# STAResNet

## A Network in Spacetime Algebra to solve Maxwell's PDEs

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**Applied Geometric Algebras in Computer Science and Engineering (AGACSE)**
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# Outline

# Table of Contents

# Motivation

**Geometric Algebra (GA) Networks** have been gaining significant momentum in the past two years[1].

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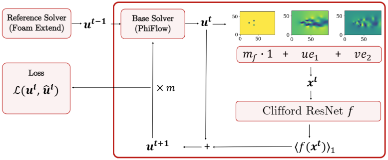


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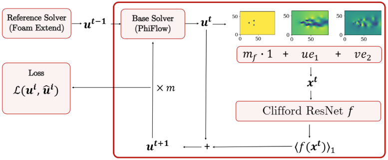


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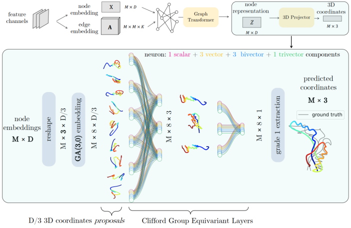


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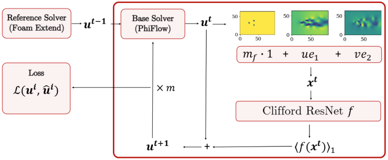


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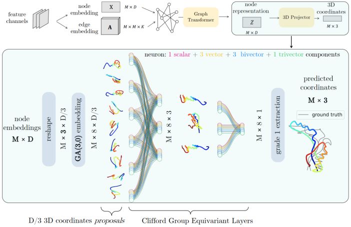


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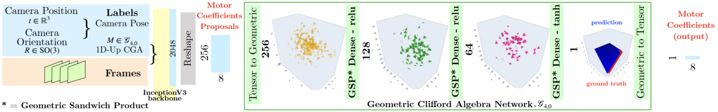


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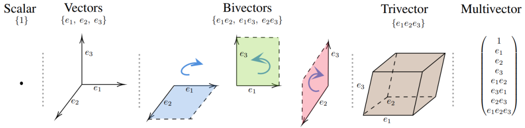
A key step in GA Networks is the **embedding** in a given algebra

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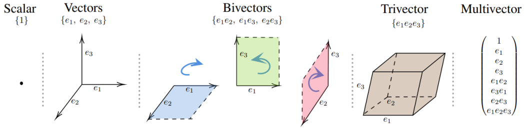


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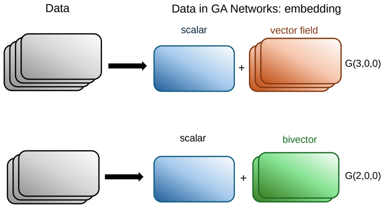


Figure 3: Two examples of GA embedding

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- First implementation of a network entirely in STA

# Table of Contents

# Problem Definition: Describing Maxwell's PDEs

## Maxwell's Equations in Differential Form

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$$

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$$\nabla \mathbf{F} = J, \text{ with}$$
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$$J = (\rho - \mathbf{J})\gamma_0$$
$$\nabla = \gamma^i \frac{\partial}{\partial x_i}$$

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  *"This led to the development of a new method for calculating EM response of conductors to incoming plane waves [...] it was possible to change the illumination in real time"*

# Problem Definition: Architecture

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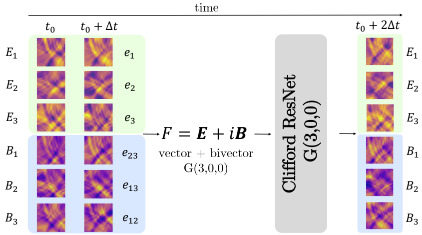


(a) Clifford ResNet (GA) [1]

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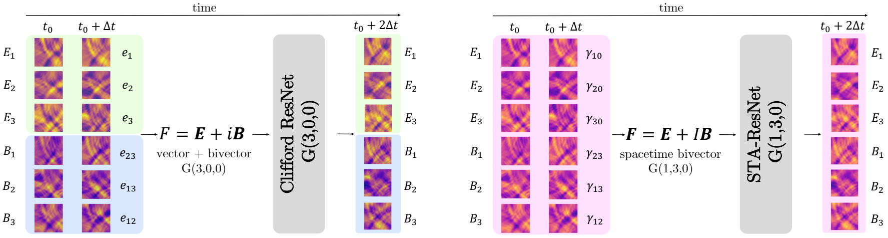


(a) Clifford ResNet (GA) [1]  (b) STAResNet (STA) [ours]

