})^{\triangle}\Big),\quad n=6.$$

These functions coincide with the determinant of a multivector defined previously. Also, they can be expressed differently i.e., there are multiple formulas for these norm functions. One reason why that is because

$$\mathrm{Det}(U)=\mathrm{Det}(\widehat{U})=\mathrm{Det}(\widetilde{U})$$

Proof: Start by noting that since $\widehat{\widetilde{(U\widehat{\widetilde{U}})}}=U\widehat{\widetilde{U}}$ and $n=2$ $\widetilde{(U\widetilde{U})}=U\widetilde{U}$, $U\widehat{\widetilde{U}}\in\bar{0}\oplus\bar{3}$ and $U\widetilde{U}\in\bar{0}\oplus\bar{1}$.

As for $n=1$, $\widehat{U}=\widehat{\widetilde{U}}$, and $\bar{0}\oplus\bar{1}=\bar{0}\oplus\bar{3}=\mathbb{R}$ for $n\leq 2$, we are done.

For $n=3$, $U\widetilde{U}=H_0+H_3$ and thus $U\widetilde{U}\widehat{U}\widehat{\widetilde{U}}\in\mathbb{R}$ and we are done.

For $n=4$,
$U\widetilde{U}=H_0+H_1+H_4$ and $(\widehat{U}\widehat{\widetilde{U}})^{\triangle}=H_0-H_1-H_4$. Now,

$$U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}=(H_0+H_1+H_4)(H_0-H_1-H_4)$$
$$=H_0^2-(H_1+H_4)^2-[H_0,H_1+H_4]$$
$$=H_0^2-H_1^2-H_4^2-\{H_1,H_4\}$$
$$=H_0^2-H_1^2-H_4^2\in\mathbb{R}\quad\text{and we are done.}$$

For $n=5$, $U\widetilde{U}=H_0+H_1+H_4+H_5$
$$(\widehat{U}\widehat{\widetilde{U}})^{\triangle}=H_0-H_1-H_4+H_5$$
$$U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}=(H_0+H_1+H_4+H_5)(H_0-H_1-H_4+H_5)$$
$$=(H_0+H_5)^2-(H_1+H_4)^2-[H_0+H_5,H_1+H_4]$$
$$=H_0^2+H_5^2+2H_0H_5-H_1^2-H_4^2-\{H_1,H_4\}$$
$$=Y_0+Y_5$$

Thus, $U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle}(U\widetilde{U}(\widehat{U}\widehat{\widetilde{U}})^{\triangle})^{\triangle}\in\mathbb{R}$ and we are done.

No theoretical proof is known for $n=6$ case.

Subalgebras like $\mathrm{Span}_{\mathbb{R}}\{1,e_{1256},e_{2345},e_{1346}\}$ cause trouble.