On Multidimensional Dirac–Hestenes Equation

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Applied Geometric Algebras in Computer Science and Engineering (AGACSE)

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Matrix formalism

Pseudo-Euclidean space $\mathbb{R}^{1,n}$

Notations:

- $\{x^\mu\}_{\mu=0}^n$ are Cartesian coordinates;
- $\partial_\mu = \partial/\partial x^\mu$ is a partial derivative.

Matrices $\{\gamma^\mu\}_{\mu=0}^n$

The metric tensor $\mathbb{R}^{1,n}$ is given by the diagonal matrix η:

$$\eta = (\eta^{\mu\nu})_{\mu,\nu=0}^n = \mathrm{diag}(1, -1, -1, \dots, -1). \tag{1}$$

Let $\{\gamma^\mu\}_\mu$ be matrices that satisfy the following anticommutation relations:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{I}, \quad \mu, \nu \in \{0, 1, \dots, n\}, \tag{2}$$

where $\mathbb{I}$ is the identity matrix of size $2^{[(n+1)/2]}$.

Matrix formalism

Dirac equation

Multidimensional Dirac equation for a particle in an electromagnetic field has the form:

$$\sum_{\mu=0}^{n} i\gamma^{\mu}(\partial_{\mu} + ia_{\mu}(x))\varphi(x) = m\varphi(x).$$

Notations

- i is the imaginary unit;
- m is the mass of a particle;
- $\mathbf{a}(x) = (a_0(x), \ldots, a_n(x)) : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ is the electromagnetic vector-potential;
- $\varphi(x) : \mathbb{R}^{1,n} \to \mathbb{C}^{n+1}$ is a Dirac spinor.

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Real geometric algebra $\mathcal{G}_{1,n}$

Generators

The generators $e^0, e^1, \ldots, e^n$ satisfy the following anticommutation relations:

$$e^\mu e^\nu + e^\nu e^\mu = 2\eta^{\mu\nu} e, \quad \mu, \nu \in \{0, 1, \ldots, n\}, \tag{3}$$

where η is diagonal matrix (1) and e is the identity element.

Note: relations (2) and (3) are similar

Basis

The basis of $\mathcal{G}_{1,n}$ consists of all possible ordered products of the generators:

$$e^{\mu_1} e^{\mu_2} \cdots e^{\mu_k} = e^{\mu_1 \mu_2 \cdots \mu_k}, \quad 0 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq n.$$

We denote multi-index by M:

$$M = \mu_1 \mu_2 \ldots \mu_k, \quad |M| = k.$$

Basis decomposition

Real geometric algebra $\mathcal{G}_{1,n}$

The basis decomposition of an element $U \in \mathcal{G}_{1,n}$ is:

$$U = \sum_M u_M e^M, \quad u_M \in \mathbb{R}.$$

The dimension of $\mathcal{G}_{1,n}$:

$$\dim \mathcal{G}_{1,n} = 2^{n+1}.$$

Complexified geometric algebra $\mathcal{G}_{1,n}$

The basis decomposition of an element $U \in \mathbb{C} \otimes \mathcal{G}_{1,n}$ is:

$$U = \sum_M u_M e^M, \quad u_M \in \mathbb{C}.$$

The dimension of $\mathbb{C} \otimes \mathcal{G}_{1,n}$:

$$\dim \mathbb{C} \otimes \mathcal{G}_{1,n} = 2^{n+2}$$

Even and odd elements

Even subalgebra

Let $\mathcal{G}^{(0)}_{1,n}$ be an even subalgebra of $\mathcal{G}_{1,n}$:

$$\mathcal{G}^{(0)}_{1,n} = \{U \in \mathcal{G}_{1,n} | U = \sum_{|M|=2k} u_M e^M\}, \quad \dim \mathcal{G}^{(0)}_{1,n} = 2^n.$$

An element of the even subalgebra $\mathcal{G}^{(0)}_{1,n}$ is called an even element.