Figure 4: The two approaches to the solution of Maxwell's PDEs

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- Rollout error

# Metrics

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- Mean squared error

$$\mathcal{L} = \frac{1}{LMN} \sum_{j+x,y,z} \sum_{l=0}^{L} \sum_{m=0}^{M} \sum_{n=0}^{N} (E_{jlmn,i+2\Delta t} - \hat{E}_{jlmn,i+2\Delta t})^2 + (B_{jlmn,i+2\Delta t} - \hat{B}_{jlmn,i+2\Delta t})^2$$

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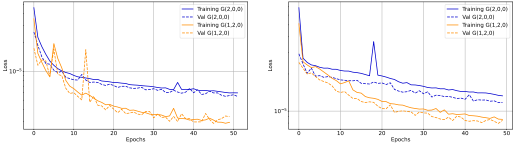
- Structural Similarity Index Measure

$$\text{SSIM}(\mathbf{F}^2, \hat{\mathbf{F}^2}) = \frac{(2\mu_{\mathbf{F}^2}\mu_{\hat{\mathbf{F}^2}} + C_1)(2\sigma_{\mathbf{F}^2\hat{\mathbf{F}^2}} + C_2)}{(\mu_{\mathbf{F}^2}^2 + \mu_{\hat{\mathbf{F}^2}}^2 + C_1)(\sigma_{\mathbf{F}^2}^2 + \sigma_{\hat{\mathbf{F}^2}}^2 + C_2)}$$

# Experiments: 2D fields, varying $\Delta t$




(a) $\Delta t = 25$s

(b) $\Delta t = 50$s

Figure 5: Training and validation losses versus number of epochs for 2D Maxwell's PDEs.

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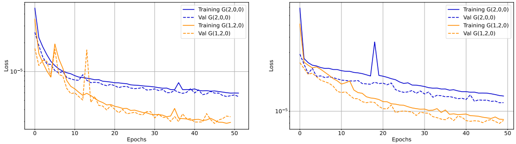


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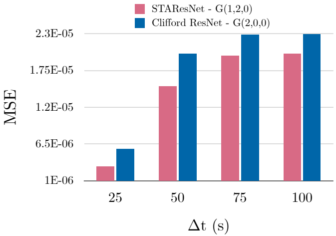


Figure 6: MSE between estimated and GT EM fields vs $\Delta t$.

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$$\mathbf{F}^2 = (\mathbf{E} + I\mathbf{B})^2 = (E_1\gamma_{10} + E_2\gamma_{20} + E_3\gamma_{30} + B_1\gamma_{13} + B_2\gamma_{13} + B_3\gamma_{12})^2$$

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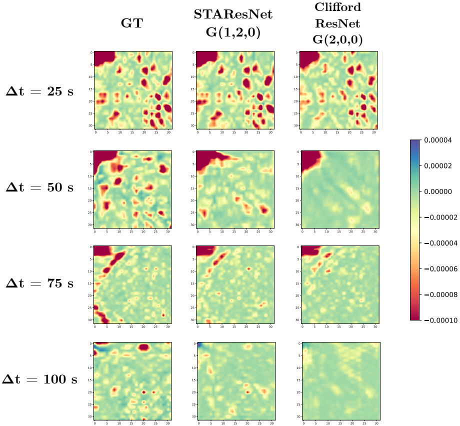


Figure 7: Squared magnitude of the Faraday bivector $\mathbf{F}^2$ for varying $\Delta t$.

# Experiments: impact of obstacles

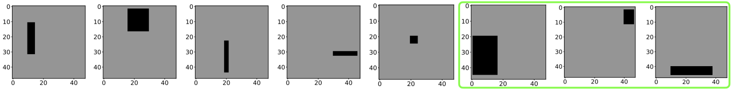


Figure 8: The 5 different obstacle configurations. The 3 unseen geometries are highlighted.

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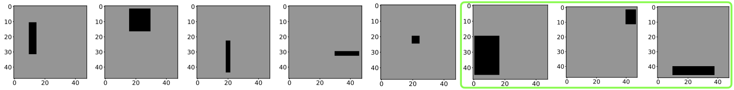

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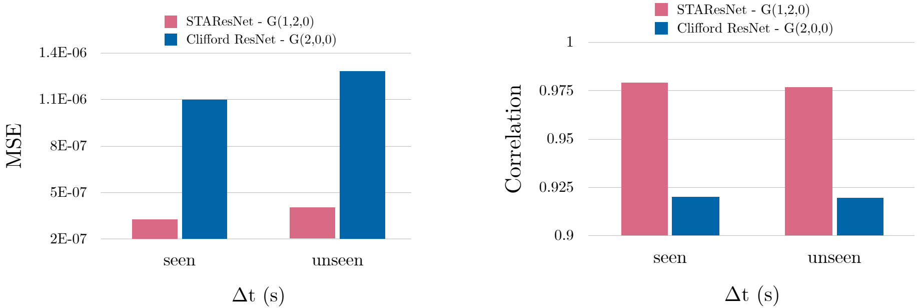


(a) MSE (↓). (b) Correlation (↑).

