Pencils of 3D CGA and set operators

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In short

I am representing 3D CGA objects as pencils of spheres, which are linear space of spheres, and introduce set operators on these set of spheres.

Algebraic curves

- Primitives of geometric algebra are usualy algebraic hypersurfaces
- The spheres of 3D CGA can be understood in that mindset, as vectors

Algebraic equation of a sphere S

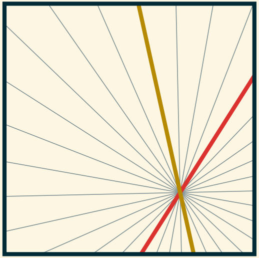
$$Eq_S : (\boldsymbol{x} - \boldsymbol{x}_c)^2 - r^2 = 0 \qquad \boldsymbol{x} \in \mathbb{R}^3 \tag{1}$$

$$: S \cdot g(\boldsymbol{x}) = 0 \qquad g(\boldsymbol{x}) = [\|\boldsymbol{x}\|^2, \boldsymbol{x}, 1]^\top \in \mathbb{R}^5 \tag{2}$$

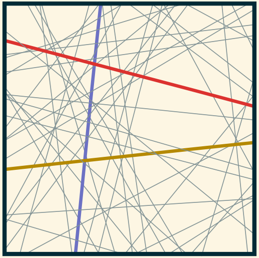
Pencils

As algebraic hypersurfaces can be represented as vectors, it is possible to build vector spaces of hypersurfaces. We call these vector spaces k-pencils, with k the dimension of the vector space.

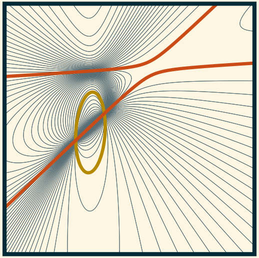
$$\mathrm{Pencil}\,(S_1, \ldots, S_k) = \left\{ \sum_{i=1}^{k} \lambda_i S_i \;\middle|\; \forall i \in [1, k], \lambda_i \in \mathbb{R} \right\} \tag{3}$$



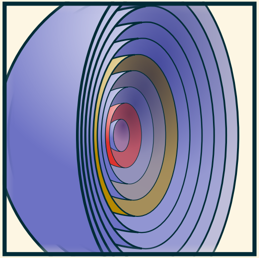
(a) A 2-pencil of lines



(b) A 3-pencil of lines

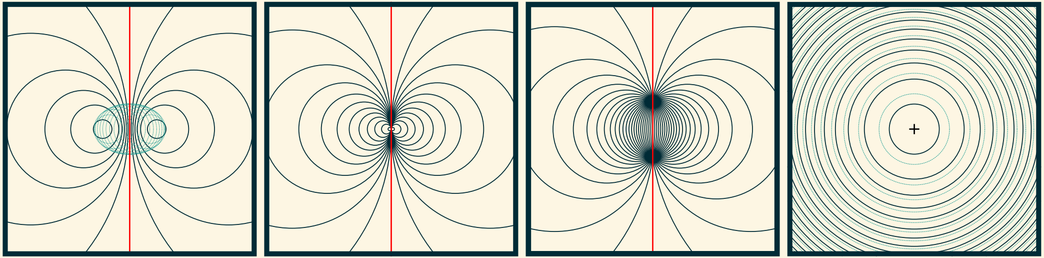


(c) A 2-pencil of conics

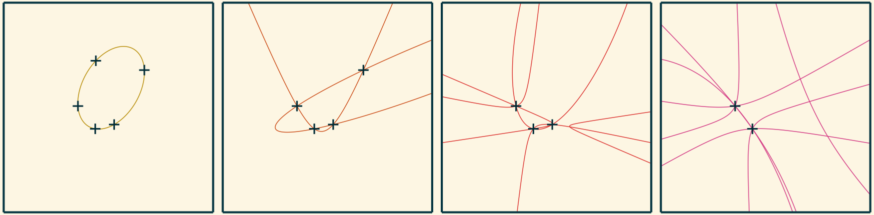


(d) A 2-pencil of concentric spheres

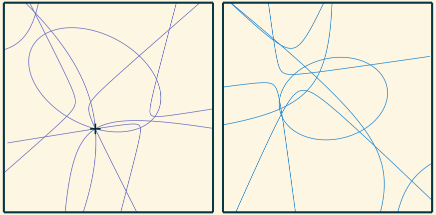
More pencils (of circles)