Odd subspace

Let $\mathcal{G}^{(1)}_{1,n}$ be an odd subspace of $\mathcal{G}_{1,n}$:

$$\mathcal{G}^{(1)}_{1,n} = \{U \in \mathcal{G}_{1,n} | U = \sum_{|M|=2k-1} u_M e^M\}, \quad \dim \mathcal{G}^{(1)}_{1,n} = 2^n.$$

An element of the odd subspace $\mathcal{G}^{(1)}_{1,n}$ is called an odd element.

Left ideal

Hermitian conjugation

In $\mathbb{C} \otimes \mathcal{G}_{1,n}$, the operation has the form:

$$U^\dagger = e^0 U^* e^0,$$

where the star denotes the superposition of reversion and complex conjugation:

$$U^* = \sum_M (-1)^{\frac{|M|(|M|-1)}{2}} \bar{u}_M e^M, \quad u_M \in \mathbb{C}$$

Idempotent

An element $t \in \mathbb{C} \otimes \mathcal{G}_{1,n}$ is called a Hermitian idempotent if:

$$t^2 = t, \quad t^\dagger = t.$$

The corresponding left ideal $L(t)$ generated by t:

$$L(t) = \{U \in \mathbb{C} \otimes \mathcal{G}_{1,n} | Ut = U\}.$$

Primitive Hermitian idempotent

Definition

If a left ideal $L(t)$ does not contain another left ideal except itself and $L(0)$, then it is called a **minimal left ideal**.

The corresponding Hermitian idempotent t is called the **primitive Hermitian idempotent**.

Idempotent in $\mathbb{C} \otimes \mathcal{G}_{1,n}$

Case $n = 3$:

$$t = \frac{1}{4}(e + e^0)(e + ie^{12}) \in \mathbb{C} \otimes \mathcal{G}_{1,3}. \tag{4}$$

Multi-dimensional case:

$$t = \frac{1}{2}(e + e^0) \prod_{\mu=1}^{[\frac{n+1}{2}]-1} \frac{1}{2}(e + ie^{2\mu-1}e^{2\mu}) \in \mathbb{C} \otimes \mathcal{G}_{1,n}. \tag{5}$$

Spinors in different cases

Matrix representation

$$\mathbb{C} \otimes \mathcal{G}_{1,n} \simeq \begin{cases} \mathrm{Mat}(2^d, \mathbb{C}), & n = 2d - 1 \\ \mathrm{Mat}(2^d, \mathbb{C}) \oplus \mathrm{Mat}(2^d, \mathbb{C}), & n = 2d \end{cases}$$

Dirac spinor

A Dirac spinor is an element of the left ideal $L(t)$:

$$\varphi(x) : \mathbb{R}^{1,n} \to L(t).$$

Cases of $\varphi(x)$:

1. Spinor: $n = 2d - 1$; (5) — a **primitive** Hermitian idempotent
2. Semi-spinor: $n = 2d$; t — a **primitive** Hermitian idempotent
3. Double spinor: $n = 2d$; (5) — a Hermitian idempotent

Matrix representation of spinors

Case $n = 2d - 1$; Spinor

$$t = \left(\begin{array}{c|c} \begin{matrix} 1 \\ 0 \\ 0 \\ \dots \end{matrix} & \Huge 0 \end{array}\right)$$

Case $n = 2d$

Semi-spinor	Double spinor
$t = \left(\begin{array}{ccc\|c} 1 & 0 & \dots & \\ 0 & 0 & \dots & 0 \\ & \dots & & \\ \hline & 0 & & 0 \end{array}\right)$	$t = \left(\begin{array}{ccc\|ccc} 1 & 0 & \dots & & & \\ 0 & 0 & \dots & & 0 & \\ & \dots & & & & \\ \hline & & & 1 & 0 & \dots \\ & 0 & & 0 & 0 & \dots \\ & & & & \dots & \end{array}\right)$

Geometric algebra formalism

Dirac equation

Dirac equation for a particle in an electromagnetic field has the form:

$$\sum_{\mu=0}^{n} ie^{\mu}(\partial_{\mu} + ia_{\mu}(x))\varphi(x) = m\varphi(x), \qquad (6)$$

where:

- e^{μ} is the generator of $\mathcal{G}_{1,n}$;
- m is the mass of a particle;
- $\mathbf{a}(x) = (a_0(x), \dots, a_n(x)) : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ is the electromagnetic vector-potential.

