Geometric algebra the mathematics of the future

James Chappell (Working with David Berkahn and Derek Abbott, the University of Adelaide)

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Roadmap

- A geometric algebra(GA) approach to projectile motion

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- Deriving Minkowski spacetime from $C\ell(\Re^3)$

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- Deriving Minkowski spacetime from $C\ell(\Re^3)$
- Exploring the property of chirality in GA

Projectile motion

We assume that gravity is a constant acceleration vector

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}. \tag{1}$$

As $\boldsymbol{a}$ is constant, we can integrate giving

$$\boldsymbol{v} - \boldsymbol{u} = \boldsymbol{a}t. \tag{2}$$

where $\boldsymbol{u}$ is the initial velocity. Defining $\boldsymbol{v} = \frac{d\boldsymbol{s}}{dt}$, and integrating gives the well known equation

$$\boldsymbol{s} - \boldsymbol{u}t = \frac{1}{2}\boldsymbol{a}t^2. \tag{3}$$

Single governing equation

Combining the last two equations we find

$$\frac{\boldsymbol{s}}{t} = \frac{\boldsymbol{u}+\boldsymbol{v}}{2}. \tag{4}$$

Then, multiplying from the left by $\boldsymbol{v} - \boldsymbol{u} = \boldsymbol{a}t$ gives

$$2\boldsymbol{as} = (\boldsymbol{v}-\boldsymbol{u})(\boldsymbol{v}+\boldsymbol{u}) = v^2 - u^2 + 2\boldsymbol{v} \wedge \boldsymbol{u}. \tag{5}$$

This is a general equation for projectile motion that has eliminated the time variable t, and relates $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{s}$ and $\boldsymbol{a}$.

Insights from the governing relation

Expanding the vector product $\boldsymbol{a}\boldsymbol{s}$, we find

$$\begin{gathered}\boldsymbol{a}\cdot\boldsymbol{s}=\frac{1}{2}v^2-\frac{1}{2}u^2,\\ \boldsymbol{a}\wedge\boldsymbol{s}=\boldsymbol{v}\wedge\boldsymbol{u}.\end{gathered} \tag{6}$$

The first equation states that the change in kinetic energy equals the work done, a statement of the work-energy theorem. We can see that no work is done orthogonal to the acceleration vector, so the velocity is constant. The second equation shows that the parallelogram area $\boldsymbol{v}\wedge\boldsymbol{u}$ equals the parallelogram area $\boldsymbol{a}\wedge\boldsymbol{s}$. The first equation magnitude of velocity, the second equation the direction. The torque equals the rate of change of angular momentum. In order to minimize the launch velocity $\boldsymbol{u}$ we clearly need $\boldsymbol{u}\perp\boldsymbol{v}$, a general result.

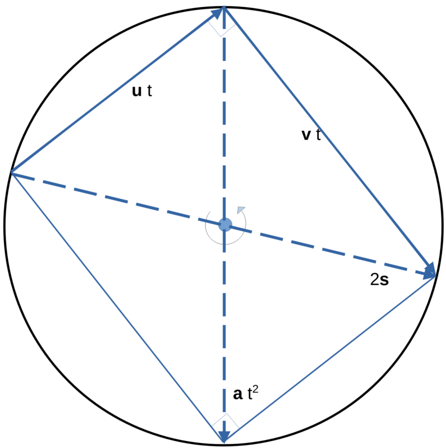
The governing equation on level ground

Considering the equations

$$\begin{aligned} 2\boldsymbol{a}\cdot\boldsymbol{s} &= v^2 - u^2, \\ \boldsymbol{a}\wedge\boldsymbol{s} &= \boldsymbol{v}\wedge\boldsymbol{u}. \end{aligned} \tag{7}$$

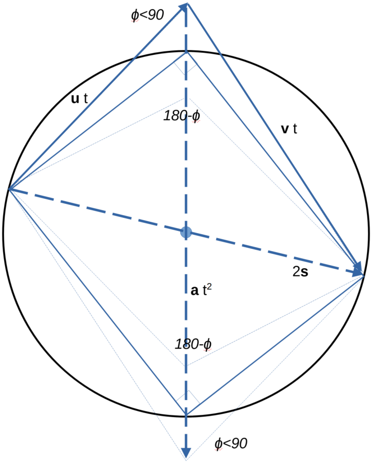
For level ground $\boldsymbol{a}\cdot\boldsymbol{s} = 0$ and hence $v = u$. The second equation then implies $as = vu\sin\phi = u^2\sin\phi$, where ϕ is defined as the angle between the initial and final velocities $\boldsymbol{u}$ and $\boldsymbol{v}$. In order to minimize u, we require $\phi = 90^\circ$, and so $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, and the initial velocity required is $u = \sqrt{as}$.

