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On Generalized Degenerate Lipschitz and Spin Groups

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Agenda

1. Geometric algebras $\mathcal{G}_{p,q,r}$
2. Centralizers and twisted centralizers in $\mathcal{G}_{p,q,r}$
3. Ordinary Clifford, Lipschitz, and spin groups in $\mathcal{G}_{p,q,0}$
4. Generalized Clifford and Lipschitz groups in $\mathcal{G}_{p,q,r}$:
   - definitions,
   - relation with the groups defined using centralizers, twisted centralizers, and norm functions,
   - examples in the cases $\mathcal{G}_{p,q,0}$ and $\mathcal{G}_{0,0,n}$
5. Degenerate Lipschitz and spin groups in $\mathcal{G}_{p,q,r}$
6. Generalized degenerate spin groups in $\mathcal{G}_{p,q,r}$

Applications

The generalized Clifford and Lipschitz groups can be interesting:

- in deep learning to construct ***neural networks*** that are ***equivariant*** with respect to the action of pseudo-orthogonal groups,
- for consideration of the ***Galilei group*** related to the ***spin groups*** in the Galilei–Clifford algebra $\mathcal{G}_{3,0,1}$,
- for the study of the ***generalized degenerate spin groups***,
- for working with ***orthogonal transformations in PGA*** $\mathcal{G}_{p,0,1}$, which are applied in computer graphics and vision, robotics, motion capture, dynamics simulation, etc.

1. Geometric algebras $\mathcal{G}_{p,q,r}$

Geometric algebras $\boldsymbol{\mathcal{G}_{p,q,r}}$

Let us consider the **geometric (Clifford) algebra** $\mathcal{G}(V) = \boldsymbol{\mathcal{G}_{p,q,r}}$, $p+q+r = n \geq 1$, over a vector space V with a symmetric bilinear form, which can be real $\mathbb{R}^{p,q,r}$ or complex $\mathbb{C}^{p+q,0,r}$. We consider both the cases of the non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$ and the degenerate geometric algebras $\mathcal{G}_{p,q,r}$, $r \neq 0$.

We denote by $\boldsymbol{\Lambda_r} := \mathcal{G}_{0,0,r}$ the subalgebra of $\mathcal{G}_{p,q,r}$, which is the Grassmann (exterior) algebra.

The identity element is denoted by $\boldsymbol{e}$, **the generators** are denoted by $\boldsymbol{e_a}$, $a = 1, \ldots, n$. The generators satisfy

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad a, b = 1, \ldots, n, \tag{1}$$

where $\eta = (\eta_{ab})$ is the diagonal matrix with p times $+1$, q times -1, and r times 0 on the diagonal in the real case $\mathcal{G}(\mathbb{R}^{p,q,r})$ and $p+q$ times $+1$ and r times 0 on the diagonal in the complex case $\mathcal{G}(\mathbb{C}^{p+q,0,r})$.

Grade involution and projections

Consider **the subspaces** $\mathcal{G}^k_{p,q,r}$ **of grades** $k = 0, \ldots, n$, which elements are linear combinations of the basis elements $e_{a_1 \ldots a_k} := e_{a_1} \cdots e_{a_k}$, $a_1 < \cdots < a_k$, with ordered multi-indices of length k. The grade-0 subspace is denoted by $\mathcal{G}^0$ without the lower indices p, q, r, since it does not depend on the Clifford algebra's signature.

The **grade involution** of an element $U \in \mathcal{G}_{p,q,r}$ is denoted by $\widehat{U}$. The grade involution defines the even $\mathcal{G}^{(0)}_{p,q,r}$ and odd $\mathcal{G}^{(1)}_{p,q,r}$ subspaces:

$$\mathcal{G}^{(k)}_{p,q,r} = \{U \in \mathcal{G}_{p,q,r} : \ \widehat{U} = (-1)^k U\} = \bigoplus_{j=k \bmod 2} \mathcal{G}^j_{p,q,r}, \quad k = 0, 1.$$

For an arbitrary subset $\mathrm{H} \subseteq \mathcal{G}_{p,q,r}$,

$$\langle \mathbf{H} \rangle_{(0)} := \mathrm{H} \cap \mathcal{G}^{(0)}_{p,q,r}, \qquad \langle \mathbf{H} \rangle_{(1)} := \mathrm{H} \cap \mathcal{G}^{(1)}_{p,q,r}.$$

2. Ordinary Clifford, Lipschitz, and spin groups

Clifford, Lipschitz, and spin groups

Consider the **adjoint representation** ad and the **twisted adjoint representations** $\check{\mathrm{ad}}$ and $\tilde{\mathrm{ad}}$ acting on the group of all invertible elements $\mathrm{ad}, \check{\mathrm{ad}}, \tilde{\mathrm{ad}} : \mathcal{G}^{\times}_{p,q,r} \to \mathrm{Aut}(\mathcal{G}_{p,q,r})$ as $T \mapsto \mathrm{ad}_T$, $T \mapsto \check{\mathrm{ad}}_T$, and $T \mapsto \tilde{\mathrm{ad}}_T$ respectively, where for $U \in \mathcal{G}_{p,q,r}$, $T \in \mathcal{G}^{\times}_{p,q,r}$,

$$\mathbf{ad}_{T}(U) := TUT^{-1}, \qquad \check{\mathbf{ad}}_{T}(U) := \widehat{T}UT^{-1}, \qquad \tilde{\mathbf{ad}}_{T}(U) := T\langle U\rangle_{(0)}T^{-1} + \widehat{T}\langle U\rangle_{(1)}T^{-1}$$

Clifford, Lipschitz, and spin groups

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Consider the well-known **Clifford and Lipschitz groups**, which are defined in the non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$ as:

$$\Gamma_{p,q,0} := \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \mathrm{ad}_T(\mathcal{G}^1_{p,q,0}) := T\mathcal{G}^1_{p,q,0}T^{-1} \subseteq \mathcal{G}^1_{p,q,0}\}, \qquad \textbf{Clifford groups}$$

$$\Gamma^{\pm\Lambda}_{p,q,0} := \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,0}) = \tilde{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,0}) := \widehat{T}\mathcal{G}^1_{p,q,0}T^{-1} \subseteq \mathcal{G}^1_{p,q,0}\}. \qquad \textbf{Lipschitz groups}$$

Clifford, Lipschitz, and spin groups

Consider the **adjoint representation** $\mathbf{ad}$ and the **twisted adjoint representations** $\check{\mathbf{ad}}$ and $\tilde{\mathbf{ad}}$ acting on the group of all invertible elements $\mathrm{ad}, \check{\mathrm{ad}}, \tilde{\mathrm{ad}} : \mathcal{G}^{\times}_{p,q,r} \to \mathrm{Aut}(\mathcal{G}_{p,q,r})$ as $T \mapsto \mathrm{ad}_T$, $T \mapsto \check{\mathrm{ad}}_T$, and $T \mapsto \tilde{\mathrm{ad}}_T$ respectively, where for $U \in \mathcal{G}_{p,q,r}$, $T \in \mathcal{G}^{\times}_{p,q,r}$,

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Consider the well-known **Clifford and Lipschitz groups**, which are defined in the non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$ as:

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Clifford groups

Lipschitz groups

Similarly, in arbitrary $\mathcal{G}_{p,q,r}$, they can be defined as:

$$\boldsymbol{\Gamma}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \mathrm{ad}_T(\mathcal{G}^1_{p,q,r}) := T\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\},$$

$$\boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,r}) = \tilde{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,r}) := \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\}.$$

Clifford groups

Lipschitz groups

Clifford, Lipschitz, and spin groups

Consider two **norm functions** widely used in the theory of spin groups:

$$\psi(T) := \widetilde{T}T, \qquad \chi(T) := \widehat{\widetilde{T}}T, \qquad \forall T \in \mathcal{G}_{p,q,r}.$$

For example, in the case of the non-degenerate geometric algebra $\mathcal{G}_{p,q,0}$, the spin groups are defined as

$$\begin{aligned}
\mathbf{Pin}(p,q,0) &:= \{T \in \boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0} : \quad \widetilde{T}T = \pm e\} = \{T \in \boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0} : \quad \widehat{\widetilde{T}}T = \pm e\},\\
\mathbf{Pin}_{+\psi}(p,q,0) &:= \{T \in \boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0} : \quad \widetilde{T}T = +e\},\\
\mathbf{Pin}_{+\chi}(p,q,0) &:= \{T \in \boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0} : \quad \widehat{\widetilde{T}}T = +e\},\\
\mathbf{Spin}(p,q,0) &:= \{T \in \langle\boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0}\rangle_{(0)} : \quad \widetilde{T}T = \pm e\} = \{T \in \langle\boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0}\rangle_{(0)} : \quad \widehat{\widetilde{T}}T = \pm e\},\\
\mathbf{Spin}_{+}(p,q,0) &:= \{T \in \langle\boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0}\rangle_{(0)} : \quad \widetilde{T}T = +e\} = \{T \in \langle\boldsymbol{\Gamma}^{\pm\Lambda}_{p,q,0}\rangle_{(0)} : \quad \widehat{\widetilde{T}}T = +e\}.
\end{aligned}$$

3. Generalized Clifford and Lipschitz groups

The subspaces determined by the grade involution and the reversion

The **reversion** is denoted by $\widetilde{U}$, the **Clifford conjugation** is denoted by $\widehat{\widetilde{U}}$.