Even more pencils (of conics)



(a) 1-pencil (b) 2-pencil (c) 3-pencil (d) 4-pencil



(e) 5-pencil (f) 6-pencil

Curves and Pencils of 3D CGA

Points and Spheres

$$\mathrm{p} = e_o + xe_1 + ye_2 + ze_3 + \frac{x^2 + y^2 + z^2}{2} e_\infty \quad (4)$$

$$S = \left(\mathrm{p}_c - \frac{r^2}{2} e_\infty\right)^* = \mathrm{p}_1 \wedge \mathrm{p}_2 \wedge \mathrm{p}_3 \wedge \mathrm{p}_4 \quad (5)$$

The $(5 - n)$-blades of CGA are n-pencils of spheres

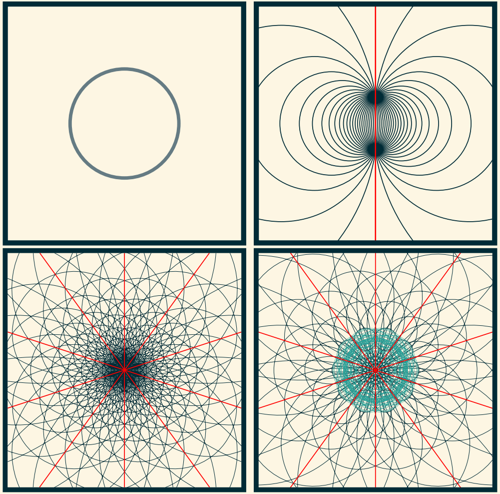
$$S_1 = \mathrm{p}_1 \wedge \mathrm{p}_2 \wedge \mathrm{p}_3 \wedge \mathrm{p}_4 = \text{1-pencil of spheres} \quad (6)$$

$$S_1 \vee S_2 = \mathrm{p}_1 \wedge \mathrm{p}_2 \wedge \mathrm{p}_3 = \text{2-pencil of spheres} \quad (7)$$

$$S_1 \vee S_2 \vee S_3 = \mathrm{p}_1 \wedge \mathrm{p}_2 = \text{3-pencil of spheres} \quad (8)$$

$$S_1 \vee S_2 \vee S_3 \vee S_4 = \mathrm{p}_1 = \text{4-pencil of spheres} \quad (9)$$

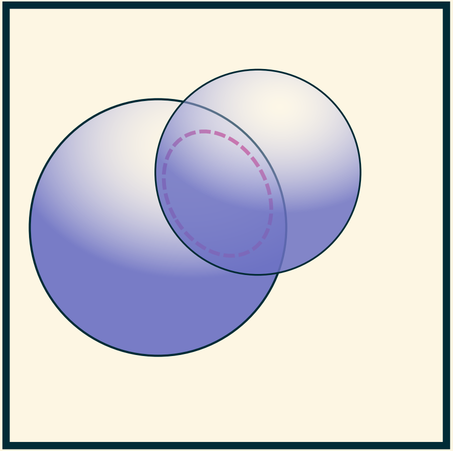
(it also works with more general 1-vectors)



Pencils are hard to draw so we just draw their zeroes

In 3D CGA:

- A circle defines a 2-pencil of spheres
- A point pair a 3-pencil of spheres
- A point a 4-pencil of spheres



Flat and round pencils

Any round n-pencil P can be decomposed into

- a flat $(n-1)$-pencil: $\mathrm{Flat}(P) = P \wedge e_\infty$
- its smallest sphere: $\mathrm{Small}(P) = P \wedge \mathrm{Flat}(P)^*$

