GEOMETRIC ALGEBRA IN SWITCHED SYSTEMS CONTROL

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What is all the story about?

The controllability of 2×2 switched systems with regular matrices is investigated by means of Geometric Algebra for Conics (GAC). The research demonstrates the efficiency of GAC in the construction of switching points and paths while minimizing the number of switches and numerical errors.

Given a real vector space V equipped with an quadratic form $Q(\mathbf{v})$, the Clifford algebra $\mathcal{C}l(V)$ associated with V is constructed as the quotient space:

$$\mathcal{C}l(V) = T(V)/I,$$

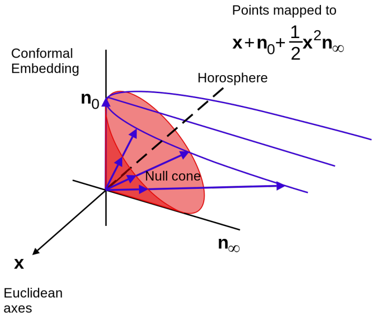
where $T(V)$ is the tensor algebra of V and I is the ideal generated by $v \otimes v - Q(\mathbf{v})\mathbf{1}$ for all $v \in V$, where $\mathbf{1}$ denotes the scalar element of the basis of $T(V)$. The outer-product is anti-commutative and associative giving $a \wedge b = -(b \wedge a)$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge b \wedge c$.

Geometric Algebra

Geometric Algebra (GA) is a Clifford algebra with a specific embedding of a Euclidean space in such a way that the intrinsic geometric primitives as well as their transformations are viewed as elements of a single vector space, precisely multivectors.
Three-dimensional Euclidean space is represented in a Clifford algebra $\mathcal{C}l(4,1)$, and the consequent geometric algebra is often denoted as $\mathbb{G}_{4,1}$ with spheres of all types as geometric primitives and Euclidean transformations. The generalisation of $\mathbb{G}_{4,1}$ is Geometric Algebra for Conics (GAC), proposed by C. Perwass.

Compass Ruler Algebra (CRA)

Notation	Meaning
e_1, e_2 e_0	2D basis vectors origin
e_∞	infinity
AB	geometric product of A and B
$A \wedge B$	outer product of A and B
$A \dot{B}$	inner product of A and B
A^*	dual of A
A^{-1}	inverse of A



Geometric Algebra for Conics (C.Perwass)

Geometric Algebra for Conics (GAC) is the Clifford algebra $\mathbb{G}_{5,3}$ together with the embedding of point $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$, given by the operator $C : \mathbb{R}^2 \to \mathcal{C} \subset \mathbb{R}^{5,3}$, which is defined by

$$C(x, y) = \bar{n}_+ + xe_1 + ye_2 + \frac{1}{2}(x^2 + y^2)n_+ + \frac{1}{2}(x^2 - y^2)n_- + xyn_\times . \quad (1)$$

in the basis

$$\bar{n}_+, \bar{n}_-, \bar{n}_\times, e_1, e_2, n_+, n_-, n_\times . \quad (2)$$

Notation	Meaning
e_1, e_2	2D basis vectors
$\bar{n}_+, \bar{n}_-, \bar{n}_\times$	origins
$n_+, n_-, n_\times$	infinities
AB	geometric product of A and B
$A \wedge B$	outer product of A and B
$A \cdot B$	inner product of A and B
A^*	dual of A
A^{-1}	inverse of A

Table: Notations of Geometric Algebra for Conics

If a point P lies on a conic C then

$$P \cdot E = 0 \quad \text{and} \quad P \wedge E^* = 0.$$

Conic C is seen as:

- 5-vector $E^* = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5$, called an **outer product null space representation (OPNS)**
- 1-vector $E = \bar{v}^+ \bar{n}_+ + \bar{v}^- \bar{n}_- + \bar{v}^\times \bar{n}_\times + v^1 e_1 + v^2 e_2 + v^+ n_+.$, called **the inner product null space (IPNS) representation**.