The grade involution and the reversion define four subspaces $\mathcal{G}^{\overline{0}}_{p,q,r}$, $\mathcal{G}^{\overline{1}}_{p,q,r}$, $\mathcal{G}^{\overline{2}}_{p,q,r}$, and $\mathcal{G}^{\overline{3}}_{p,q,r}$ (they are called the subspaces of quaternion types $0, 1, 2$, and 3 respectively):

$$\mathcal{G}^{\overline{k}}_{p,q,r} = \{U \in \mathcal{G}_{p,q,r} : \widehat{U} = (-1)^k U, \ \widetilde{U} = (-1)^{\frac{k(k-1)}{2}} U\}, \quad k = 0, 1, 2, 3.$$

The Clifford algebra $\mathcal{G}_{p,q,r}$ can be represented as a direct sum $\mathcal{G}_{p,q,r} = \mathcal{G}^{\overline{0}}_{p,q,r} \oplus \mathcal{G}^{\overline{1}}_{p,q,r} \oplus \mathcal{G}^{\overline{2}}_{p,q,r} \oplus \mathcal{G}^{\overline{3}}_{p,q,r}$.

We denote the direct sum of these subspaces by $\mathcal{G}^{\overline{kl}}_{p,q,r} := \mathcal{G}^{\overline{k}}_{p,q,r} \oplus \mathcal{G}^{\overline{l}}_{p,q,r}$.

Generalized Clifford and Lipschitz groups

Consider setwise stabilizers of the subspaces $\mathcal{G}^{\overline{k}}_{p,q,r}$ and $\mathcal{G}^{\overline{kl}}_{p,q,r}$, $k,l=0,1,2,3$, in the group $\mathcal{G}^{\times}_{p,q,r}$ under the group actions ad, $\check{\mathrm{ad}}$, and $\tilde{\mathrm{ad}}$:

$$\boldsymbol{\Gamma}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \mathrm{ad}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) := T\mathcal{G}^{\overline{k}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{k}}_{p,q,r}\},$$

$$\check{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) := \widehat{T}\mathcal{G}^{\overline{k}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{k}}_{p,q,r}\},$$

$$\tilde{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \tilde{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) \subseteq \mathcal{G}^{\overline{k}}_{p,q,r}\}.$$

$$\boldsymbol{\Gamma}^{\overline{\boldsymbol{kl}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \mathrm{ad}_T(\mathcal{G}^{\overline{kl}}_{p,q,r}) := T\mathcal{G}^{\overline{kl}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{kl}}_{p,q,r}\},$$

$$\check{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{kl}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^{\overline{kl}}_{p,q,r}) := \widehat{T}\mathcal{G}^{\overline{kl}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{kl}}_{p,q,r}\},$$

$$\tilde{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{kl}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \tilde{\mathrm{ad}}_T(\mathcal{G}^{\overline{kl}}_{p,q,r}) \subseteq \mathcal{G}^{\overline{kl}}_{p,q,r}\}.$$

generalized degenerate Clifford and Lipschitz groups

Generalized Clifford and Lipschitz groups

Note that the groups $\boldsymbol{\Gamma}^{\overline{\mathbf{02}}}_{\boldsymbol{p,q,r}}$, $\check{\boldsymbol{\Gamma}}^{\overline{\mathbf{02}}}_{\boldsymbol{p,q,r}}$, $\tilde{\boldsymbol{\Gamma}}^{\overline{\mathbf{02}}}_{\boldsymbol{p,q,r}}$, $\boldsymbol{\Gamma}^{\overline{\mathbf{13}}}_{\boldsymbol{p,q,r}}$, $\check{\boldsymbol{\Gamma}}^{\overline{\mathbf{13}}}_{\boldsymbol{p,q,r}}$, and $\tilde{\boldsymbol{\Gamma}}^{\overline{\mathbf{13}}}_{\boldsymbol{p,q,r}}$ (preserving the even $\mathcal{G}^{\overline{02}}_{p,q,r} = \mathcal{G}^{(0)}_{p,q,r}$ and odd $\mathcal{G}^{\overline{13}}_{p,q,r} = \mathcal{G}^{(1)}_{p,q,r}$ subspaces under ad, $\check{\mathrm{ad}}$, and $\tilde{\mathrm{ad}}$ respectively) are considered in details in the paper

> Filimoshina E., Shirokov D.: On Some Lie Groups in Degenerate Clifford Geometric Algebras.
> Advances in Applied Clifford Algebras, 33(44) (2023), arXiv:2301.06842

Centralizers and twisted centralizers

Consider the **centralizers** $\mathbf{Z}^m_{\boldsymbol{p,q,r}}$ and **twisted centralizers** $\check{\mathbf{Z}}^m_{\boldsymbol{p,q,r}}$ of the fixed grade subspaces $\mathcal{G}^m_{p,q,r}$, $m = 0, \ldots, n$:

$$\mathbf{Z}^m_{\boldsymbol{p,q,r}} := \{X \in \mathcal{G}_{p,q,r} : \quad XV = VX, \quad \forall V \in \mathcal{G}^m_{p,q,r}\},$$

$$\check{\mathbf{Z}}^m_{\boldsymbol{p,q,r}} := \{X \in \mathcal{G}_{p,q,r} : \quad \widehat{X}V = VX, \quad \forall V \in \mathcal{G}^m_{p,q,r}\}.$$

The center of the geometric algebra $\mathcal{G}_{p,q,r}$ is the centralizer of the grade-1 subspace $\mathcal{G}^1_{p,q,r}$ and of the entire geometric algebra $\mathcal{G}_{p,q,r}$ as well.

Explicit forms of $\mathbf{Z}^m_{\boldsymbol{p,q,r}}$ and $\check{\mathbf{Z}}^m_{\boldsymbol{p,q,r}}$ in the case of arbitrary $m = 0, \ldots, n$ are presented in the paper

> Filimoshina E., Shirokov D.: A Note on Centralizers and Twisted Centralizers in Clifford Algebras, Adv. Appl. Clifford Algebras, 2024 (to appear), arXiv:2404.15169

Examples of centralizers and twisted centralizers

$$\mathbf{Z}^{1}_{p,q,r} = \{X \in \mathcal{G}_{p,q,r} : \quad XV = VX, \quad \forall V \in \mathcal{G}^{1}_{p,q,r}\} = \begin{cases} \Lambda^{(0)}_{r} \oplus \mathcal{G}^{n}_{p,q,r}, & n \text{ is odd,} \\ \Lambda^{(0)}_{r}, & n \text{ is even;} \end{cases} \quad (1)$$

$$\check{\mathbf{Z}}^{1}_{p,q,r} = \{X \in \mathcal{G}_{p,q,r} : \quad \widehat{X}V = VX, \quad \forall V \in \mathcal{G}^{1}_{p,q,r}\} = \Lambda_{r}; \quad (2)$$

$$\mathbf{Z}^{2}_{p,q,r} = \{X \in \mathcal{G}_{p,q,r} : \quad XV = VX, \quad \forall V \in \mathcal{G}^{2}_{p,q,r}\} = \begin{cases} \Lambda_{r} \oplus \mathcal{G}^{n}_{p,q,r}, & r \neq n, \\ \Lambda_{r}, & r = n; \end{cases} \quad (3)$$

$$\mathbf{Z}^{3}_{p,q,r} = \begin{cases} \Lambda^{(0)}_{r} \oplus \Lambda^{n-2}_{r} \oplus \{\mathcal{G}^{1}_{p,q,0}(\Lambda^{n-3}_{r} \oplus \Lambda^{n-2}_{r})\} \oplus \{\mathcal{G}^{2}_{p,q,0}\Lambda^{n-3}_{r}\} \oplus \mathcal{G}^{n}_{p,q,r}, & n \text{ is odd,} \\ \Lambda^{(0)}_{r} \oplus \Lambda^{n-1}_{r} \oplus \{\mathcal{G}^{1}_{p,q,0}\Lambda^{\geq n-2}_{r}\} \oplus \{\mathcal{G}^{2}_{p,q,0}\Lambda^{n-2}_{r}\}, & n \text{ is even.} \end{cases} \quad (4)$$