Graphical solution: minimal energy trajectories



We first draw a circle of radius s, and then draw the diameter with the same slope as the ground. A vertical diameter for $\boldsymbol{a}t$ is then drawn through the centre of the circle. As the two diameters are equal, we have $2s = at^2$, allowing us to calculate $t = \sqrt{2s/a}$. The sides of the parallelogram are of length $\boldsymbol{u}t$ and $\boldsymbol{v}t$, which can now be measured. The diameter $2\boldsymbol{s}$ can be rotated around the centre, which shows the effect of a different ground slope.

Graphical solution for the general trajectory



We relax the restriction $\boldsymbol{u} \perp \boldsymbol{v}$, for energy optimal trajectories. The intersection point of $\boldsymbol{u}t$ and $\boldsymbol{v}t$, can now be moved up or down the vertical centre line $\boldsymbol{a}t^2$, forming an ellipse, from which a new time of flight, based on the measured length of $\boldsymbol{a}t^2$, can be found. The magnitude of the initial and final velocities can now be measured. All angles can be measured straight from the diagram.

Projectile motion in $C\ell(\Re^2)$

- Geometric Algebra allows a pure vector based approach to projectile motion, without the need for a coordinate system, thus satisfying the relativity principle.
- The scalars, vectors and bivectors in $C\ell(\Re^2)$ provide additional insights, as energy, momentum and torque.
- In GA, we can write down a single governing equation for projectile motion.
- A pure vector-based approach allows a single diagram solving the general projectile problem, on sloping ground.

Physical space

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[1] Coxeter H S M 1973 Regular Polytopes (Dover Pubns)

[2] Hoyle C D, Schmidt U, Heckel B R, Adelberger E G, Gundlach J H, Kapner D J and Swanson H E 2001 Phys. Rev. Lett. 86(8) 1418–1421

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- Therefore, it is natural to adopt the formalism of Clifford geometric algebra (GA) of three dimensions $C\ell(\Re^3)$ in order to describe physical space.

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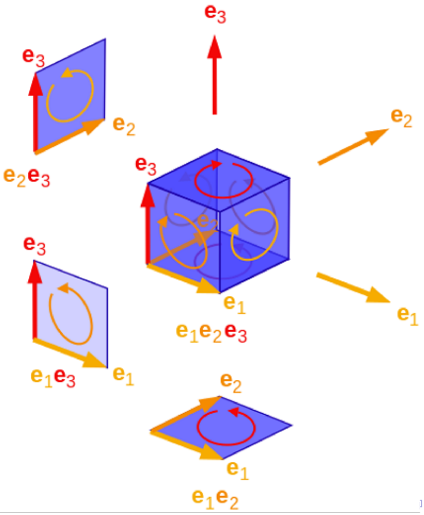
Physical space as $C\ell(\Re^3)$

Algebraically, we write a general multivector in $C\ell(\Re^3)$ as

$$M = a + \boldsymbol{x} + j\boldsymbol{n} + jb, \tag{8}$$

where $\boldsymbol{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$ a vector, $j\boldsymbol{n}$ a bivector, where $\boldsymbol{n} = n_1 e_1 + n_2 e_2 + n_3 e_3$, and $j = e_1 e_2 e_3$ trivector, with $a, b, x_1, x_2, x_3, n_1, n_2, n_3$ real scalars. We have defined the three unit elements e_1, e_2, e_3, having a unit square $e_1^2 = e_2^2 = e_3^2 = 1$, and are anticommuting, with $e_1 e_2 = -e_2 e_1$, $e_1 e_3 = -e_3 e_1$ and $e_2 e_3 = -e_3 e_2$. This eight-dimensional object, has complex numbers, quaternions, polar vectors and axial vectors as subspaces. The four grades of scalars, vectors, bivector and trivectors represent the four common geometrical entities of points, lines, areas and volumes, respectively.