It is well known, that the type of a given unknown conic can be read off its matrix representation Q, which in our case reads

$$Q = \begin{pmatrix} -\frac{1}{2}(\bar{v}^{+} + \bar{v}^{-}) & -\frac{1}{2}\bar{v}^{\times} & \frac{1}{2}v^{1} \\ -\frac{1}{2}\bar{v}^{\times} & -\frac{1}{2}(\bar{v}^{+} - \bar{v}^{-}) & \frac{1}{2}v^{2} \\ \frac{1}{2}v^{1} & \frac{1}{2}v^{2} & -v^{+} \end{pmatrix}.$$

The entries can be easily computed by means of the inner product:

$$
\begin{aligned}
q_{11} &= Q_I \cdot \tfrac{1}{2}(n_+ - n_-),\\
q_{22} &= Q_I \cdot \tfrac{1}{2}(n_+ + n_-),\\
q_{33} &= Q_I \cdot \bar{n}_+,\\
q_{12} = q_{21} &= Q_I \cdot \tfrac{1}{2} n_\times,\\
q_{13} = q_{31} &= Q_I \cdot \tfrac{1}{2} e_1,\\
q_{23} = q_{32} &= Q_I \cdot \tfrac{1}{2} e_2.
\end{aligned}
$$

Example

Let us consider the IPNS representation of the axis-aligned ellipse with the semi-axes $a = 2, b = 4$ centred in $(u, v) = (0, 2)$:

$$E = 0.8e_2 - 0.5e_3 - 0.3e_4 + 0.5e_6 + 0.3e_7$$

The rotor for a rotation around the origin by the angle $\frac{\pi}{3}$ is given by $R = R_+(R_1 \wedge R_2)$, where

$$R_+ = \cos(\tfrac{\pi}{6}) + \sin(\tfrac{\pi}{6})e_1 \wedge e_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}e_1 \wedge e_2, \tag{3}$$

$$R_1 = \cos(\tfrac{\pi}{3}) + \sin(\tfrac{\pi}{3})\bar{n}_\times \wedge n_- = \frac{1}{2} + \frac{\sqrt{3}}{2}\bar{n}_\times \wedge n_-, \tag{4}$$

$$R_2 = \cos(\tfrac{\pi}{3}) - \sin(\tfrac{\pi}{3})\bar{n}_- \wedge n_\times = \frac{1}{2} - \frac{\sqrt{3}}{2}\bar{n}_- \wedge n_\times. \tag{5}$$

Rotated ellipse has equation:

$$E_{rotated} = -0.69282e_1 + 0.4e_2 - 0.5e_3 +$$

$$0.15e_4 - 0.25981e_5 + 0.5e_6 - 0.15e_7 + 0.25981e_8.$$

The scalor for scaling a conic by $\alpha \in \mathbb{R}^+$ is given by $S = S_+ S_- S_\times$, where

$$S_+ = \frac{\alpha+1}{2\sqrt{\alpha}} + \frac{\alpha-1}{2\sqrt{\alpha}} \bar{n}_+ \wedge n_+,$$

$$S_- = \frac{\alpha+1}{2\sqrt{\alpha}} + \frac{\alpha-1}{2\sqrt{\alpha}} \bar{n}_- \wedge n_-,$$

$$S_\times = \frac{\alpha+1}{2\sqrt{\alpha}} + \frac{\alpha-1}{2\sqrt{\alpha}} \bar{n}_\times \wedge n_\times.$$

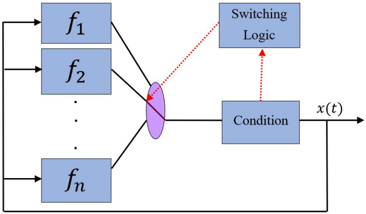
$$E_{scaled} = S_+ S_- S_\times E \bar{S}_\times \bar{S}_- \bar{S}_+.$$

Intersections and contact points

Introduction to switched systems

By a switched system we mean the following system

$$\dot{\boldsymbol{x}}(t) = f_{\sigma(t)}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$



where $\boldsymbol{x} \in \mathbb{R}^m$ is *a continuous state*, σ stands for *a discrete state* with values from an index set $M := \{1, \ldots, n\}$, and $f_{\sigma(t)}$, $\sigma(t) \in M$, are the vector fields.

Definition

We say that the switched system

$$\dot{\boldsymbol{x}}(t) = f_{\sigma(t)}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

is *controllable* if for any two points A, B there exists a switching signal generating a continuous path from A to B.