3.1. Generalized Clifford and Lipschitz groups $\boldsymbol{\Gamma^{\overline{kl}}_{p,q,r}}$, $\boldsymbol{\check{\Gamma}^{\overline{kl}}_{p,q,r}}$, $\boldsymbol{\tilde{\Gamma}^{\overline{kl}}_{p,q,r}}$

$$\boldsymbol{\Gamma^{\overline{kl}}_{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \mathrm{ad}_T(\mathcal{G}^{\overline{kl}}_{p,q,r}) := T\mathcal{G}^{\overline{kl}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{kl}}_{p,q,r}\},$$

$$\boldsymbol{\check{\Gamma}^{\overline{kl}}_{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^{\overline{kl}}_{p,q,r}) := \widehat{T}\mathcal{G}^{\overline{kl}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{kl}}_{p,q,r}\},$$

$$\boldsymbol{\tilde{\Gamma}^{\overline{kl}}_{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \tilde{\mathrm{ad}}_T(\mathcal{G}^{\overline{kl}}_{p,q,r}) \subseteq \mathcal{G}^{\overline{kl}}_{p,q,r}\}.$$

Generalized Clifford and Lipschitz groups

Theorem 1. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\Gamma^{\overline{01}}_{p,q,r} = \mathbf{A}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} \subseteq \Gamma^{\overline{23}}_{p,q,r} = \mathbf{A}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}},$$

$$\Gamma^{\overline{12}}_{p,q,r} = \mathbf{B}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} \subseteq \Gamma^{\overline{03}}_{p,q,r} = \mathbf{B}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}}.$$

Consider the following groups:

$$\mathbf{A}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \boldsymbol{\psi(T)} = \widetilde{T}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$

$$\mathbf{A}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\mathrm{Z}^{2}_{p,q,r} \cap \mathrm{Z}^{3}_{p,q,r})^{\times}\},$$

$$\mathbf{B}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \boldsymbol{\chi(T)} = \widehat{\widetilde{T}}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$

$$\mathbf{B}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{3\times}_{p,q,r}\},$$

Generalized Clifford and Lipschitz groups

Theorem 1. In arbitrary $\mathcal{G}_{p,q,r}$,

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Theorem 2. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\check{\Gamma}^{\overline{12}}_{p,q,r} = \check{\mathbf{A}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}}, \qquad \check{\Gamma}^{\overline{03}}_{p,q,r} = \check{\mathbf{A}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}},$$

$$\check{\Gamma}^{\overline{01}}_{p,q,r} = \check{\mathbf{B}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} \subseteq \check{\Gamma}^{\overline{23}}_{p,q,r} = \check{\mathbf{B}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}}.$$

Consider the following groups:

$$\mathbf{A}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \psi(T) = \widetilde{T}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$

$$\mathbf{A}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\mathrm{Z}^{2}_{p,q,r} \cap \mathrm{Z}^{3}_{p,q,r})^{\times}\},$$

$$\mathbf{B}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \chi(T) = \widehat{\widetilde{T}}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$

$$\mathbf{B}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{3\times}_{p,q,r}\},$$

$$\check{\mathbf{A}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\check{\mathrm{Z}}^{1}_{p,q,r} \cap \check{\mathrm{Z}}^{2}_{p,q,r})^{\times}\},$$

$$\check{\mathbf{A}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\mathrm{Z}^{3}_{p,q,r} \cap \mathcal{G}^{(0)}_{p,q,r})^{\times}\},$$

$$\check{\mathbf{B}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in (\mathrm{Z}^{1}_{p,q,r} \cap \mathcal{G}^{(0)}_{p,q,r})^{\times}\},$$

$$\check{\mathbf{B}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in (\check{\mathrm{Z}}^{2}_{p,q,r} \cap \check{\mathrm{Z}}^{3}_{p,q,r})^{\times}\},$$

Generalized Clifford and Lipschitz groups

Theorem 1. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\Gamma^{\overline{01}}_{p,q,r} = \mathbf{A}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} \subseteq \Gamma^{\overline{23}}_{p,q,r} = \mathbf{A}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}},$$
$$\Gamma^{\overline{12}}_{p,q,r} = \mathbf{B}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} \subseteq \Gamma^{\overline{03}}_{p,q,r} = \mathbf{B}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}}.$$

Theorem 2. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\check{\Gamma}^{\overline{12}}_{p,q,r} = \check{\mathbf{A}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}}, \qquad \check{\Gamma}^{\overline{03}}_{p,q,r} = \check{\mathbf{A}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}},$$
$$\check{\Gamma}^{\overline{01}}_{p,q,r} = \check{\mathbf{B}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} \subseteq \check{\Gamma}^{\overline{23}}_{p,q,r} = \check{\mathbf{B}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}}.$$

Theorem 3. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\tilde{\Gamma}^{\overline{01}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}}, \qquad \tilde{\Gamma}^{\overline{23}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}},$$
$$\tilde{\Gamma}^{\overline{12}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} \subseteq \tilde{\Gamma}^{\overline{03}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}}.$$

Consider the following groups:

$$\mathbf{A}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \boldsymbol{\psi(T)} = \widetilde{T}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$
$$\mathbf{A}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\mathrm{Z}^{2}_{p,q,r} \cap \mathrm{Z}^{3}_{p,q,r})^{\times}\},$$
$$\mathbf{B}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \boldsymbol{\chi(T)} = \widehat{\widetilde{T}}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$
$$\mathbf{B}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{3\times}_{p,q,r}\},$$
$$\check{\mathbf{A}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\check{\mathrm{Z}}^{1}_{p,q,r} \cap \check{\mathrm{Z}}^{2}_{p,q,r})^{\times}\},$$
$$\check{\mathbf{A}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in (\mathrm{Z}^{3}_{p,q,r} \cap \mathcal{G}^{(0)}_{p,q,r})^{\times}\},$$
$$\check{\mathbf{B}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in (\mathrm{Z}^{1}_{p,q,r} \cap \mathcal{G}^{(0)}_{p,q,r})^{\times}\},$$
$$\check{\mathbf{B}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T \in (\check{\mathrm{Z}}^{2}_{p,q,r} \cap \check{\mathrm{Z}}^{3}_{p,q,r})^{\times}\},$$
$$\tilde{\mathbf{Q}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \mathrm{Z}^{4\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}\},$$
$$\tilde{\mathbf{Q}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \mathrm{Z}^{2\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{3\times}_{p,q,r}\},$$
$$\tilde{\mathbf{Q}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{2\times}_{p,q,r}\},$$
$$\tilde{\mathbf{Q}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \check{\mathrm{Z}}^{3\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{4\times}_{p,q,r}\}.$$