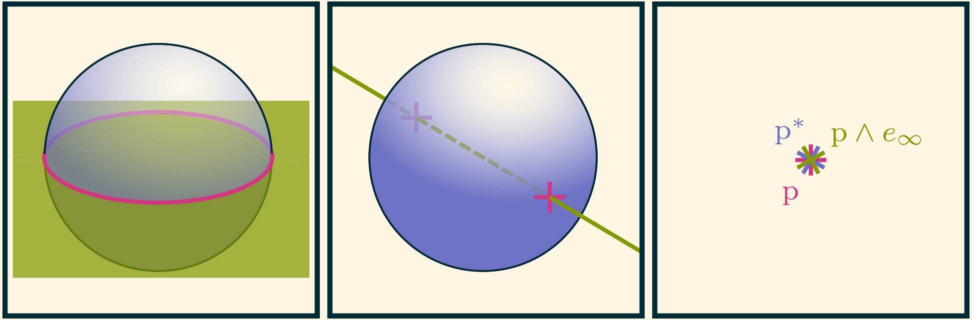
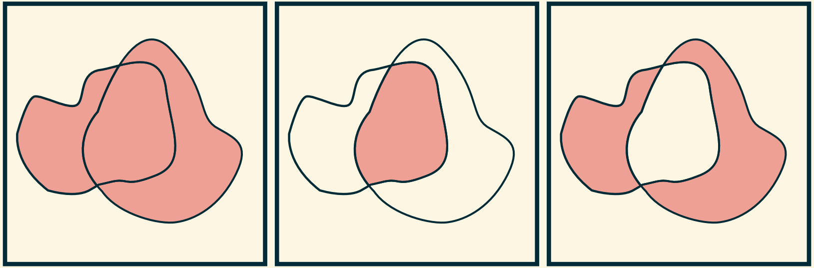


Figure 4: Examples of $\mathrm{Flat}(P)$ and $\mathrm{Small}(P)$ for several P.

Set operators on 2-pencils

We want to introduce some set operators on pencils as sets of spheres, described by these Venn diagrams:

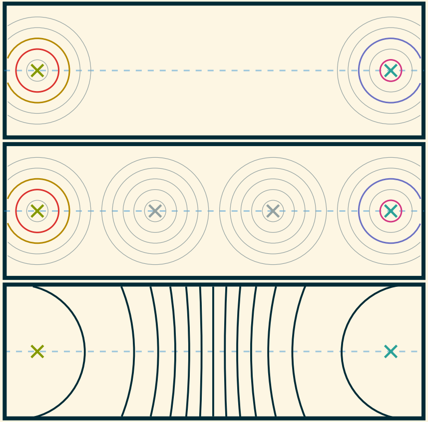


(a) $A \cup B$ **(b)** $A \cap B$ **(c)** $A \triangle B$

Set operator	Geometric signification	example (with S_1, S_2, S_3 independent spheres)
Union	Intersection	$(S_1 \wedge S_2) \cup (S_1 \wedge S_3) = S_1 \wedge S_2 \wedge S_3$
Intersection	Union	$(S_1 \wedge S_2) \cap (S_1 \wedge S_3) = S_1$
Symetric difference	circle perpendicular to both	$(S_1 \wedge S_2) \triangle (S_1 \wedge S_3) = S_2 \wedge S_3$

Example

Consider two pencils of concentric spheres A and B (sliced for visibility).



$$A = S_1 \vee S_2 = \mathrm{p_1}^* \vee e_\infty^* \tag{10}$$

$$B = S_3 \vee S_4 = \mathrm{p_2}^* \vee e_\infty^* \tag{11}$$

$$A \cup B = S_1 \vee S_2 \vee S_3 = I^* \tag{12}$$

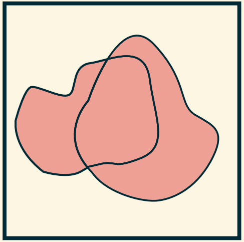
$$A \cap B = e_\infty^* \tag{13}$$

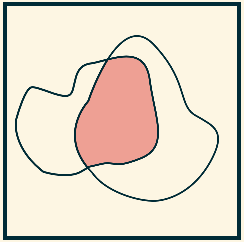
$$A \triangle B = \mathrm{p_1}^* \vee \mathrm{p_2}^* \tag{14}$$

Set operators: easy cases

- Two pencils are independent iff they share no spheres.
- $A \cup B = A \vee B$ iif A and B are independents
- $A \cap B = A \wedge B$ iff A^* and B^* are independent

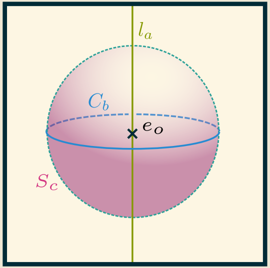
Problematic case: $S1 \vee S2$ and $S1 \vee S3$
Neither them or their dual are independent.
How then?