The above definition corresponds to the concept of controllability for control systems of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

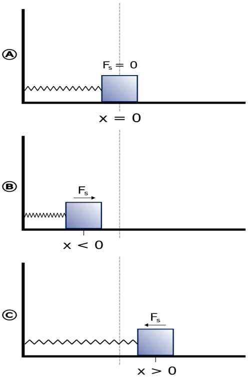
where the control $\boldsymbol{u}(t)$ plays the role of a switching signal.

The problem of oscillation of a spring pendulum under the condition of absence of external and friction forces

$$\ddot{\boldsymbol{x}} = -k\boldsymbol{x},$$

with a switchable stiffness coefficient $k > 0$, that changes value from k_1 by joining and removing an additional spring with a stiffness coefficient k_2.

$$\dot{\boldsymbol{x}}(t) = A_i \boldsymbol{x}(t), \quad A_i \in \mathcal{M}at_2(\mathbb{R}), \quad i = 1, 2.$$



Linear switched systems

We are interested in linear switched systems

$$\dot{\boldsymbol{x}}(t) = A_{\sigma(t)}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \neq 0,$$

where $A_1 \dots A_n$ are given 2×2 matrices with both subsystems having purely imaginary eigenvalues.

Goal: Algorithm for constructing a controlling switching signal.

Tool: Geometric Algebra for Conics(GAC)

Table: Classification of equilibrium points in the case when $\det A \neq 0$

Roots of characteristic Equation	Point Type	Trajectory
λ_1, λ_2 are real numbers of the same sign $\lambda_1\lambda_2 > 0$	Node	Parabola
λ_1, λ_2 are real numbers of the opposite sign $\lambda_1\lambda_2 < 0$	Saddle	Hyperbola
λ_1, λ_2 are complex numbers $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 \neq 0$	Focus	Spiral
λ_1, λ_2 are complex numbers $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 = 0$	Center	Ellipse

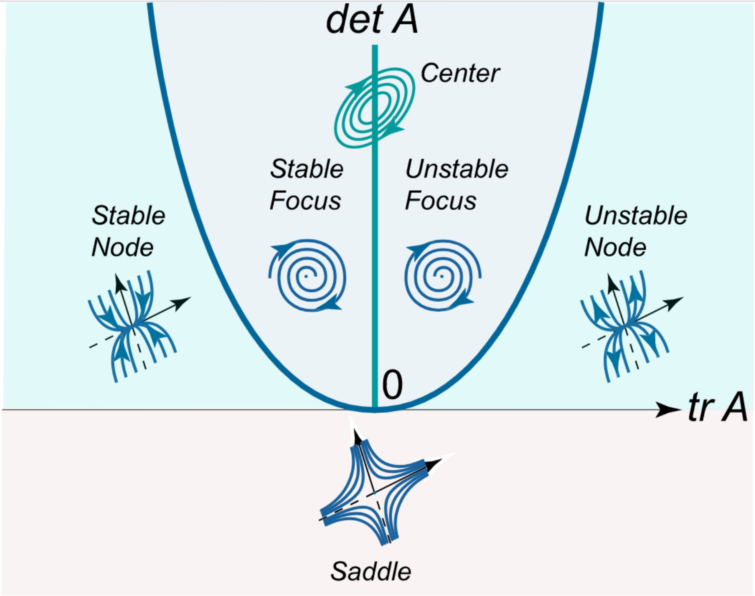
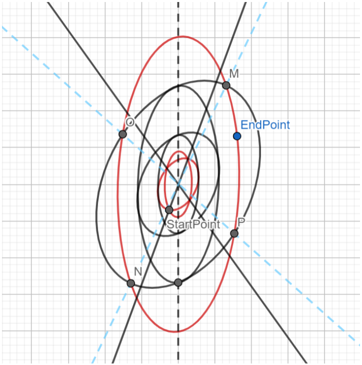


Figure: Poincare diagram

Algorithm Algorithm for a switching path construction

1: Get A, B, the starting and final point, respectively, i.e. get their conformal embedding $\mathcal{C}(A), \mathcal{C}(B)$ to GAC.
2: Find the IPNS representation of the initial ellipse E_1^1 by conic fitting algorithm. Denote its semiaxis by a and b.
3: Find the final ellipse E_f.
4: Find the first intermediate ellipse E_2^1. Note that lower index indicates, to which system the ellipse belongs.
5: If $E_f \cap E_2^i \neq \emptyset$ then find the intersection points of all ellipses, get the path from A to B by choosing the nearest point with respect to the path evolution. This will switch to final ellipse.
6: If $E_f \cap E_2^i = \emptyset$ then calculate the scaling parameter SP. By scaling E_1^i, E_2^i using SP get new pair of circumscribed ellipses

$$E_2^{i+1} := \text{scale}(E_2^i, SP), \quad E_1^{i+1} := \text{scale}(E_1^i, SP).$$