What does $C\ell(\Re^3)$ look like?



Invariants in $C\ell(\Re^3)$ and the laws of physics

We define the involution of *Clifford conjugation* of a multivector M as

$$\bar{M} = a - \mathbf{x} - j\mathbf{n} + jb. \tag{9}$$

For two multivectors $M, N \in C\ell\left(\Re^3\right)$, $\overline{MN} = \bar{N}\bar{M}$.
We define the *amplitude squared* of a multivector M as

$$|M|^2 = M\bar{M} = a^2 - \mathbf{x}^2 + \mathbf{n}^2 - b^2 + 2j\left(ab - \mathbf{x}\cdot\mathbf{n}\right) \tag{10}$$

forming a complex-like number, commuting with the rest of the algebra. We can define the multivector *amplitude* as $|M| = \sqrt{|M|^2}$, which may be two-valued and complex.

A norm in $C\ell(\Re^3)$

For two multivectors $M_1, M_2 \in C\ell\left(\Re^3\right)$ the amplitude squared has the property

$$|M_1 M_2|^2 = M_1 M_2 \bar{M}_2 \bar{M}_1 = M_1 \bar{M}_1 M_2 \bar{M}_2 = |M_1|^2 |M_2|^2. \qquad (11)$$

We can therefore write a norm relation

$$|M_1 M_2| = |M_1||M_2|, \qquad (12)$$

provided that the appropriate branch is used when finding the complex square roots.

Group transformations

We define a general bilinear transformation on a multivector M as

$$M' = KML, \tag{13}$$

where $M, K, L \in C\ell(\Re^3)$. We find the transformed multivector amplitude

$$|M'|^2 = KML\,\overline{KML} = KML\bar{L}\bar{M}\bar{K} = |K|^2|L|^2|M|^2, \tag{14}$$

using the crucial commuting property of the amplitude. Hence, provided we specify a unitary condition $|K|^2|L|^2 = \pm 1$ for these transformations, then the amplitude $|M|$ will be invariant. This transformation is then the most general bilinear transformation that preserves the multivector amplitude and so produces an invariant distance.

The general transformations over $C\ell(\Re^3)$

We can therefore write the general transformation operation

$$M' = \mathrm{e}^{\boldsymbol{p}+j\boldsymbol{q}} M \mathrm{e}^{\boldsymbol{r}+j\boldsymbol{s}}, \tag{15}$$

which will leave the multivector amplitude invariant. The four three-vectors $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ illustrate that the set of transformations is a twelve dimensional manifold, thus generalizing the conventional six dimensional Lorentz group, consisting of boosts and rotations. For comparison, the conventional Lorentz transformations can be written as

$$M' = \mathrm{e}^{-\boldsymbol{p}-j\boldsymbol{q}} M \mathrm{e}^{-\boldsymbol{p}+j\boldsymbol{q}}. \tag{16}$$

Fields

Multivectors formed from a product of two multivectors $\bar{A}B$ transform as

$$\bar{A}'B' = \overline{KAL}KBL = \bar{L}\bar{A}\bar{K}KBL = \bar{L}\bar{A}BL. \tag{17}$$

These have a distinct transformation law

$$F' = \bar{L}FL. \tag{18}$$

We will refer to such quantities as "fields", as we find this transformation applies to the electromagnetic field, for example.

An invariant dot product

Since $M\bar{M}$ is invariant, then $(A+B)(\overline{A+B})$ must also be invariant, where $A, B \in C\ell(\Re^3)$. We have

$$(A+B)(\overline{A+B}) = A\bar{A} + B\bar{B} + A\bar{B} + B\bar{A}. \tag{19}$$

Hence, as $A\bar{A}, B\bar{B}$ are known to be invariant, then we can define a multivector dot product with the final two terms

$$A \cdot \bar{B} = \frac{1}{2}\left(A\bar{B} + B\bar{A}\right) = B \cdot \bar{A}. \tag{20}$$

This is also an invariant, being in the form of a complex-like number. The invariant dot product thus provides a mechanism to combine two distinct multivectors, as in the electromagnetic Lagrangian $A \cdot \bar{J}$, for example.