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Dirac–Hestenes equation

Lemma on unique decomposition

$$\forall \varphi \in L(t) \quad \exists ! \Psi \in \mathcal{G}^{(0)}_{1,3} : \quad \varphi = \Psi t.$$

The real dimensions of the minimal left ideal $L(t)$ that is generated by (4) and the even real subalgebra $\mathcal{G}^{(0)}_{1,3}$ are the same:

$$\dim L(t) = \dim \mathcal{G}^{(0)}_{1,3} = 8.$$

Dirac–Hestenes equation

$$\sum_{\mu=0}^{3} e^{\mu}(\partial_{\mu}\Psi(x) + \Psi(x)a_{\mu}(x)I)E + m\Psi(x)I = 0, \quad \Psi(x) \in \mathcal{G}^{(0)}_{1,3}.$$

$$I = -e^{12}, \quad E = e^{0}$$

Question

Will the Dirac–Hestenes equation be similar in multidimensional case?

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Answer: No

1^{st} reason: dimensions

If t is fixed as (5), then:

$$\dim L(t) = 2^{[(n+2)/2]} + 1, \quad \dim \mathcal{G}^{(0)}_{1,n} = 2^n.$$

$$[\frac{n+2}{2}] + 1 < n, \quad \forall n > 4$$

2^{nd} reason: lemma on unique decomposition

- Case $n = 3$:
  If $Y \in \mathcal{G}^{(0)}_{1,3}$ and $Yt = 0$, then $Y = 0$.
- Case $n = 5$:
  If $Y = e^{12} - e^{34} \in \mathcal{G}^{(0)}_{1,5}$ and t is fixed as (5), then $Yt = -it + it = 0$.

Theorem in case $n = 2d - 1$

Conditions

Let fix the primitive Hermitian idempotent t as in formula (5) and the real algebra Q' be:

$$Q' = \mathcal{G}(e^0, e^1, e^2, e^3, e^5, e^7, \dots, e^{2d-1}) \subset \mathcal{G}_{1,2d-1}.$$

Dirac–Hestenes equation

$$\sum_{\mu=0,1,2,3,5,7,\dots,2d-1} e^\mu(\partial_\mu \Psi(x) + \Psi(x) a_\mu(x) I) E +$$

$$+ \sum_{\mu=3,5,\dots,2d-3} (\partial_{\mu+1}\Psi(x) I - \Psi(x) a_{\mu+1}(x)) e^\mu E + m\Psi(x) I = 0, \qquad (7)$$

where $I = -e^{12}$ and $E = e^0$.

$\varphi(x) \in L(t)$ is a solution of multidimensional Dirac equation (6)

$\Updownarrow$

$\Psi(x) \in Q'^{(0)} : \ \varphi(x) = \Psi(x)t$ is a solution of multidimensional Dirac–Hestenes equation (7)

Theorem in case $n = 2d$. Semi-spinor

Conditions

Let fix the primitive Hermitian idempotent t as:

$$t = \frac{1}{2}(e + e^0) \prod_{\mu=1}^{[\frac{2d+1}{2}]} \frac{1}{2}(e + ie^{2\mu-1}e^{2\mu}) \in \mathbb{C} \otimes \mathcal{G}_{1,2d}. \tag{8}$$

and the real algebra Q' be:

$$Q' = \mathcal{G}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1}) \subset \mathcal{G}_{1,2d}.$$