Algorithm Final ellipse construction

Inputs: Two co–centric ellipses, where one is the scaled copy of another

Output: Final ellipse E_f

1: Construct a line l passing through the points $\bar{n}^+$ and e_2:

$$l = e_2 \wedge n_+ \wedge \bar{n}_+ \wedge n_- \wedge n_\times .$$

2: Find the intersection points $C = E_1 \cap l$, $B = E_2 \cap l$ of the line and both ellipses, i.e. solve a quadratic equation in a Euclidean space.

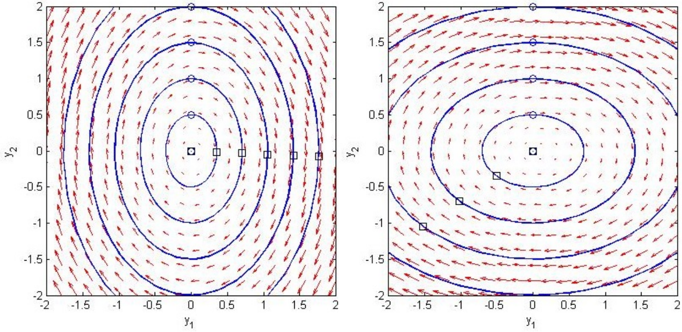
3: The scale parameter between ellipses is

$$SP = \frac{|\bar{n}_+ \cdot \mathcal{C}(B)|}{|\bar{n}_+ \cdot \mathcal{C}(C)|}.$$

4: Construct final ellipse: $E_f = S_+ S_- S_\times E_1^1 \bar{S}_\times \bar{S}_- \bar{S}_+$.

Preparation and assumptions

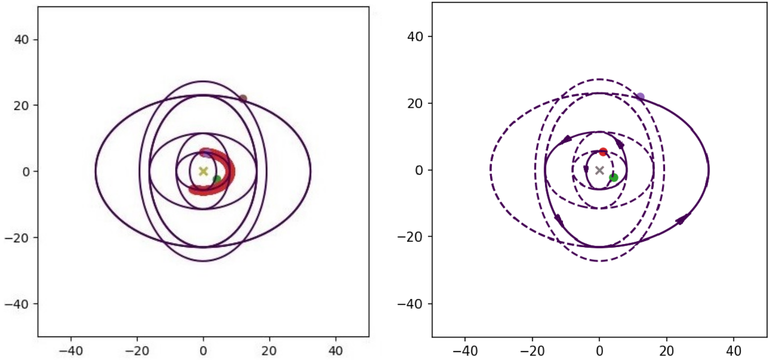
- Solve ODE numerically (e.g. by Runge-Kutta method).
- Start and end with same family of ellipses.



Examples

Example (Oscillatory system without damping)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where $A_1 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$. The initial point is $(2, 5)$, and we need to find the way to the point $(12, 22)$



Examples

Example (Oscillatory system with damping)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where $A_1 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$.
The initial point is $(2, 5)$, and we need to find the way to the point $(30, 22)$

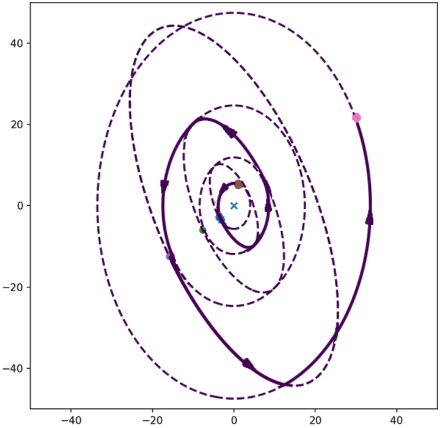


Table: Controllability of the switched 2x2 systems

A_1 | A_2		Center	Saddle	Node Stable	Node Unstable	Focus Stable	Focus Unstable
Center		+					
Saddle							
Node	Stable						
	Unstable						
Focus	Stable						
	Unstable						