Generalized Clifford and Lipschitz groups

Table 1: **Generalized Clifford and Lipschitz groups**

Lie group	$\psi(T) = \widetilde{T}T$	$\chi(T) = \widehat{\widetilde{T}}T$
$\mathbf{A}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} = \Gamma^{\overline{01}}_{p,q,r}$	$\mathrm{Z}^{1\times}_{p,q,r}$	
$\mathbf{A}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} = \Gamma^{\overline{23}}_{p,q,r}$	$(\mathrm{Z}^{2}_{p,q,r} \cap \mathrm{Z}^{3}_{p,q,r})^{\times}$	
$\mathbf{B}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} = \Gamma^{\overline{12}}_{p,q,r}$		$\mathrm{Z}^{1\times}_{p,q,r}$
$\mathbf{B}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} = \Gamma^{\overline{03}}_{p,q,r}$		$\mathrm{Z}^{3\times}_{p,q,r}$
$\check{\mathbf{A}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} = \check{\Gamma}^{\overline{12}}_{p,q,r}$	$(\check{\mathrm{Z}}^{1}_{p,q,r} \cap \check{\mathrm{Z}}^{2}_{p,q,r})^{\times}$	
$\check{\mathbf{A}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} = \check{\Gamma}^{\overline{03}}_{p,q,r}$	$(\mathrm{Z}^{3}_{p,q,r} \cap \mathcal{G}^{(0)}_{p,q,r})^{\times}$	
$\check{\mathbf{B}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} = \check{\Gamma}^{\overline{01}}_{p,q,r}$		$(\mathrm{Z}^{1}_{p,q,r} \cap \mathcal{G}^{(0)}_{p,q,r})^{\times}$
$\check{\mathbf{B}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} = \check{\Gamma}^{\overline{23}}_{p,q,r}$		$(\check{\mathrm{Z}}^{2}_{p,q,r} \cap \check{\mathrm{Z}}^{3}_{p,q,r})^{\times}$
$\tilde{\mathbf{Q}}^{\overline{\mathbf{01}}}_{\boldsymbol{p,q,r}} = \tilde{\Gamma}^{\overline{01}}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$
$\tilde{\mathbf{Q}}^{\overline{\mathbf{23}}}_{\boldsymbol{p,q,r}} = \tilde{\Gamma}^{\overline{23}}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$
$\tilde{\mathbf{Q}}^{\overline{\mathbf{12}}}_{\boldsymbol{p,q,r}} = \tilde{\Gamma}^{\overline{12}}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$
$\tilde{\mathbf{Q}}^{\overline{\mathbf{03}}}_{\boldsymbol{p,q,r}} = \tilde{\Gamma}^{\overline{03}}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$

Generalized Clifford and Lipschitz groups

Table 1: **Generalized Clifford and Lipschitz groups**

Lie group	$\psi(T)=\widetilde{T}T$	$\chi(T)=\widehat{\widetilde{T}}T$
$\mathbf{A}^{\overline{01}}_{p,q,r}=\Gamma^{\overline{01}}_{p,q,r}$	$\mathrm{Z}^{1\times}_{p,q,r}$	
$\mathbf{A}^{\overline{23}}_{p,q,r}=\Gamma^{\overline{23}}_{p,q,r}$	$(\mathrm{Z}^{2}_{p,q,r}\cap\mathrm{Z}^{3}_{p,q,r})^{\times}$	
$\mathbf{B}^{\overline{12}}_{p,q,r}=\Gamma^{\overline{12}}_{p,q,r}$		$\mathrm{Z}^{1\times}_{p,q,r}$
$\mathbf{B}^{\overline{03}}_{p,q,r}=\Gamma^{\overline{03}}_{p,q,r}$		$\mathrm{Z}^{3\times}_{p,q,r}$
$\check{\mathbf{A}}^{\overline{12}}_{p,q,r}=\check{\Gamma}^{\overline{12}}_{p,q,r}$	$(\check{\mathrm{Z}}^{1}_{p,q,r}\cap\check{\mathrm{Z}}^{2}_{p,q,r})^{\times}$	
$\check{\mathbf{A}}^{\overline{03}}_{p,q,r}=\check{\Gamma}^{\overline{03}}_{p,q,r}$	$(\mathrm{Z}^{3}_{p,q,r}\cap\mathcal{G}^{(0)}_{p,q,r})^{\times}$	
$\check{\mathbf{B}}^{\overline{01}}_{p,q,r}=\check{\Gamma}^{\overline{01}}_{p,q,r}$		$(\mathrm{Z}^{1}_{p,q,r}\cap\mathcal{G}^{(0)}_{p,q,r})^{\times}$
$\check{\mathbf{B}}^{\overline{23}}_{p,q,r}=\check{\Gamma}^{\overline{23}}_{p,q,r}$		$(\check{\mathrm{Z}}^{2}_{p,q,r}\cap\check{\mathrm{Z}}^{3}_{p,q,r})^{\times}$
$\tilde{\mathbf{Q}}^{\overline{01}}_{p,q,r}=\tilde{\Gamma}^{\overline{01}}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$
$\tilde{\mathbf{Q}}^{\overline{23}}_{p,q,r}=\tilde{\Gamma}^{\overline{23}}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$
$\tilde{\mathbf{Q}}^{\overline{12}}_{p,q,r}=\tilde{\Gamma}^{\overline{12}}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$
$\tilde{\mathbf{Q}}^{\overline{03}}_{p,q,r}=\tilde{\Gamma}^{\overline{03}}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$

$$\mathbf{Z}^{1}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\mathcal{G}^n_{p,q,r}, & n \textit{ is odd},\\ \Lambda^{(0)}_r, & n \textit{ is even};\end{cases}$$

$$\mathbf{Z}^{2}_{p,q,r}\cap\mathbf{Z}^{3}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\Lambda^{n-2}_r\oplus\mathcal{G}^n_{p,q,r}, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r, & n \text{ is even};\end{cases}$$

$$\mathbf{Z}^{3}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\Lambda^{n-2}_r\oplus\{\mathcal{G}^1_{p,q,0}(\Lambda^{n-3}_r\oplus\Lambda^{n-2}_r)\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-3}_r\}\oplus\mathcal{G}^n_{p,q,r}, & n \textit{ is odd},\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{\geq n-2}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-2}_r\}, & n \textit{ is even};\end{cases}$$

$$\check{\mathbf{Z}}^{1}_{p,q,r}\cap\check{\mathbf{Z}}^{2}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\Lambda^{n}_r, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r, & n \text{ is even};\end{cases}\qquad \mathbf{Z}^{1}_{p,q,r}\cap\boldsymbol{\mathcal{G}}^{(0)}_{p,q,r}=\Lambda^{(0)}_r,$$

$$\mathbf{Z}^{3}_{p,q,r}\cap\boldsymbol{\mathcal{G}}^{(0)}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-2}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-3}_r\}, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-1}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-2}_r\}, & n \text{ is even};\end{cases}$$

$$\check{\mathbf{Z}}^{2}_{p,q,r}\cap\check{\mathbf{Z}}^{3}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\Lambda^{n}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-1}_r\}, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r\oplus\{\mathcal{G}^1_{p,q,0}(\Lambda^{n-2}_r\oplus\Lambda^{n-1}_r)\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-2}_r\}, & n \text{ is even};\end{cases}$$

$$\mathbf{Z}^{4}_{p,q,r}=\begin{cases}\Lambda_r\oplus\{\mathcal{G}^1_{p,q,0}(\Lambda^{n-3}_r\oplus\Lambda^{n-2}_r)\}\oplus\{\mathcal{G}^2_{p,q,0}(\Lambda^{n-4}_r\oplus\Lambda^{n-3}_r)\}\oplus\mathcal{G}^n_{p,q,r}, & r\neq n,\\ \Lambda_r, & r=n;\end{cases}$$

$$\check{\mathbf{Z}}^{1}_{p,q,r}=\Lambda_r,\qquad \check{\mathbf{Z}}^{3}_{p,q,r}=\Lambda_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{\geq n-2}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{\geq n-3}_r\},$$

$$\mathbf{Z}^{2}_{p,q,r}=\begin{cases}\Lambda_r\oplus\mathcal{G}^n_{p,q,r}, & r\neq n,\\ \Lambda_r, & r=n;\end{cases}$$

Examples

Example 1.

In the **non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$**,

$$\check{\mathrm{A}}^{\overline{12}}_{p,q,0} = \check{\mathrm{A}}^{\overline{03}}_{p,q,0} = \mathrm{A}_{\pm} = \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widetilde{T}T \in \mathcal{G}^{0\times}\},$$

$$\check{\mathrm{B}}^{\overline{01}}_{p,q,0} = \check{\mathrm{B}}^{\overline{23}}_{p,q,0} = \mathrm{B}_{\pm} = \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widehat{\widetilde{T}}T \in \mathcal{G}^{0\times}\},$$

$$\mathrm{A}^{\overline{01}}_{p,q,0} = \mathrm{A}^{\overline{23}}_{p,q,0} = \mathrm{A} = \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widetilde{T}T \in \mathrm{Z}^{\times}_{p,q,0}\},$$

$$\mathrm{B}^{\overline{12}}_{p,q,0} = \mathrm{B}^{\overline{03}}_{p,q,0} = \mathrm{B} = \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{\times}_{p,q,0}\},$$

where

$$\mathrm{Z}_{p,q,0} := \begin{cases} \mathcal{G}^0 \oplus \mathcal{G}^n_{p,q,0}, & n \text{ is odd}, \\ \mathcal{G}^0, & n \text{ is even}. \end{cases}$$

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Shirokov D.: On inner automorphisms preserving fixed subspaces of Clifford algebras. Adv. Appl. Clifford Algebras 31(30), (2021)

Examples

Example 2.

