Notes on GA in geometric control theory

Conformal symmetries in control of vertical rolling disc

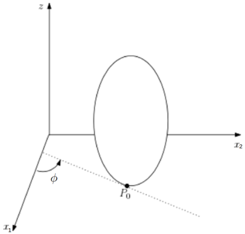
Jaroslav Hrdina

AGACSE 2024 - Amsterdam
August 27-29

BRNO UNIVERSITY OF TECHNOLOGY

Controlability of vertical rolling disc

Let the considered mechanism be the homogeneous rolling disc of mass $m = 1$ and radius $r = 1$ in the plane. Assume that the disc moves without tilting and slipping



Obviously, we got two vector fields

$$X_1 = \cos\phi \partial_{x_1} + \sin\phi \partial_{x_2}, \quad X_2 = \partial_\phi,$$

where $(x_1, x_2) \in \mathbb{R}^2,\ \theta \in \mathbb{S}^1$ which forms the first-order control system

$$\dot{q} = u_1 X_1 + u_2 X_2. \tag{1}$$

Controlability of vertical rolling disc

Lie bracked shows the remaining vector field

$$X_3 = [X_1, X_2] = \sin\phi\partial_{x_1} - \cos\phi\partial_{x_2},$$

which completes the algebra. In these coordinates Lie algebra of controllability $\mathfrak{g}$ is equipped with the following multiplication table

$\mathfrak{g}$	X_1	X_2	X_{12}
X_1	0	X_{12}	0
X_2	$-X_{12}$	0	X_1
X_{12}	0	$-X_1$	0

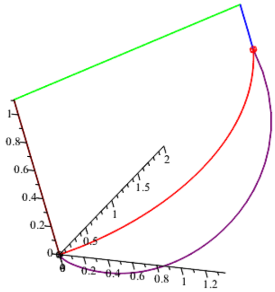
Table: The multiplication table of Lie algebra of controlabilty of vertical rolling disc

Controlability of vertical rolling disc

Thanks to the Chow-Rashevskii theorem, the disk is controllable on the complete configuration space $\mathbb{R}^2 \times \mathbb{S}^1$

$$\begin{vmatrix} \cos\phi & +\sin\phi & 0 \\ 0 & 0 & 1 \\ \sin\phi & -\cos\phi & 0 \end{vmatrix} = - \begin{vmatrix} \cos\phi & +\sin\phi \\ \sin\phi & -\cos\phi \end{vmatrix} = 1$$

How to control it?



We have to pick some control!

Geometric control theory (Sub-Riemannian geometry)

Sub-Riemannian geometry (also known as Carnot geometry in France, and non-holonomic geometry in Russia) is a geometric structure studied intensively over the last decades, because it plays an important role in the Geometric control theory.

In a sub-Riemannian space we can neither move nor send information in all the directions, nor can we receive information from everywhere. There are constraints (imposed by God, by a moral imperative, by a government, or just by the laws of Nature).

Geometric control theory (Sub-Riemannian geometry)

Normal extremals of the rolling disc problem are solutions of the ODE system

$$\dot{h}_1 = -h_3 h_2, \quad \dot{h}_2 = h_3 h_1, \quad \dot{h}_3 = -h_1 h_2,$$
$$\dot{x} = h_2 \cos\theta, \quad \dot{y} = h_2 \sin\theta, \quad \dot{\theta} = h_1,$$

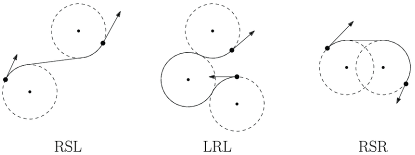
based on Hamiltonian system of PMP

$$\dot{h}_i = \{H, h_i\}$$
$$\dot{g} = h_1 X_1(g) + h_2 X_2(g)$$

where

$$H = \frac{1}{2}(h_1^2 + h_2^2)$$

Dubins paths:



Nilpotent approximation (NA)

The nilpotent approximation (NA) is a classical technique to approximate the algebra of controllability by a nilpotent Lie algebra and thus simplify the finding of the optimal solution. For the construction of nilpotent coordinates, we use the Bellaiche algorithm.

$$X_1 = \cos\phi\partial_{x_1} + \sin\phi\partial_{x_2} \Longrightarrow \phi\partial_{x_1} + \partial_{x_2}$$
$$X_2 = \partial_\phi, \Longrightarrow \partial_\phi$$

Nilpotent approximation (NA)

This new coordinates (y_1, y_2, y_3) yields a nilpotent Lie algebra $\mathfrak{n}$ with the following multiplication table,