Figure 9: (a) MSE and (b) correlation between estimated and GT fields with obstacles.

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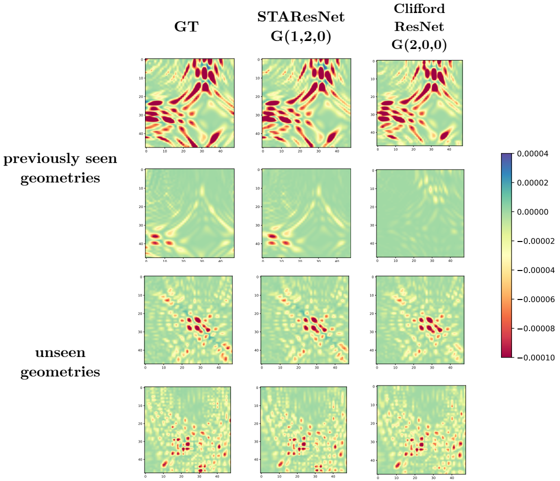


Figure 10: Squared magnitude of the Faraday bivector $\mathbf{F}^2$ over the test set with seen and unseen obstacles configurations.

# Experiments: impact of trainable parameters

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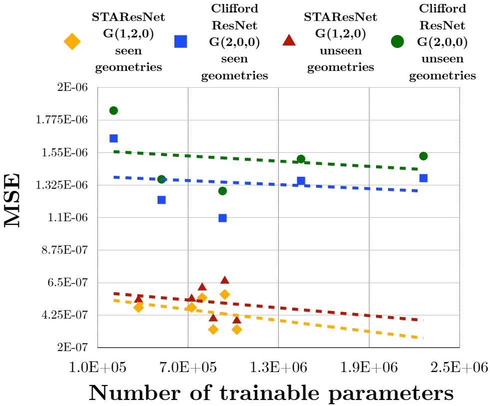


Figure 11: Test error over the estimated EM fields in the presence of seen and unseen obstacle geometries versus the number of trainable parameters.

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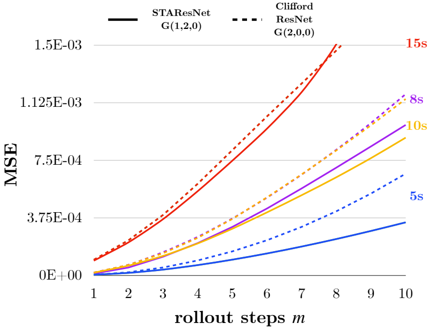
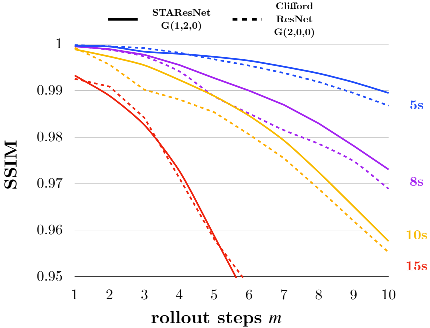


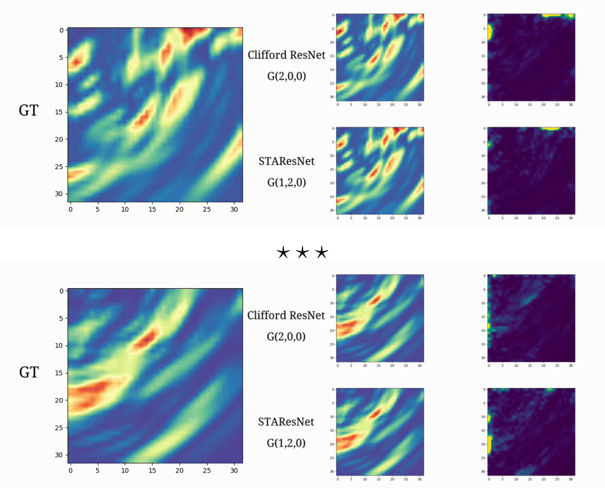

(a) MSE (↓) versus rollout steps.




(b) SSIM (↑) versus rollout steps.