Dirac–Hestenes equation

$$\sum_{\mu=0,1,2,3,5,7,\ldots,2d-1} e^\mu(\partial_\mu \Psi(x) + \Psi(x) a_\mu(x) I)E +$$

$$+ \sum_{\mu=3,5,\ldots,2d-3,2d-1} (\partial_{\mu+1}\Psi(x) I - \Psi(x) a_{\mu+1}(x)) e^\mu E + m\Psi(x) I = 0. \tag{9}$$

Theorem in case $n = 2d$. Semi-spinor

$\varphi(x) \in L(t)$ is a solution (semi-spinor) of multidimensional Dirac equation (6)

$$\Updownarrow$$

$\Psi(x) \in Q'^{(0)} : \varphi(x) = \Psi(x)t$ is a solution of the multidimensional Dirac–Hestenes equation (9)

Theorem in case $n = 2d$. Double spinor

Conditions

Let fix the Hermitian idempotent t as in formula (5) and the real algebra Q' be:

$$Q' = \mathcal{G}(e^0, e^1, e^2, e^3, e^5, e^7, \dots, e^{2d-3}, e^{2d-1}, e^{2d}) \subset \mathcal{G}_{1,2d}.$$

Dirac–Hestenes equation

$$\sum_{\mu=0,1,2,3,5,7,\dots,2d-1,2d} e^\mu(\partial_\mu\Psi(x) + \Psi(x)a_\mu(x)I)E +$$

$$+ \sum_{\mu=3,5,\dots,2d-3} (\partial_{\mu+1}\Psi(x)I - \Psi(x)a_{\mu+1}(x))e^\mu E + m\Psi(x)I = 0. \quad (10)$$

$\varphi(x) \in L(t)$ is a solution (double spinor) of multidimensional Dirac equation (6)

$$\Updownarrow$$

$\Psi(x) \in Q'^{(0)}$: $\varphi(x) = \Psi(x)t$ is a solution of the multidimensional Dirac–Hestenes equation (10)

Dirac–Hestenes equation for graphene

Matrix formalism

$$\varphi(x) = \begin{pmatrix} \varphi_{A,1}(x) + i\varphi_{A,2}(x) \\ \varphi_{B,1}(x) + i\varphi_{B,2}(x) \end{pmatrix} \in \mathbb{C}^2 \Rightarrow \varphi(x) \text{ is a semi-spinor}$$

Geometric algebra formalism

We consider Dirac equation (6) in $\mathbb{C} \otimes \mathcal{G}_{1,2}$. Let us fix the idempotent t as (8):

$$t = \frac{1}{4}(e + e^0)(e + ie^1e^2) \in \mathbb{C} \otimes \mathcal{G}_{1,2}.$$

The real algebra Q' is:

$$Q' = \mathcal{G}(e^0, e^1, e^2) = \mathcal{G}_{1,2}.$$

The Dirac–Hestenes equation	The explicit form of a solution
$\sum_{\mu=0}^{3} e^\mu(\partial_\mu \Psi + \Psi a_\mu I)E + m\Psi I = 0.$	$\Psi = 2\left(\varphi_{A,1}e + \varphi_{A,2}I\right) + 2\left(\varphi_{B,1}e^{01} + \varphi_{B,2}e^{01}I\right) \in \mathcal{G}^{(0)}_{1,2}.$

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Conclusions

Advantage of considering Dirac–Hestenes equation

The Dirac–Hestenes equation gives a deeper understanding of geometry in various tasks, since the considering wave function is entirely real.

Need to consider auxiliary algebra Q'

$$\dim L(t) \neq \dim \mathcal{G}^{(0)}_{1,n}.$$

Dependence on parity of n

Cases:

1. n is odd (spinor);
2. n is even (semi-spinor);
3. n is even (double spinor).

Thank you for your attention!

On Multidimensional Dirac–Hestenes Equation

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