Constructing simple equations

The product of a multivector with a field XF will transform the same as a general multivector, $X'F' = KXL\bar{L}FL = K(XF)L$. Hence, we can write an invariant equation

$$XF = X(\bar{B}A) = Y, \tag{21}$$

where X, Y transform as multivectors, defined in Eq. (15), and $F = \bar{B}A$ transforms as a field. Selecting $X = \partial$, $F = \bar{\partial}A$ and $Y = J$, we produce the general form of Maxwell's equations

$$\partial F = \partial\bar{\partial}A = J, \tag{22}$$

where J represents the sources.
Another elementary equation we could write is $\partial F = YF^*$, which is equivalent to the Dirac equation. The eight-dimensional multivector F, naturally corresponding with the eight-dimensional Dirac spinor.

The field as the gradient of a potential

The electromagnetic field was defined as $F = \bar{\partial} A$, as proposed by Fermi[3], as opposed to the conventional $F = \partial \wedge A$.

$$\begin{aligned} F &= \left(\frac{\partial}{\partial t} - \nabla\right)(\phi - \boldsymbol{A}) \\ &= \frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{A} - \nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \wedge \boldsymbol{A} \\ &= \ell + \boldsymbol{E} + j\boldsymbol{B}, \end{aligned} \tag{23}$$

where $\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$, $j\boldsymbol{B} = \nabla \wedge \boldsymbol{A} = j\nabla \times \boldsymbol{A}$ and $\ell = \frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{A}$. In order to recover the standard electromagnetic field $F = \boldsymbol{E} + j\boldsymbol{B}$ we need to adopt the Lorenz gauge with $\ell = 0$. The Lorenz gauge produces a Lorentz invariant form of electromagnetism, enforcing causality and charge conservation, which is generally assumed to be a requirement of a physical theory.

[3] Van Oosten A 2000 The European Physical Journal D 8 9–12

Spin in Spacetime

If we consider a Lorentz boost $M' = \mathrm{e}^{-\hat{\boldsymbol{v}}\phi/2} M \mathrm{e}^{-\hat{\boldsymbol{v}}\phi/2}$, on the multivector $M = a + \boldsymbol{x}_{\|} + \boldsymbol{x}_{\perp} + j\boldsymbol{n}_{\|} + j\boldsymbol{n}_{\perp} + jb$, where we split the spatial coordinate into components perpendicular and parallel to the boost direction $\hat{\boldsymbol{v}}$

$$\begin{aligned} M' &= a\mathrm{e}^{-\hat{\boldsymbol{v}}\phi} + \boldsymbol{x}_{\|}\mathrm{e}^{-\hat{\boldsymbol{v}}\phi} + \boldsymbol{x}_{\perp} \\ &\quad + j\boldsymbol{n}_{\|}\mathrm{e}^{-\hat{\boldsymbol{v}}\phi} + j\boldsymbol{n}_{\perp} + bj\mathrm{e}^{-\hat{\boldsymbol{v}}\phi} \\ &= \gamma\left(a - vx_{\|}\right) + \gamma\left(\boldsymbol{x}_{\|} - \boldsymbol{v}a\right) + \boldsymbol{x}_{\perp} \\ &\quad + j\gamma\left(\boldsymbol{n}_{\|} - \boldsymbol{v}b\right) + j\boldsymbol{n}_{\perp} + j\gamma\left(b - vn_{\|}\right). \end{aligned} \tag{24}$$

We have two disjoint subspaces, where $a + \boldsymbol{x}$ can be identified as conventional spacetime provided we identify the scalar a with the time t and the second four-vector $j(\boldsymbol{n} + b)$ as four-spin.

Eight dimensional spacetime

We have a generalised spacetime event X, in differential form, as

$$dX = dt + d\boldsymbol{x} + jd\boldsymbol{n} + jdb. \quad (25)$$

This has amplitude

$$|dX|^2 = dt^2 - d\boldsymbol{x}^2 + d\boldsymbol{n}^2 - db^2 + 2j\left(dtdb - d\boldsymbol{x} \cdot d\boldsymbol{n}\right). \quad (26)$$

Hence, solely based on the requirement for the most general invariant quantity in $C\ell(\Re^3)$ we see that the Minkowski line element $dt^2 - d\boldsymbol{x}^2$ has arisen, as well as EM fields and their transformations.