In the case of the **Grassmann (exterior) algebra** $\boldsymbol{\mathcal{G}_{0,0,n} = \Lambda_n}$,

$$\check{\mathbf{A}}^{\overline{03}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widetilde{T}T \in \Lambda^{(0)\times}_n\},$$

$$\check{\mathbf{A}}^{\overline{12}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widetilde{T}T \in \begin{cases} (\Lambda^{(0)}_n \oplus \Lambda^n_n)^{\times}, & n \text{ is odd}, \\ (\Lambda^{(0)}_n \oplus \Lambda^{n-1}_n)^{\times}, & n \text{ is even}; \end{cases}\},$$

$$\check{\mathbf{B}}^{\overline{01}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widehat{\widetilde{T}}T \in \Lambda^{(0)\times}_n\},$$

$$\check{\mathbf{B}}^{\overline{23}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widehat{\widetilde{T}}T \in \begin{cases} (\Lambda^{(0)}_n \oplus \Lambda^n_n)^{\times}, & n \text{ is odd}, \\ (\Lambda^{(0)}_n \oplus \Lambda^{n-1}_n)^{\times}, & n \text{ is even}; \end{cases}\},$$

$$\mathbf{A}^{\overline{01}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widetilde{T}T \in \begin{cases} (\Lambda^{(0)}_n \oplus \Lambda^n_n)^{\times}, & n \text{ is odd}, \\ \Lambda^{(0)\times}_n, & n \text{ is even}; \end{cases}\},$$

$$\mathbf{A}^{\overline{23}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widetilde{T}T \in \begin{cases} (\Lambda^{(0)}_n \oplus \Lambda^{n-2}_n \oplus \Lambda^n_n)^{\times}, & n \text{ is odd}, \\ (\Lambda^{(0)}_n \oplus \Lambda^{n-1}_n)^{\times}, & n \text{ is even}; \end{cases}\},$$

$$\mathbf{B}^{\overline{12}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widehat{\widetilde{T}}T \in \begin{cases} (\Lambda^{(0)}_n \oplus \Lambda^n_n)^{\times}, & n \text{ is odd}, \\ \Lambda^{(0)\times}_n, & n \text{ is even}; \end{cases}\},$$

$$\mathbf{B}^{\overline{03}}_{\mathbf{0,0,}\boldsymbol{n}} = \{T \in \Lambda^{\times}_n : \quad \widehat{\widetilde{T}}T \in \begin{cases} (\Lambda^{(0)}_n \oplus \Lambda^{n-2}_n \oplus \Lambda^n_n)^{\times}, & n \text{ is odd}, \\ (\Lambda^{(0)}_n \oplus \Lambda^{n-1}_n)^{\times}, & n \text{ is even}; \end{cases}\},$$

$$\tilde{\mathbf{Q}}^{\overline{01}}_{\mathbf{0,0,}\boldsymbol{n}} = \tilde{\mathbf{Q}}^{\overline{23}}_{\mathbf{0,0,}\boldsymbol{n}} = \tilde{\mathbf{Q}}^{\overline{12}}_{\mathbf{0,0,}\boldsymbol{n}} = \tilde{\mathbf{Q}}^{\overline{03}}_{\mathbf{0,0,}\boldsymbol{n}} = \Lambda^{\times}_n.$$

3.2. Generalized Clifford and Lipschitz groups $\Gamma^{\overline{k}}_{p,q,r}$, $\check{\Gamma}^{\overline{k}}_{p,q,r}$, $\tilde{\Gamma}^{\overline{k}}_{p,q,r}$

$$\Gamma^{\overline{k}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \mathrm{ad}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) := T\mathcal{G}^{\overline{k}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{k}}_{p,q,r}\},$$

$$\check{\Gamma}^{\overline{k}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) := \widehat{T}\mathcal{G}^{\overline{k}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{k}}_{p,q,r}\},$$

$$\tilde{\Gamma}^{\overline{k}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \tilde{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) \subseteq \mathcal{G}^{\overline{k}}_{p,q,r}\}.$$

Generalized Clifford and Lipschitz groups

The groups $\tilde{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$, $k=0,1,2,3$, are related to the groups $\boldsymbol{\Gamma}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$ and $\check{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$ as follows:

$$\tilde{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} = \begin{cases} \check{\boldsymbol{\Gamma}}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}, & k=1,3, \\ \boldsymbol{\Gamma}^{\overline{\boldsymbol{k}}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}, & k=0,2, \end{cases}$$

since $\tilde{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) = \mathrm{ad}_T(\mathcal{G}^{\overline{k}}_{p,q,r})$ in the cases $k=0,2$ and $\tilde{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r}) = \check{\mathrm{ad}}_T(\mathcal{G}^{\overline{k}}_{p,q,r})$ in the cases $k=1,3$.

$$\mathbf{ad}_{\boldsymbol{T}}(\boldsymbol{U}) := TUT^{-1}, \qquad \check{\mathbf{ad}}_{\boldsymbol{T}}(\boldsymbol{U}) := \widehat{T}UT^{-1}, \qquad \tilde{\mathbf{ad}}_{\boldsymbol{T}}(\boldsymbol{U}) := T\langle U\rangle_{(0)}T^{-1} + \widehat{T}\langle U\rangle_{(1)}T^{-1}$$

Generalized Clifford and Lipschitz groups

Theorem 4. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\Gamma^{\overline{1}}_{p,q,r} = \mathbf{Q}^{\overline{1}}_{p,q,r},$$

$$\Gamma^{\overline{2}}_{p,q,r} = \mathbf{Q}^{\overline{2}}_{p,q,r},$$

$$\Gamma^{\overline{3}}_{p,q,r} = \mathbf{Q}^{\overline{3}}_{p,q,r},$$

$$\Gamma^{\overline{0}}_{p,q,r} = \mathbf{Q}^{\overline{0}}_{p,q,r},$$

where

$$\Gamma^{\overline{1}}_{p,q,r} \subseteq \Gamma^{\overline{m}}_{p,q,r} \subseteq \Gamma^{\overline{0}}_{p,q,r}, \qquad m = 0, 1, 2, 3.$$

Consider the following groups:

$$\mathbf{Q}^{\overline{1}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \psi(T) = \widetilde{T}T \in \mathrm{Z}^{1\times}_{p,q,r}, \ \chi(T) = \widehat{\widetilde{T}}T \in \mathrm{Z}^{1\times}_{p,q,r}\}$$

$$\mathbf{Q}^{\overline{2}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \mathrm{Z}^{2\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{2\times}_{p,q,r}\},$$

$$\mathbf{Q}^{\overline{3}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \mathrm{Z}^{3\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{3\times}_{p,q,r}\},$$

$$\mathbf{Q}^{\overline{0}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \mathrm{Z}^{4\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{4\times}_{p,q,r}\},$$

Generalized Clifford and Lipschitz groups

Theorem 4. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\Gamma^{\overline{1}}_{p,q,r} = \mathbf{Q}^{\overline{1}}_{p,q,r},$$
$$\Gamma^{\overline{2}}_{p,q,r} = \mathbf{Q}^{\overline{2}}_{p,q,r},$$
$$\Gamma^{\overline{3}}_{p,q,r} = \mathbf{Q}^{\overline{3}}_{p,q,r},$$
$$\Gamma^{\overline{0}}_{p,q,r} = \mathbf{Q}^{\overline{0}}_{p,q,r},$$

where

$$\Gamma^{\overline{1}}_{p,q,r} \subseteq \Gamma^{\overline{m}}_{p,q,r} \subseteq \Gamma^{\overline{0}}_{p,q,r}, \qquad m = 0, 1, 2, 3.$$