Examples

Union



$l_a = e_1^* \vee e_2^*$

$C_b = \left(e_o - \frac{1}{2}e_\infty\right)^* \vee e_3^*$

$S_c = l_a \cup C_b = \left(e_{o123} + \frac{1}{2}e_{123\infty}\right)^* = e_o + \frac{1}{2}e_\infty$

Intersection

no 2-pencils so that their dual are independent !

The symetric difference

The operator $\triangle$ is defined so: $C_1 \triangle C_2 = (C_1 \cup C_2) \setminus (C_1 \cap C_2)$

Independent 2-pencils (no common sphere)

$C_1 \triangle C_2 = C_1 \cup C_2 = C_1 \vee C_2$

Dependent 2-pencils (one common sphere)

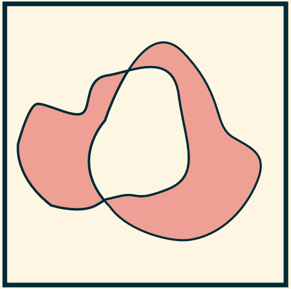
$C_1 \triangle C_2 = (C_1 \times C_2)^*$

Identical 2-pencils (same spheres sets)

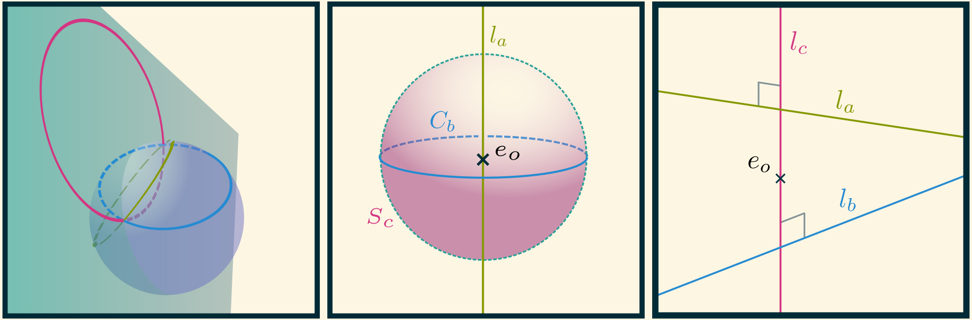
$C_1 \triangle C_2 = I$

All cases put together

$C_1 \triangle C_2 = (C_1 \times C_2)^* + C_1 \vee C_2 + C_1 \wedge C_2^*$



Examples



$$C_a = S_1 \vee S_2$$
$$C_b = S_1 \vee S_3$$
$$C_c = C_a \triangle C_b = S_2 \vee S_3$$
$$P = \mathrm{Flat}(C_c)$$

$$l_a = e_1^* \vee e_2^*$$
$$C_b = \left(e_o - \frac{1}{2} e_\infty\right)^* \vee e_3^*$$
$$S_c = l_a \triangle C_b = e_o + \frac{1}{2} e_\infty$$

$$\boldsymbol{u} \perp \boldsymbol{v} \qquad \boldsymbol{u} \perp \boldsymbol{w}$$
$$l_a = (\boldsymbol{u} - e_\infty)^* \vee \boldsymbol{v}^*$$
$$l_b = (\boldsymbol{u} + e_\infty)^* \vee \boldsymbol{w}^*$$
$$l_c = l_a \triangle l_b = \boldsymbol{v}^* \vee \boldsymbol{w}^* = e_o \wedge \boldsymbol{u}$$

Pencil union and intersection

Round 2-pencils

$$C_1 \cup C_2 = C_1 \vee \mathrm{Flat}(C_1 \triangle C_2) \tag{15}$$

Flat 2-pencils

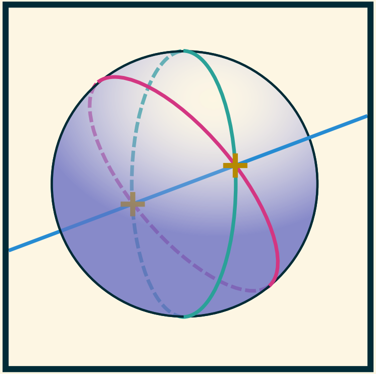
$$l_1 \cup l_2 = l_1 \wedge ((l1 \triangle l2) \vee e123) \tag{16}$$