Examples

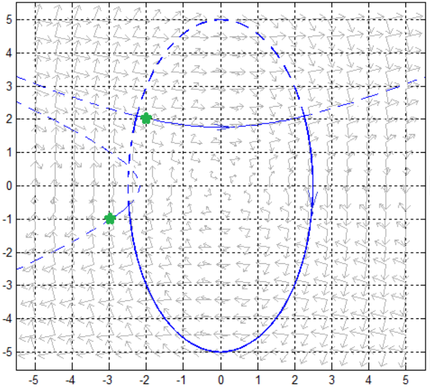
Example (System with Saddle and Centrum type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}.$$

Starting point (-2;2), ending point (-3;-1)



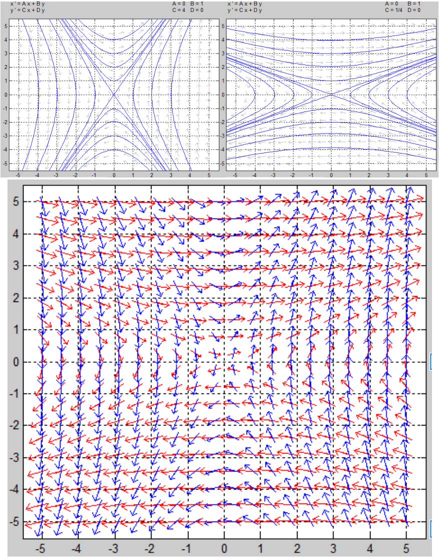
Examples

Example (System with Saddle and Saddle type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}.$$

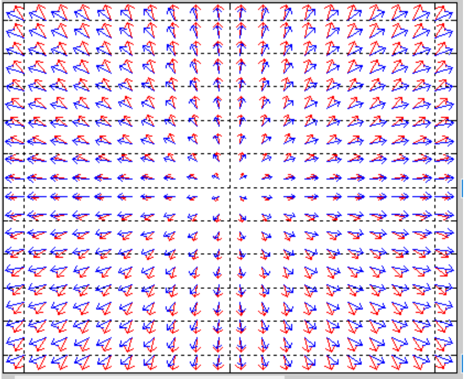


Example (System with both Unstable Node type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}.$$

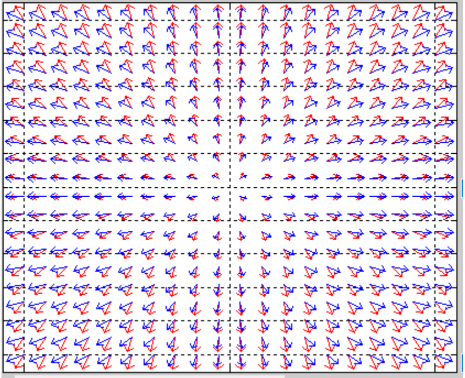


Example (System with both Unstable Node type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}.$$

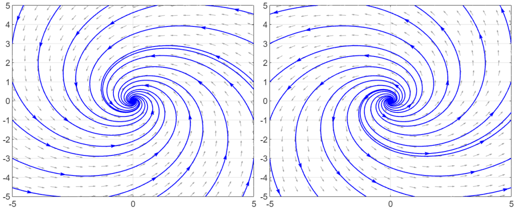


Example (System with both Stable Focus type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} -1 & \alpha \\ -\alpha & -1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & \frac{1}{\alpha} \\ -\frac{1}{\alpha} & -1 \end{pmatrix}.$$



Examples

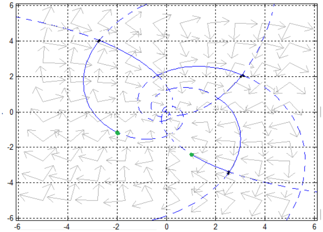
Example (System with Stable and Unstable Focus type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & \frac{1}{\alpha} \\ -\frac{1}{\alpha} & -1 \end{pmatrix}$$

.



Controlled under condition

Example (Saddle and Stable Focus type subsystems)

Consider the switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$ where

$$A_1 = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & \frac{1}{\alpha} \\ -\frac{1}{\alpha} & -1 \end{pmatrix}.$$

where $\alpha > 1$, the system will be not controlled (not possible to reach 2 and 4 quadrants). When $0 < \alpha < 1$, we have controllability outside the neighbourhood of zero.