$\mathfrak{n}$	n_1	n_2	n_{12}
n_1	0	n_{12}	0
n_2	$-n_{12}$	0	0
n_{12}	0	0	0

Table: The multiplication table of nilpotent Lie algebra

where the new vector fields look like

$$n_1 = \partial_{y_1} - y_2\partial_{y_3}, \quad n_2 = \partial_{y_2}, \quad n_{12} = [n_1, n_2] = \partial_{y_3}, \tag{2}$$

i.e. forms so-called Heisenberg algebra $\mathfrak{h}_3$, together with the first-order nilpotent control system

$$\dot{q} = u_1 n_1 + u_2 n_2. \tag{3}$$

Nilpotent approximation (NA)

The method suggests the use of an iterative method of the Newton type. At first, the method solves the motion planning problem for a nilpotent approximation of the system, given by an initial point x^{init} and a final point x^{final}. Then, the resulting input control $\hat{u}$ is applied to the original system and the procedure is iterated from the resulting point x_k. The algorithm is designed in the following algorithm.

Input: $x^{\text{init}}, x^{\text{final}}, e$

$k := 0$

$x^k := x^{\text{init}}$

while $d(x^k, x^{\text{final}}) > e$ **do**

Compute $\hat{u}^k$ such that $x^{\text{final}} = \hat{\gamma}(T; x^k, \hat{u}^k)$

Set $x^{k+1} := \gamma(T; x^k, \hat{u}^k)$

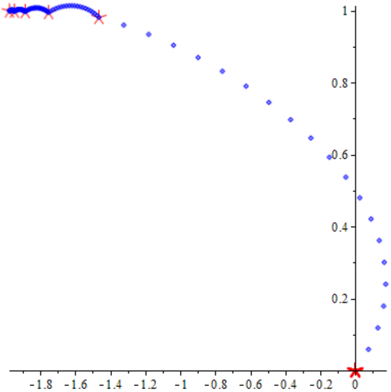
$k := k + 1$

Where d is the sub-Riemannian distance of the original system.

Nilpotent approximation (NA)

Normal extremals of the rolling disc problem are solutions of the ODE system

$$\dot{h}_1 = -h_3 h_2, \quad \dot{h}_2 = h_3 h_1, \quad \dot{h}_3 = 0,$$
$$\dot{x} = h_2 \theta, \quad \dot{y} = h_2 \theta, \quad \dot{\theta} = h_1,$$



Symmetric three dimensional Heisenberg group $\mathbb{H}_3$

With a straightforward change of coordinates, we can transform them into a vector field represented in a symmetric form

$$X = \partial_{y_1} - 2y_2\partial_\tau, \quad Y = \partial_{y_2} - 2y_1\partial_\tau, \quad T = 4\partial_\tau,$$

where the group operation on $\mathbb{R}^3$ will then be of the form

$$(y_1, y_2, \tau) \circ (\bar{y}_1, \bar{y}_2, \bar{\tau}) = (y_1 + \bar{y}_1, y_2 + \bar{y}_2, \tau + \bar{\tau} - 2(y_1\bar{y}_2 - y_2\bar{y}_1)).$$

The Lie group $(\mathbb{R}^3, \circ)$ is called *symmetric three dimensional Heisenberg group* $\mathbb{H}_3$, the element $(0, 0, 0)$ is origin and the inverse of (y_1, y_2, τ) is $(-y_1, -y_2, -\tau)$.

$\mathcal{G}r(2)$ model of $\mathbb{H}_3$

The symmetric three dimensional Heisenberg group $\mathbb{H}_3$ can be realized in the Grassmannian algebra $\mathcal{G}r(2)$

$$\alpha : \mathbb{H}_3 \hookrightarrow \mathcal{G}r(2) = \mathbb{R} \oplus \mathbb{R}^2 \oplus \wedge^2\mathbb{R}^2 \cong \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathfrak{so}(2)$$

by the identification

$$(y_1, y_2, t) \mapsto 1 + y_1 e_1 + y_2 e_2 + t e_1 \wedge e_2 \cong 1 + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$$

where the multiplication coincides with wedge operation

$$(1 + y_1 e_1 + y_2 e_2 + t e_1 \wedge e_2) \wedge (1 + \bar{y}_1 e_1 + \bar{y}_2 e_2 + \bar{t} e_1 \wedge e_2)$$
$$= 1 + (y_1 + \bar{y}_1) e_1 + (y_2 + \bar{y}_2) e_2 + (t + \bar{t} + (y_1 \bar{y}_2 - y_2 \bar{y}_1)) e_1 \wedge e_2).$$