Theorem 5. In arbitrary $\mathcal{G}_{p,q,r}$,

$$\check{\Gamma}^{\overline{1}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{1}}_{p,q,r} \subseteq \check{\Gamma}^{\overline{3}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{3}}_{p,q,r},$$
$$\check{\Gamma}^{\overline{2}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{2}}_{p,q,r},$$
$$\check{\Gamma}^{\overline{0}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{0}}_{p,q,r}.$$

Consider the following groups:

$$\mathbf{Q}^{\overline{1}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \ \psi(T) = \widetilde{T}T \in \mathrm{Z}^{1\times}_{p,q,r},\ \chi(T) = \widehat{\widetilde{T}}T \in \mathrm{Z}^{1\times}_{p,q,r}\},$$
$$\mathbf{Q}^{\overline{2}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \ \widetilde{T}T \in \mathrm{Z}^{2\times}_{p,q,r},\ \widehat{\widetilde{T}}T \in \mathrm{Z}^{2\times}_{p,q,r}\},$$
$$\mathbf{Q}^{\overline{3}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \ \widetilde{T}T \in \mathrm{Z}^{3\times}_{p,q,r},\ \widehat{\widetilde{T}}T \in \mathrm{Z}^{3\times}_{p,q,r}\},$$
$$\mathbf{Q}^{\overline{0}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \ \widetilde{T}T \in \mathrm{Z}^{4\times}_{p,q,r},\ \widehat{\widetilde{T}}T \in \mathrm{Z}^{4\times}_{p,q,r}\},$$

$$\check{\mathbf{Q}}^{\overline{1}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \widetilde{T}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r},\ \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}\},$$
$$\check{\mathbf{Q}}^{\overline{2}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \widetilde{T}T \in \check{\mathrm{Z}}^{2\times}_{p,q,r},\ \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{2\times}_{p,q,r}\},$$
$$\check{\mathbf{Q}}^{\overline{3}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \widetilde{T}T \in \check{\mathrm{Z}}^{3\times}_{p,q,r},\ \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{3\times}_{p,q,r}\},$$
$$\check{\mathbf{Q}}^{\overline{0}}_{p,q,r} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \widetilde{T}T \in \langle \mathrm{Z}^{4}_{p,q,r}\rangle^{\times}_{(0)},\ \widehat{\widetilde{T}}T \in \langle \mathrm{Z}^{4}_{p,q,r}\rangle^{\times}_{(0)}\}.$$

Generalized Clifford and Lipschitz groups

Table 2: **Generalized Clifford and Lipschitz groups**

Lie group	$\psi(T)=\widetilde{T}T$	$\chi(T)=\widehat{\widetilde{T}}T$
$\mathbf{Q}^{\overline{1}}_{p,q,r}=\Gamma^{\overline{1}}_{p,q,r}$	$\mathrm{Z}^{1\times}_{p,q,r}$	$\mathrm{Z}^{1\times}_{p,q,r}$
$\mathbf{Q}^{\overline{2}}_{p,q,r}=\Gamma^{\overline{2}}_{p,q,r}=\tilde{\Gamma}^{\overline{2}}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$
$\mathbf{Q}^{\overline{3}}_{p,q,r}=\Gamma^{\overline{3}}_{p,q,r}$	$\mathrm{Z}^{3\times}_{p,q,r}$	$\mathrm{Z}^{3\times}_{p,q,r}$
$\mathbf{Q}^{\overline{0}}_{p,q,r}=\Gamma^{\overline{0}}_{p,q,r}=\tilde{\Gamma}^{\overline{0}}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{1}}_{p,q,r}=\check{\Gamma}^{\overline{1}}_{p,q,r}=\tilde{\Gamma}^{\overline{1}}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{2}}_{p,q,r}=\check{\Gamma}^{\overline{2}}_{p,q,r}$	$\check{\mathrm{Z}}^{2\times}_{p,q,r}$	$\check{\mathrm{Z}}^{2\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{3}}_{p,q,r}=\check{\Gamma}^{\overline{3}}_{p,q,r}=\tilde{\Gamma}^{\overline{3}}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{0}}_{p,q,r}=\check{\Gamma}^{\overline{0}}_{p,q,r}$	$\langle \mathrm{Z}^{4}_{p,q,r}\rangle^{\times}_{(0)}$	$\langle \mathrm{Z}^{4}_{p,q,r}\rangle^{\times}_{(0)}$

Generalized Clifford and Lipschitz groups

Table 2: **Generalized Clifford and Lipschitz groups**

Lie group	$\psi(T)=\widetilde{T}T$	$\chi(T)=\widehat{\widetilde{T}}T$
$\mathbf{Q}^{\overline{1}}_{p,q,r}=\Gamma^{\overline{1}}_{p,q,r}$	$\mathrm{Z}^{1\times}_{p,q,r}$	$\mathrm{Z}^{1\times}_{p,q,r}$
$\mathbf{Q}^{\overline{2}}_{p,q,r}=\Gamma^{\overline{2}}_{p,q,r}=\tilde{\Gamma}^{\overline{2}}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$	$\mathrm{Z}^{2\times}_{p,q,r}$
$\mathbf{Q}^{\overline{3}}_{p,q,r}=\Gamma^{\overline{3}}_{p,q,r}$	$\mathrm{Z}^{3\times}_{p,q,r}$	$\mathrm{Z}^{3\times}_{p,q,r}$
$\mathbf{Q}^{\overline{0}}_{p,q,r}=\Gamma^{\overline{0}}_{p,q,r}=\tilde{\Gamma}^{\overline{0}}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$	$\mathrm{Z}^{4\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{1}}_{p,q,r}=\check{\Gamma}^{\overline{1}}_{p,q,r}=\tilde{\Gamma}^{\overline{1}}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$	$\check{\mathrm{Z}}^{1\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{2}}_{p,q,r}=\check{\Gamma}^{\overline{2}}_{p,q,r}$	$\check{\mathrm{Z}}^{2\times}_{p,q,r}$	$\check{\mathrm{Z}}^{2\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{3}}_{p,q,r}=\check{\Gamma}^{\overline{3}}_{p,q,r}=\tilde{\Gamma}^{\overline{3}}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$	$\check{\mathrm{Z}}^{3\times}_{p,q,r}$
$\check{\mathbf{Q}}^{\overline{0}}_{p,q,r}=\check{\Gamma}^{\overline{0}}_{p,q,r}$	$\langle \mathrm{Z}^{4}_{p,q,r}\rangle^{\times}_{(0)}$	$\langle \mathrm{Z}^{4}_{p,q,r}\rangle^{\times}_{(0)}$

$$\mathbf{Z}^{1}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\mathcal{G}^n_{p,q,r}, & n \text{ is odd},\\ \Lambda^{(0)}_r, & n \text{ is even};\end{cases}\qquad \mathbf{Z}^{2}_{p,q,r}=\begin{cases}\Lambda_r\oplus\mathcal{G}^n_{p,q,r}, & r\neq n,\\ \Lambda_r, & r=n;\end{cases}$$

$$\mathbf{Z}^{3}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\Lambda^{n-2}_r\oplus\{\mathcal{G}^1_{p,q,0}(\Lambda^{n-3}_r\oplus\Lambda^{n-2}_r)\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-3}_r\}\oplus\mathcal{G}^n_{p,q,r}, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{\geq n-2}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-2}_r\}, & n \text{ is even};\end{cases}$$

$$\mathbf{Z}^{4}_{p,q,r}=\begin{cases}\Lambda_r\oplus\{\mathcal{G}^1_{p,q,0}(\Lambda^{n-3}_r\oplus\Lambda^{n-2}_r)\}\oplus\{\mathcal{G}^2_{p,q,0}(\Lambda^{n-4}_r\oplus\Lambda^{n-3}_r)\}\oplus\mathcal{G}^n_{p,q,r}, & r\neq n,\\ \Lambda_r, & r=n;\end{cases}$$

$$\check{\mathbf{Z}}^{1}_{p,q,r}=\Lambda_r,\qquad \check{\mathbf{Z}}^{3}_{p,q,r}=\Lambda_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{\geq n-2}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{\geq n-3}_r\},$$

$$\check{\mathbf{Z}}^{2}_{p,q,r}=\begin{cases}\Lambda^{(0)}_r\oplus\Lambda^n_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-1}_r\}, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-2}_r\}\oplus\mathcal{G}^n_{p,q,r}, & n \text{ is even},\quad r\neq n,\\ \Lambda^{(0)}_r\oplus\Lambda^{n-1}_r, & n \text{ is even},\quad r=n;\end{cases}$$

$$\langle\mathbf{Z}^{4}_{p,q,r}\rangle_{(0)}=\begin{cases}\Lambda^{(0)}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-2}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-3}_r\}, & n \text{ is odd},\\ \Lambda^{(0)}_r\oplus\{\mathcal{G}^1_{p,q,0}\Lambda^{n-3}_r\}\oplus\{\mathcal{G}^2_{p,q,0}\Lambda^{n-4}_r\}\oplus\mathcal{G}^n_{p,q,r}, & n \text{ is even},\quad r\neq n,\\ \Lambda^{(0)}_r, & n \text{ is even},\quad r=n.\end{cases}$$

Examples

Example 3.

