Symmetries of the Boundary Theorem and Electrodynamics

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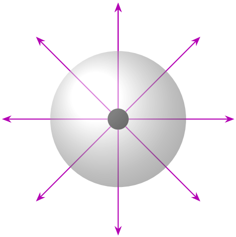
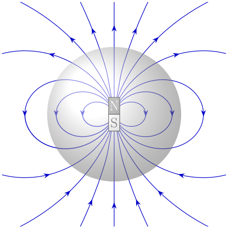
Integral Form of Maxwell's Equations

$$\oiint_{\partial N} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_N \rho \, \mathrm{d}V$$

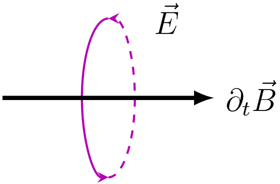


$$\oiint_{\partial N} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

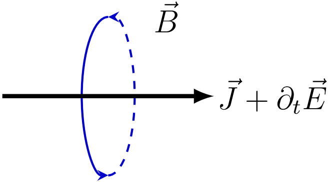


Integral Form of Maxwell's Equations

$$\oint_{\partial M} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_M \mathbf{B} \cdot \mathrm{d}\mathbf{S},$$



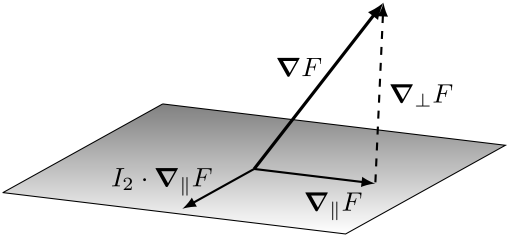
$$\oint_{\partial M} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_M \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_M \mathbf{E} \cdot \mathrm{d}\mathbf{S},$$



The GA Vector Derivative

$$\boldsymbol{\nabla} F = \boldsymbol{\nabla}_{\|} F + \boldsymbol{\nabla}_{\perp} F$$

$$\boldsymbol{\partial} F = \boldsymbol{\nabla}_{\|} F = I_m^{-1}(I_m \cdot \boldsymbol{\nabla} F)$$



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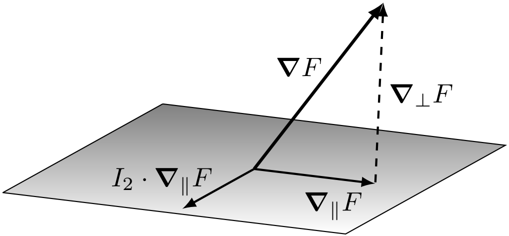
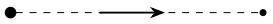


For the special case of $m = n$ for M^n in $\mathbb{R}^n$, then $\boldsymbol{\nabla}_{\perp} = 0$ and thus $\boldsymbol{\partial} = \boldsymbol{\nabla}$.

The Boundary Theorem in Geometric Calculus

$$\oint_{\partial M} d\mathbf{x}^{m-1}\, F = \int_M \mathrm{d}\mathbf{x}^m\, \boldsymbol{\partial} F$$
$$= \int_M \mathrm{d}\mathbf{x}^m\, (\boldsymbol{\partial} \cdot F + \boldsymbol{\partial} \wedge F)$$

The Boundary Theorem in GA, 1D and 2D

$dim(F) = dim(\partial\mathcal{M})$	$\oint_{\partial\mathcal{M}} d\mathbf{x}^{n-1} \cdot F = \int_{\mathcal{M}} \mathrm{d}\mathbf{x}^{n} \cdot (\nabla \wedge F)$	$\oint_{\partial\mathcal{M}} d\mathbf{x}^{n-1} \wedge F = \int_{\mathcal{M}} \mathrm{d}\mathbf{x}^{n} \wedge (\nabla \cdot F)$
0		Undefined