Figure 12: (a) Mean squared error and (b) correlation between estimated and ground truth EM fields over test set versus rollout steps $m$ for the 2D case.

# Experiments: rollout error - visualizing $\mathbf{F}^2$



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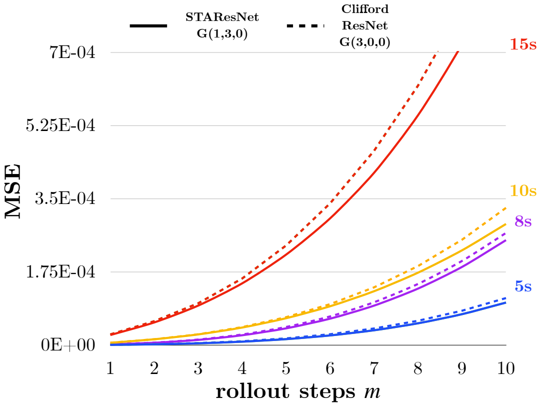


Figure 13: MSE between estimated and GT EM fields vs rollout steps $m$ for the 3D case.

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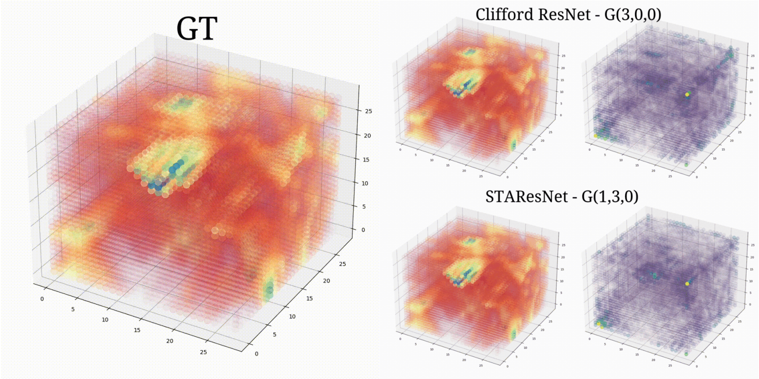

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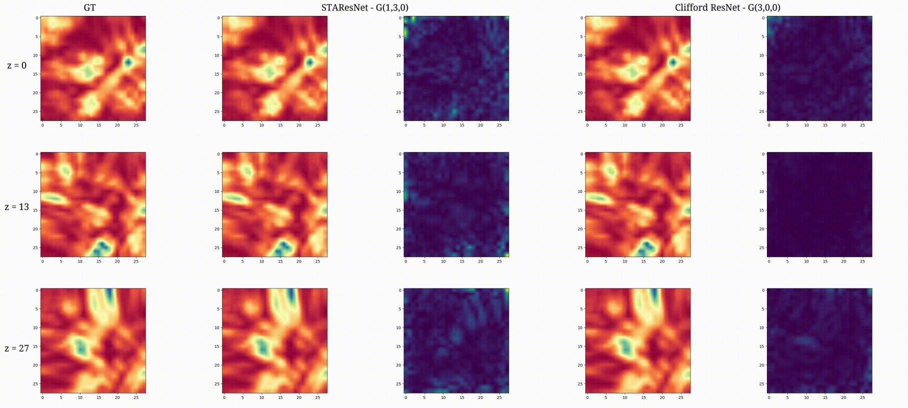

# Table of Contents

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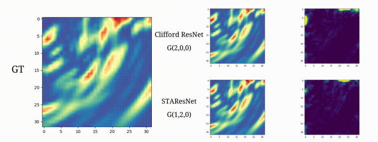


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paper

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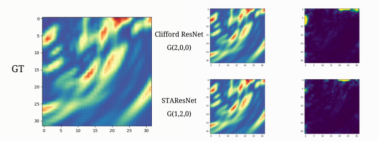


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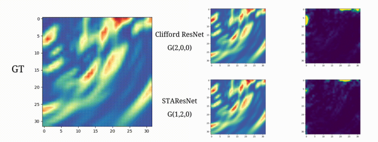


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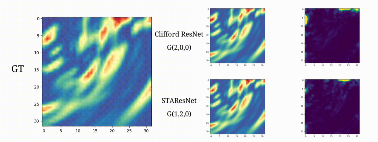


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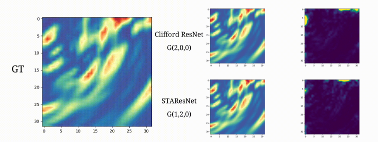


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- Growing interest in industry: Huawei, Microsoft, ...

**Acknowledgements:** Joan Lasenby, Sven Buchholz, Christian Hockey, David Bowie