In the **non-degenerate geometric algebras** $\mathcal{G}_{p,q,0}$,

If $n \neq 2, 3$, then

$$\mathbf{Q}^{\overline{1}}_{p,q,0} = \mathbf{Q}^{\overline{3}}_{p,q,0} = \mathrm{Q}, \quad \mathbf{Q}^{\overline{0}}_{p,q,0} = \mathbf{Q}^{\overline{2}}_{p,q,0} = \begin{cases} \mathrm{Q}, & n = 1,2,3 \bmod 4, \\ \mathrm{Q}', & n = 0 \bmod 4; \end{cases}$$

If $n = 2, 3$, then $\mathbf{Q}^{\overline{1}}_{p,q,0} = \mathbf{Q}^{\overline{2}}_{p,q,0} = \mathrm{Q}$;

If $n \neq 1, 2$, then

$$\check{\mathbf{Q}}^{\mathbf{1}}_{p,q,0} = \check{\mathbf{Q}}^{\mathbf{3}}_{p,q,0} = \mathrm{Q}^{\pm}, \quad \check{\mathbf{Q}}^{\overline{0}}_{p,q,0} = \check{\mathbf{Q}}^{\overline{2}}_{p,q,0} = \begin{cases} \mathrm{Q}^{\pm}, & n = 1,2,3 \bmod 4, \\ \mathrm{Q}', & n = 0 \bmod 4; \end{cases}$$

If $n = 1, 2$, then $\check{\mathbf{Q}}^{\overline{1}}_{p,q,0} = \check{\mathbf{Q}}^{\overline{0}}_{p,q,0} = \mathrm{Q}^{\pm}$;

where

$$\mathrm{Q} := \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widetilde{T}T \in \mathrm{Z}^{\times}_{p,q,0}, \quad \widehat{\widetilde{T}}T \in \mathrm{Z}^{\times}_{p,q,0}\}- = \mathrm{A} \cap \mathrm{B},$$

$$\mathrm{Q}' := \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widetilde{T}T \in (\mathcal{G}^0 \oplus \mathcal{G}^n_{p,q,0})^{\times}, \quad \widehat{\widetilde{T}}T \in (\mathcal{G}^0 \oplus \mathcal{G}^n_{p,q,0})^{\times}\},$$

$$\mathrm{Q}^{\pm} := \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \widetilde{T}T \in \mathcal{G}^{0\times}, \quad \widehat{\widetilde{T}}T \in \mathcal{G}^{0\times}\} = \mathrm{A}_{\pm} \cap \mathrm{B}_{\pm},$$

and

$$\mathrm{Z}_{p,q,0} = \begin{cases} \mathcal{G}^0 \oplus \mathcal{G}^n_{p,q,0}, & n \text{ is odd}, \\ \mathcal{G}^0, & n \text{ is even}. \end{cases}$$

Shirokov D.: On inner automorphisms preserving fixed subspaces of Clifford algebras.
Adv. Appl. Clifford Algebras 31(30), (2021)

Filimoshina E., Shirokov D.: On generalization of Lipschitz groups and spin groups.
Mathematical Methods in the Applied Sciences, 47(3), 1375–1400 (2024)

4. Degenerate Lipschitz and spin groups and generalized degenerate spin groups

Degenerate Lipschitz groups

Consider the well-known **Lipschitz groups**, which are defined in the non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$ as:

$$\boldsymbol{\Gamma}^{\pm\Lambda}_{\boldsymbol{p,q,0}} := \{T \in \mathcal{G}^{\times}_{p,q,0} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,0}) = \tilde{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,0}) := \widehat{T}\mathcal{G}^1_{p,q,0}T^{-1} \subseteq \mathcal{G}^1_{p,q,0}\}. \quad (1)$$

Similarly, in arbitrary $\mathcal{G}_{p,q,r}$, they can be defined as:

$$\boldsymbol{\Gamma}^{\pm\Lambda}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \check{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,r}) = \tilde{\mathrm{ad}}_T(\mathcal{G}^1_{p,q,r}) := \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\}. \quad (2)$$

Brooke J.: A Galileian formulation of spin. I. Clifford algebras and spin groups.
J. Math. Phys., volume 19 (1978)

Brooke J.: Spin Groups Associated with Degenerate Orthogonal Spaces.
Clifford Algebras and Their Applications in Mathematical Physics, Part of the NATO ASI Series, volume 183 (1986)

Crumeyrolle A.: Orthogonal and Symplectic Clifford Algebras.
1st edition. Springer, Netherlands, 1990.

Degenerate Lipschitz groups

The **Lipschitz group**:

$$\boldsymbol{\Gamma}^{\boldsymbol{\pm\Lambda}}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\}.$$

The upper index $^{\pm\Lambda}$ is due to the equivalent definition that we prove using Theorems 6 and 7:

$$\boldsymbol{\Gamma}^{\boldsymbol{\pm\Lambda}}_{\boldsymbol{p,q,r}} = \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\}.$$

Degenerate Lipschitz groups

The upper index $^{\pm\Lambda}$ is due to the equivalent definition that we prove using Theorems 6 and 7:

$$\mathbf{\Gamma}^{\pm\Lambda}_{p,q,r} = \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{T}\mathcal{G}^{1}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{1}_{p,q,r}\}. \qquad (1)$$

Theorem 6. The generalized and ordinary Lipschitz groups $\check{\mathbf{Q}}^{\overline{\mathbf{1}}}_{p,q,r}$ and $\mathbf{\Gamma}^{\pm\Lambda}_{p,q,r}$ resectively are related in the following way:

$$\mathbf{\Gamma}^{\pm\Lambda}_{p,q,r} \subseteq \check{\mathbf{Q}}^{\overline{\mathbf{1}}}_{p,q,r}, \quad \forall n; \qquad \mathbf{\Gamma}^{\pm\Lambda}_{p,q,r} = \check{\mathbf{Q}}^{\overline{\mathbf{1}}}_{p,q,r}, \quad n \leq 4.$$

Theorem 7. We have the following inclusion:

$$\check{\mathbf{Q}}^{\overline{\mathbf{1}}}_{p,q,r} \subseteq (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r}$$

Filimoshina E., Shirokov D.: On Some Lie Groups in Degenerate Clifford Geometric Algebras. Advances in Applied Clifford Algebras, 33(44) (2023)

$$\check{\mathbf{Q}}^{\overline{\mathbf{1}}}_{p,q,r} = \{T \in \mathcal{G}^{\times}_{p,q,r} : \widehat{T}\mathcal{G}^{\overline{1}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{1}}_{p,q,r}\} \qquad (2)$$

$$= \{T \in \mathcal{G}^{\times}_{p,q,r} : \widetilde{T}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}\} \qquad (3)$$

Degenerate Lipschitz groups

The **Lipschitz group**:

$$\Gamma^{\pm\Lambda}_{p,q,r} = \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{T}\mathcal{G}^{1}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{1}_{p,q,r}\}.$$

We also consider the subgroup $\Gamma^{\pm}_{p,q,r}$ of the Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$:

$$\Gamma^{\pm}_{p,q,r} := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r} : \quad T\mathcal{G}^{1}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{1}_{p,q,r}\} \subseteq \Gamma^{\pm\Lambda}_{p,q,r}.$$

Ruhe D., Brandstetter J., and Forré P.: Clifford Group Equivariant Neural Networks. 2023, arXiv:2305.11141.