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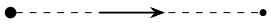
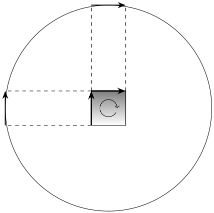
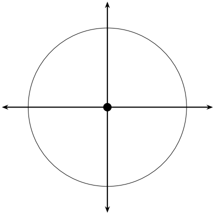
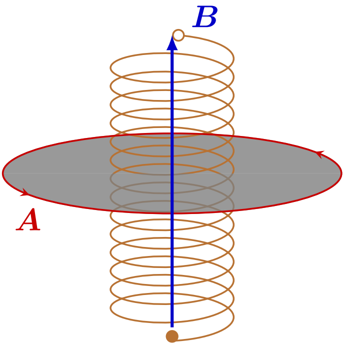
0		Undefined
1		

The Magnetic Vector Potential (MVP)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = -I\boldsymbol{\nabla} \wedge \boldsymbol{A}$$



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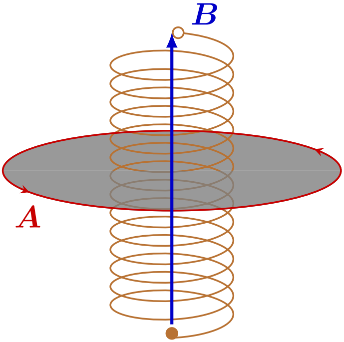
$$\oint_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{x}^1 = \iint_S \boldsymbol{\nabla} \wedge \boldsymbol{A} \cdot d\boldsymbol{x}^2$$



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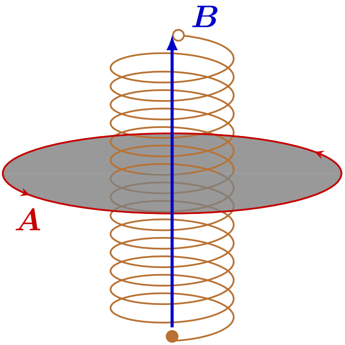

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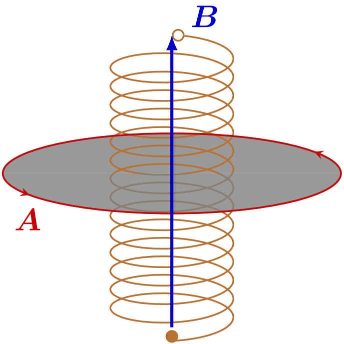

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$$\oint_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{x}^1 = -\frac{d}{dt} \iint_S \widehat{\boldsymbol{B}} \cdot d\boldsymbol{x}^2$$

since

$$\boldsymbol{E} = -\frac{d\boldsymbol{A}}{dt}.$$

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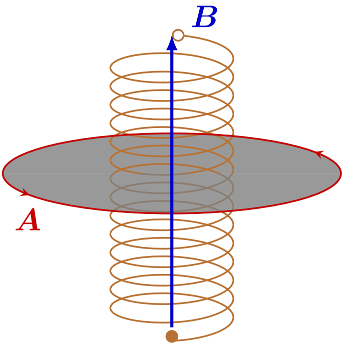

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since

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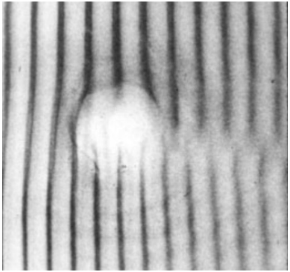
Equivalently,

$$\frac{\boldsymbol{F}}{q} = -\frac{d}{dt}\frac{\boldsymbol{p}}{q}.$$

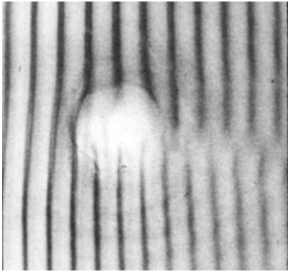
Hydrodynamics Analogy



[a]

[a] M. V. Berry, R. G. Chambers, M. D. Large, C. Upstill, and J. C. Walmsley, European Journal of Physics 1, 154–162 (1980).