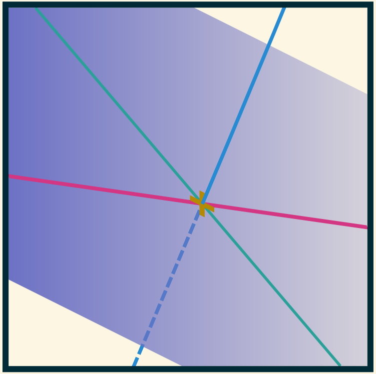
Intersection

Intersection can be derived from symetric difference and union.

$$C_1 \cap C_2 = (C_1 \cup C_2) \setminus (C_1 \triangle C_2) = (C_1 \cup C_2) \wedge (C_1 \triangle C_2)^* \tag{17}$$

Examples





$C_a = S_1 \vee S_2 \qquad C_b = S_1 \vee S_3$

$C_c = C_a \triangle C_b = (C_a \times C_b)^* = S_2 \vee S_3$

$\mathrm{Pp} = C_a \cup C_b = \mathrm{Flat}(C_a) \vee C_b$

$S_1 = C_a \cap C_b = \mathrm{Pp} \wedge C_c{}^*$

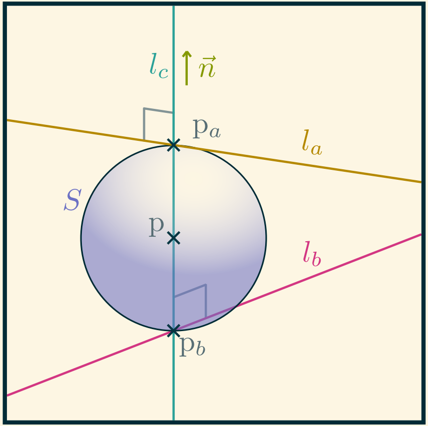
$l_a = P_1 \vee P_2 \qquad l_b = P_1 \vee P_3$

$l_c = l_a \triangle l_b = (l_a \times l_b)^* = P_2 \vee P_3$

$\mathrm{Fp} = l_a \cup l_b = l_a \wedge (l_c \vee e_{123})$

$P_1 = l_a \cap l_b = \mathrm{Fp} \wedge l_c{}^*$

Applicative case



Algorithm 1: Find smallest tangent sphere of two skew lines

Function *skew_lines_sphere*

Input: l_a, l_b

Output: $\mathrm{p} \wedge e_\infty$

$l_c \leftarrow l_a \triangle l_b$

$\boldsymbol{n} \leftarrow l_c \vee e_{123}$ // Euclidean vector

$P_{\parallel,a} \leftarrow l_a \wedge (l_a \wedge \boldsymbol{n})^*$

$P_{\parallel,b} \leftarrow l_b \wedge (l_b \wedge \boldsymbol{n})^*$

$\mathrm{Fp} \leftarrow (P_{\parallel,a} + P_{\parallel,b}) \vee l_c$

$\mathrm{Fp}_a \leftarrow P_{\parallel,a} \vee l_c$

$x_a \leftarrow -(e_{o\infty} \cdot (\mathrm{Fp}_a \wedge e_o))/(e_{o\infty} \cdot \mathrm{Fp}_a)$

$\mathrm{p}_a \leftarrow e_o + x_a + \frac{1}{2}x_a^2 e_\infty$

return $\mathrm{Fp}^* \wedge \mathrm{p}_a$

Figure 9: The algorithm and an illustration of the use case

Conclusion

- geometric primitives of CGA can the understood as pencil of spheres
- brought the Flat and Small operators on pencils of spheres
- brought the $\cup$, $\cap$ and $\triangle$ operators on 2-pencils (circles)
- results should be extendable to any pencil of CGA (to be done)

Thank you for your attention

Questions?