In the case of the non-degenerate geometric algebra $\mathcal{G}_{p,q,0}$, these groups coincide:

$$\Gamma^{\pm}_{p,q,0} = \Gamma^{\pm\Lambda}_{p,q,0}.$$

In the particular case of the Grassmann algebra $\mathcal{G}_{0,0,n} = \Lambda_n$,

$$\Gamma^{\pm}_{0,0,n} = \Lambda^{(0)\times}_{n} \subset \Gamma^{\pm\Lambda}_{0,0,n} = \Lambda^{\times}_{n} = \ker(\tilde{\mathrm{ad}}).$$

Degenerate Lipschitz groups

The **Lipschitz group**:

$$\boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p,q,r}} = \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{T}\mathcal{G}^{1}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{1}_{p,q,r}\}.$$

We also consider the subgroup $\boldsymbol{\Gamma}^{\pm}_{\boldsymbol{p,q,r}}$ of the Lipschitz group $\boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p,q,r}}$:

$$\boldsymbol{\Gamma}^{\pm}_{\boldsymbol{p,q,r}} := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r} : \quad T\mathcal{G}^{1}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{1}_{p,q,r}\} \subseteq \boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p,q,r}}.$$

Ruhe D., Brandstetter J., and Forré P.: Clifford Group Equivariant Neural Networks. 2023, arXiv:2305.11141.

Theorem 8. The Lipschitz group $\boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p,q,r}}$ can be represented as a product of the groups:

$$\boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p,q,r}} = \boldsymbol{\Gamma}^{\pm}_{\boldsymbol{p,q,r}}\Lambda^{\times}_{r}.$$

Degenerate spin groups

We define **the ordinary degenerate spin groups** as normalized subgroups of the Lipschitz group

$$\boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} = \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_r : \quad \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\} \qquad (1)$$

and its even subgroup

$$\boldsymbol{\Gamma}^{+}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad T\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\} \subseteq \boldsymbol{\Gamma}^{\pm}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} \qquad (2)$$

in the following way:

$$\mathbf{Pin}_{\psi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widetilde{T}T = \pm e\}, \quad \mathbf{Pin}_{\chi}(p,q,r) := \{T \in \boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widehat{\widetilde{T}}T = \pm e\}, \qquad (3)$$

$$\mathbf{Pin}_{+\psi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widetilde{T}T = +e\}, \quad \mathbf{Pin}_{+\chi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \boldsymbol{\Gamma}^{\pm\boldsymbol{\Lambda}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widehat{\widetilde{T}}T = +e\}, \qquad (4)$$

$$\mathbf{Spin}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \boldsymbol{\Gamma}^{+}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widetilde{T}T = \pm e\} = \{T \in \boldsymbol{\Gamma}^{+}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widehat{\widetilde{T}}T = \pm e\}, \qquad (5)$$

$$\mathbf{Spin}_{+}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \boldsymbol{\Gamma}^{+}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widetilde{T}T = +e\} = \{T \in \boldsymbol{\Gamma}^{+}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} : \quad \widehat{\widetilde{T}}T = +e\}. \qquad (6)$$

Generalized degenerate spin groups

In a similar way, we can define **the generalized degenerate spin groups** as normalized subgroups of the generalized degenerate Lipschitz group

$$\check{\mathbf{Q}}^{\overline{1}}_{p,q,r} = \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widehat{T}\mathcal{G}^{\overline{1}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{1}}_{p,q,r}\} \qquad (1)$$

$$= \{T \in \mathcal{G}^{\times}_{p,q,r} : \quad \widetilde{T}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}, \quad \widehat{\widetilde{T}}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}\} \qquad (2)$$

$$= \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widetilde{T}T \in \check{\mathrm{Z}}^{1\times}_{p,q,r}\} \qquad (3)$$

and its even subgroup $\mathcal{G}^{(0)\times}_{p,q,r}$:

$$\mathbf{Pin}^{\mathbf{Q}}_{\psi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widetilde{T}T = \pm e\}, \qquad (4)$$

$$\mathbf{Pin}^{\mathbf{Q}}_{\chi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{\widetilde{T}}T = \pm e\}, \qquad (5)$$

$$\mathbf{Pin}^{\mathbf{Q}}_{+\psi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widetilde{T}T = +e\}, \qquad (6)$$

$$\mathbf{Pin}^{\mathbf{Q}}_{+\chi}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{\widetilde{T}}T = +e\}, \qquad (7)$$

$$\mathbf{Spin}^{\mathbf{Q}}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widetilde{T}T = \pm e\} = \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T = \pm e\}, \qquad (8)$$

$$\mathbf{Spin}^{\mathbf{Q}}_{+}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widetilde{T}T = +e\} = \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T = +e\}. \qquad (9)$$

Conclusions

In this work, we consider the **generalized Clifford and Lipschitz groups**

$$\Gamma^{\overline{k}}_{p,q,r},\quad \check{\Gamma}^{\overline{k}}_{p,q,r},\quad \tilde{\Gamma}^{\overline{k}}_{p,q,r},\quad \Gamma^{\overline{kl}}_{p,q,r},\quad \check{\Gamma}^{\overline{kl}}_{p,q,r},\quad \tilde{\Gamma}^{\overline{kl}}_{p,q,r} \qquad (1)$$

preserving the subspaces $\mathcal{G}^{\overline{k}}_{p,q,r}$, $k=0,1,2,3$, determined by the grade involution and the reversion and their direct sums $\mathcal{G}^{\overline{kl}}_{p,q,r}$, $k,l=0,1,2,3$, under the adjoint representation **ad** and twisted adjoint representations $\check{\mathbf{ad}}$ and $\tilde{\mathbf{ad}}$.

We prove that these groups are defined using the norm functions ψ and χ and the centralizers $\mathbf{Z}^m_{p,q,r}$ and twisted centralizers $\check{\mathbf{Z}}^m_{p,q,r}$ of the subspaces of fixed grades.

$$\Gamma^{\overline{01}}_{p,q,r} = \mathbf{A}^{\overline{01}}_{p,q,r} \subseteq \Gamma^{\overline{23}}_{p,q,r} = \mathbf{A}^{\overline{23}}_{p,q,r},\quad \check{\Gamma}^{\overline{12}}_{p,q,r} = \check{\mathbf{A}}^{\overline{12}}_{p,q,r},\quad \check{\Gamma}^{\overline{03}}_{p,q,r} = \check{\mathbf{A}}^{\overline{03}}_{p,q,r},\quad \tilde{\Gamma}^{\overline{01}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{01}}_{p,q,r},\quad \tilde{\Gamma}^{\overline{23}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{23}}_{p,q,r},$$

$$\Gamma^{\overline{12}}_{p,q,r} = \mathbf{B}^{\overline{12}}_{p,q,r} \subseteq \Gamma^{\overline{03}}_{p,q,r} = \mathbf{B}^{\overline{03}}_{p,q,r}.\quad \check{\Gamma}^{\overline{01}}_{p,q,r} = \check{\mathbf{B}}^{\overline{01}}_{p,q,r} \subseteq \check{\Gamma}^{\overline{23}}_{p,q,r} = \check{\mathbf{B}}^{\overline{23}}_{p,q,r}.\quad \tilde{\Gamma}^{\overline{12}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{12}}_{p,q,r} \subseteq \tilde{\Gamma}^{\overline{03}}_{p,q,r} = \tilde{\mathbf{Q}}^{\overline{03}}_{p,q,r}.$$

$$\Gamma^{\overline{1}}_{p,q,r} = \mathbf{Q}^{\overline{1}}_{p,q,r},\quad \Gamma^{\overline{2}}_{p,q,r} = \mathbf{Q}^{\overline{2}}_{p,q,r},\quad \Gamma^{\overline{3}}_{p,q,r} = \mathbf{Q}^{\overline{3}}_{p,q,r},\quad \Gamma^{\overline{0}}_{p,q,r} = \mathbf{Q}^{\overline{0}}_{p,q,r},$$

$$\check{\Gamma}^{\overline{1}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{1}}_{p,q,r} \subseteq \check{\Gamma}^{\overline{3}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{3}}_{p,q,r},\quad \check{\Gamma}^{\overline{2}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{2}}_{p,q,r},\quad \check{\Gamma}^{\overline{0}}_{p,q,r} = \check{\mathbf{Q}}^{\overline{0}}_{p,q,r}.$$

The generalized Clifford and Lipschitz groups contain the **ordinary Clifford and Lipschitz groups** as subgroups and are closely related to the **degenerate spin groups**.

The groups (1) are useful for the study of the **generalized degenerate spin groups**, and that is why they are interesting for consideration.

Thank you!

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