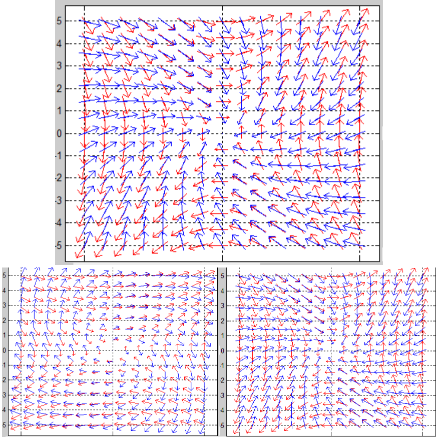
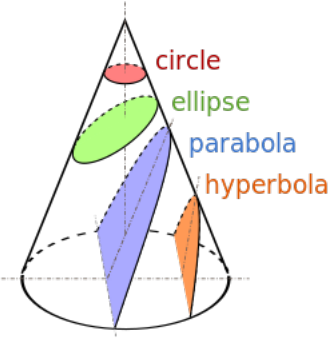


Table: Controllability of the switched 2x2 systems

$A_1 \mid A_2$		**Center**	**Saddle**	**Node**		**Focus**	
				Stable	**Unstable**	**Stable**	**Unstable**
Center		+	+	-	-	-	-
Saddle		+	-	-	-	Controlled under condition	Controlled under condition
Node	**Stable**	-	-	-	-	-	-
	Unstable	-	-	-	-	-	-
Focus	**Stable**	-	Controlled under condition	-	-	-	+
	Unstable	-	Controlled under condition	-	-	+	-

Why GAC?

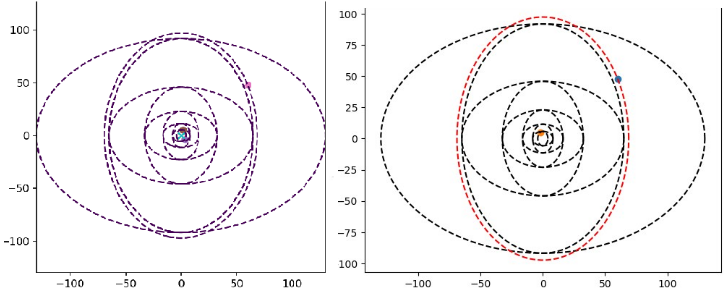


Reasons to use GAC:

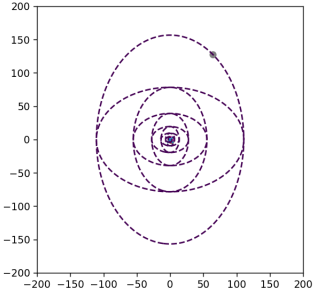
- simple representation of transformed objects;
- circumscribed conic construction;
- there is no need for a solver;
- minimization of numerical errors

Numerical error increases with the increasing number of the intermediate ellipses

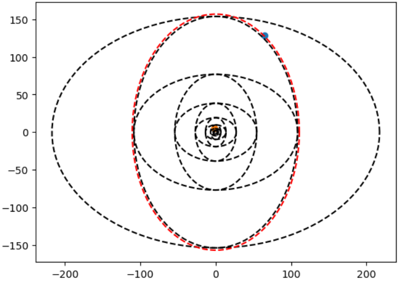
NUMERIC [0.0, 5.744562646461688], [8.12404, 0.0], [0.0, 11.48913], [16.24808, 0.0], [0.0, 22.97825], [32.49615, 0.0], [0.0, 45.9565], [64.99231, 0.0], [0.0, 91.913], FNUM=[-101.93109159592173, 29.564380018122797]
GAC [0.0, 5.74543306], [8.12526935, 0.0, [0.0, 11.49086611], [16.2505387, 0.0], [0.0, 22.98173222], [32.5010774, 0.0], [0.0, 45.96346444], [65.00215479, 0.0], [0.0, 91.92692889], FGAC=[-89.9967425686796,26.5008143923407]



In some cases even the number of switches can be different.



(a) GAC solution



(b) Numeric solution

THANK YOU FOR ATTENTION!

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