Hydrodynamics Analogy



a

Electromagnetism	**Hydrodynamics**
magnetic vector potential [$\mathbf{A}$: ML/TQ]	velocity [$\mathbf{v}$: L/T]
magnetic field [$\mathbf{B}$: M/TQ]	vorticity [$\boldsymbol{\omega}$: 1/T]
electric field [$\mathbf{E}$: ML/T^{2}Q]	acceleration [$\mathbf{L}$: L/T^2]
electric scalar potential [ϕ : ML^2/T^{2}Q]	kinematic pressure [ϕ : ML^2/T^{2}Q]
phase function [χ: ML^2/TQ]	velocity potential [Φ: L^2/T]
charge [q: Q]	mass [m: M]
charge density [ρ_q: Q/L^3]	fluid density [ρ_f: M/L^3]
current density [$\mathbf{J}$: Q/TL^2]	mass flux [$\mathbf{j}_m$: M/TL^2]

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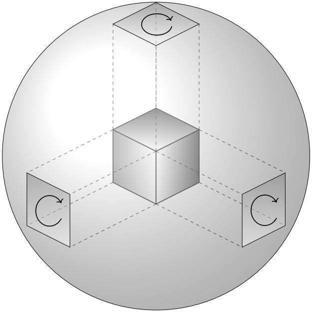
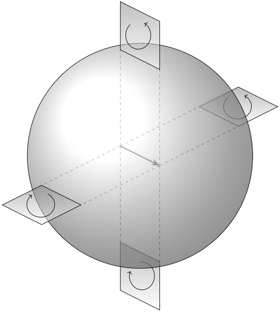
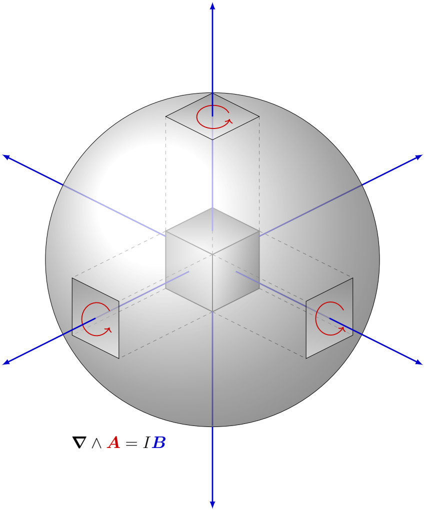
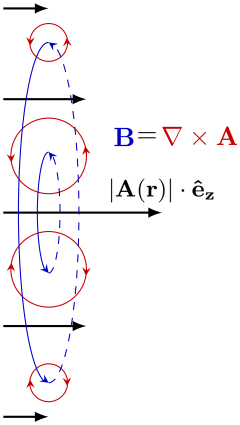
The Boundary Theorem in GA, 3D

$dim(F) = dim(\partial\mathcal{M})$	$\oiint_{\partial M} d\mathbf{x}^2 \cdot F = \iiint_M d\mathbf{x}^3 \cdot (\boldsymbol{\nabla} \wedge F)$	$\oiint_{\partial M} d\mathbf{x}^2 \times F = \iiint_M d\mathbf{x}^3 \cdot (\boldsymbol{\nabla} \cdot F)$
2		

MVP of a Magnetic Monopole



MVP of a Current Element - Ring Vortex



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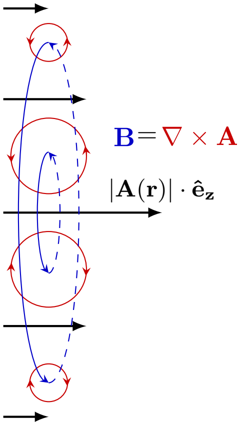

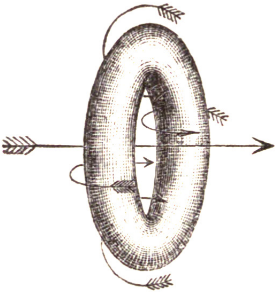

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[a] P. G. Tait, Lectures on some recent advances in physical science (London, Macmillan and co., 1876).

Summary

- The integral form of Maxwell's equations was compared to corresponding instances of the boundary theorem in geometric calculus.

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- A more direct correspondence shown to be obtainable by considering the electromagnetic potential.
- This naturally leads to consideration of the boundary theorem for more complex gauge symmetries and internal degrees of freedom, worthy of further investigation.

GA PostDoc opportunity in Iceland: Effective gauge theory of spin dynamics

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