The difference is caused by slightly different definitions $[a, b] = ab - ba = -2a \wedge b$. If we would like total equality, we can define the multiplication as $a \circ b = -2a \wedge b$, but for our purposes, the multiplication factor -2 is irrelevant. Finally, the corresponding Lie algebra then corresponds to

$$\mathfrak{h}_3 = \{y_1 e_1 + y_2 e_2 + t e_1 \wedge e_2\}$$

$\mathbb{G}_2$ model of $\mathbb{H}_3$

By control we mean the invariant control problems on Heisenberg group $G := \mathbb{H}_3$. Straightforward calculations

$$(1 + y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2) \wedge e_1 = e_1 + y_2 e_2 \wedge e_1 = n_1,$$
$$(1 + y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2) \wedge e_2 = e_2 - y_1 e_2 \wedge e_1 = n_2$$

shows that the infinitesimal group multiplication maps the vectors e_1, e_2 to vectors n_1, n_2. If we denote the local coordinates by $(y, t) \in \mathbb{R}^2 \oplus \wedge^2 \mathbb{R}^m$, we can model the corresponding Lie algebra $\mathfrak{g} := \mathfrak{h}_3$ as

$$\begin{aligned} n_1 &= e_1 + y_2 \; e_1 \wedge e_2, \qquad n_2 = e_2 - y_1 \; e_1 \wedge e_2, \\ n_{12} &= n_1 \wedge n_2 = e_1 \wedge e_2 \end{aligned} \tag{4}$$

and discuss the related optimal control problem

$$\dot{q}(t) = u_1 n_1 + u_2 n_2 \tag{5}$$

Fiber (vertical) system

In the geometric algebra $\mathbb{G}_2$ the solutions of fiber system

$$\dot{h}_1 = w, \qquad \dot{h}_2 = -w, \qquad \dot{w} = 0, \tag{6}$$

are

$$h(t) = g(Kt)h(0)\tilde{g}(Kt), \quad \text{where } h(0) = h_1(0)e_1 + h_2(0)e_2 \in \mathbb{G}_2$$
$$g(t) = \cos(t/2) + \sin(t/2)e_1 \wedge e_2 \in Spin(2), K \in \mathbb{R}$$

Proof.

The solutions w of (6) is constants that we denote by $w = K$. If K is non–zero, the first part of the fiber system (6) forms a homogeneous system of ODEs $\dot{h} = -\Omega h = -\begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} h$ with constant coefficients for $h = (h_1, h_2)^T$ and the system matrix $\Omega \in \mathfrak{so}(2)$. Its solution is given by

$$h(t) = g(t)h(0)\tilde{g}(t) = e^{-Kt/2e_1 \wedge e_2} h(0) e^{Kt/2e_1 \wedge e_2}.$$

□

Base (horizontal) system

The base system takes the form of

$$\dot{x}_i = h_i, \quad i = 1, 2$$
$$\dot{z} = -x_1 \wedge \dot{x}_2,$$

for $q = (x_i, z) \in \mathbb{H}_3$.

$$x(t) = \frac{1}{K}(e_2 \wedge e_1)(g(Kt)h_0\tilde{g}(Kt) - h_0),$$

$$z(t) = \frac{1}{K^2}[g(t)((e_2 e_1 h_0) \wedge h_0)\tilde{g}(t) - (g(t)(e_2 e_1 h_0)\tilde{g}(t)) \wedge (e_2 e_1 h_0)] + (h_0 \wedge (e_2 e_1 h_0))$$

Proposition 3.8. *In the case $h_3 \neq 0$, the horizontal system (28) has solutions satisfying $x(0) = y(0) = \theta(0) = 0$*

$$x = \frac{1}{C_1}\big(C_3 - C_2 \sin(C_1 t) - C_3 \cos(C_1 t)\big),$$

$$y = \frac{1}{4C_1^2}\big(2C_1(C_2^2 + C_3^2)t - 4C_2C_3 \cos(C_1 t) + 2C_2C_3 \cos(2C_1 t) - 4C_2^2 \sin(C_1 t) + (C_2^2 - C_3^2)\sin(2C_1 t) + 2C_2C_3\big), \tag{30}$$

$$\theta = \frac{1}{C_1}\big(C_2 - C_2 \cos(C_1 t) + C_3 \sin(C_1 t)\big)$$

for constants C_1, C_2, C_3 from Proposition 3.7. In the degenerate case $h_3 = 0$ we get $x = C_2 t$, $y = \frac{1}{2}C_2 C_1 t^2$, $\theta = C_1 t$ for C_1, C_2 from Proposition 3.7.

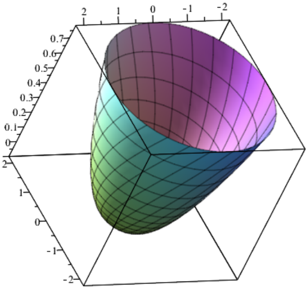
Symmetries

We live in a geometric algebra $\mathbb{G}_2$ we can see the symmetries as rotations around the origin, i.e. elements of the group $Spin(2)$

$$\begin{aligned} &g(t)(1 + y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2)\bar{g}(t) \\ &= 1 + g(t)(y_1 e_1 + y_2 e_2)\bar{g}(t) + \tau e_1 \wedge e_2. \end{aligned}$$

and we can use symmetries to transfer the control from one point to another within the sets

$$M_{k,\tau} = \{q = 1 + y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2 | (y_1 e_1 + y_2 e_2)^2 = k\}.$$



More symmetries

Our goal is to expand the set of symmetries so that we are able to simplify the motion planning even more. We add one Witt pair $\{e_0, e_\infty\}$ to $\mathbb{G}_2$ and understand the points in configuration space by elements in compas ruller algebra (CRA).

$$1 + y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2 \in \mathbb{G}_{3,1}. \tag{7}$$

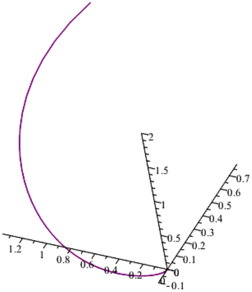
If we represent $\mathbb{H}_3$ in geometric algebra *CRA*, then the following sets are equivariant with respect to control:

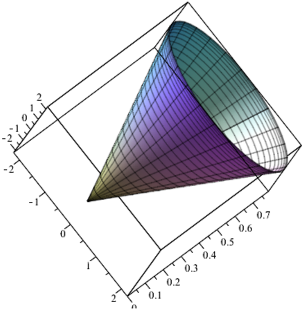
$$\bar{M}_{k,\tau} = \{q = a_0 + I y_1 e_1 + I y_2 e_2 + I \tau e_1 \wedge e_2 | (y_1 e_1 + y_2 e_2)^2 = k\}$$

If $d(k) = \exp(k/2 e_0 \wedge e_\infty) = 2 - \frac{1}{k} e_0 \wedge e_\infty$. then

$$\begin{aligned} & d(k)(e_0 + y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2)\bar{d}(k) \\ & = e_0 + d(k)(y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2)\bar{d}(k) \\ & = e_0 + k(y_1 e_1 + y_2 e_2 + \tau e_1 \wedge e_2). \end{aligned}$$

Conformal cone





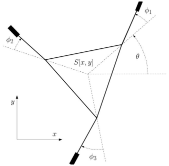
Is it optimal?

$$d(t)n_1\bar{d}(t) = d(kt)(e_1 + y_2 e_2 \wedge e_1)\bar{d}(kt)$$
$$= ke_1 + ky_2 e_2 \wedge e_1$$

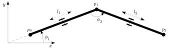
The future work

New mechanisms - now questions

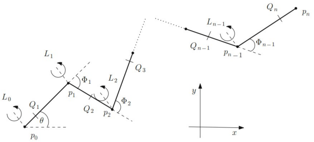
- Next free two-step mechanisms - trident snake (3,6)



- Non free two-step mechanisms - 2-link robotic worm (4,6)



- general two-step mechanisms
- mechanisms with higher filtration - 3 (n)-link robotic snake (2,3,5)



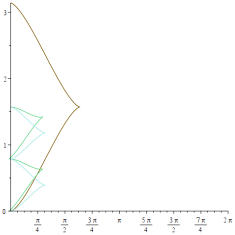
The future work

Control issues - open problems

- Fixed points of symmetries - Maxwel points

$$M_0 = \{1 + \tau e_1 \wedge e_2\}$$

- Cusp points - control of boats



- More and more symmetries ...

Papers

- J. H., L. Zalabová, *Local geometric control of a certain mechanism with the growth vector (4,7)*, **J. Dyn. Control Syst.** 26 (2020) 199–216
- J. H., A. Návrat, L. Zalabová, *Symmetries in geometric control theory using Maple.* **Math. Comput. Simul.** 190 (2021) 474–493
- J. H., A. Návrat, L., P. Vašík, L. Zalabová, *Note on geometric algebras and control problems with* $SO(3)$*–symmetries* **Math Meth Appl Sci.** 47(3), pp. 1257–1273 (2024)
- J. H., A. Návrat, L. Zalabová, *On symmetries of a sub-Riemannian structure with growth vector (4, 7)* **Annali di Matematica** 202, 293–306 (2023).
- Frolík S., J. H. : *Local control of 2-link robotic worms based on additional symmetries*. **Journal of the Franklin Institute**, Vol. 360 (16), 12280-12298 (2023)

Thank you for your attention!