Clifford conjugation reverses the linear motion and spin directions, thus appears equivalent to a time reversal on the space. Hence, spacetime combined with its time reversed copy $dXd\bar{X}$, is what creates the Minkowski spacetime structure.

Proper time is two-dimensional

Spacetime has an invariant interval

$$d\tau^2 = dt^2 - d\boldsymbol{x}^2 + d\boldsymbol{n}^2 - db^2 + 2j\left(dtdb - d\boldsymbol{x}\cdot d\boldsymbol{n}\right). \qquad (27)$$

Hence, proper time is two-dimensional, combining scalar and pseudoscalar properties. Recently, two-dimensional time has been created within a quantum computer[4] and theoretical work has shown it to be a physically meaningful hypothesis[5].
The invariant distance $dt^2 - d\boldsymbol{x}^2$ means that each observer will see a spherically expanding light shell allowing a single number t to describe its radius. Also, we notice in that an additional non-squared time factor $dtdb$ arises in the imaginary component, which breaks the normal symmetry of time.

[4] Dumitrescu P et al, 2022 Nature 607(7919) 463–467

[5] Bars I 2001 Classical and Quantum Gravity 18 3113

Lightlike particles in the general metric

If we specify a null condition $|dX|^2 = 0$, with a light speed particle $dt^2 - d\boldsymbol{x}^2 = 0$, we require $db^2 = d\boldsymbol{n}^2$ and $dbdt - d\boldsymbol{x} \cdot d\boldsymbol{n} = 0$. Combining these two results gives

$$\boldsymbol{v} \cdot \hat{\boldsymbol{n}} = \pm c, \tag{28}$$

where for clarity we introduce the speed of light c. Hence, due to the nature of the dot product, we can see that it is only satisfied by a velocity $||\boldsymbol{v}|| = c$, parallel to the spin axis $\hat{\boldsymbol{n}}$. That is, based on the eight-dimensional structure of $C\ell(\Re^3)$ alone, we find that a null particle, if traveling at the speed of light c, is required to have its spin axis parallel to its direction of motion, exactly as observed for electromagnetic radiation.

The action

The invariant distance provides a suitable action integral $S = \int |dX|$, extremizing the proper time in order to find the geodesics. Defining $\dot{t} = \frac{dt}{d\tau}$, $\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{d\tau}$, $\dot{\boldsymbol{n}} = \frac{d\boldsymbol{n}}{d\tau}$ and $\dot{b} = \frac{db}{d\tau}$ we write the action as $S = \int \frac{|dX|}{d\tau} d\tau$ implying a Lagrangian

$$\mathcal{L} = \frac{|dX|}{d\tau} = |V| = \sqrt{\dot{t}^2 - \dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{n}}^2 - \dot{b}^2} = 1. \tag{29}$$

As we have no explicit coordinate dependence, $\frac{\partial \mathcal{L}}{\partial \dot{t}}$, $\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}$, $\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{n}}}$ and $\frac{\partial \mathcal{L}}{\partial \dot{b}}$ are constants of the motion. The four grades will give the four conservation laws of energy, linear momentum, angular momentum and helicity for inertial particles.

Particle in an electromagnetic field

We found the Lagrangian for inertial particles $\mathcal{L} = |V| = \frac{|dX|}{d\tau}$. The simplest extension of this Lagrangian, while maintaining invariance is possibly $\mathcal{L} = |V + U|$, where the multivector U conceptually represents a 'flow' in the background spacetime, perturbing particle inertial motion V. We thus produce a generalised Lagrangian

$$\mathcal{L} = \frac{1}{2}|V + U|^2. \tag{30}$$

Note that we are permitted to use either $\mathcal{L} = |V + U|$ or $\mathcal{L} = \frac{1}{2}|V + U|^2$, because if a Lagrangian $\mathcal{L}$ satisfies the Euler-Lagrange equations, then in general any function $F(\mathcal{L})$ of the Lagrangian also satisfies the Euler-Lagrange equations. We can also add an $A \cdot \bar{V}$ term for electromagnetic forces.

A simple flow field for space

Special case, using a four-vector $V = c\dot{t} + \dot{\boldsymbol{x}}$ but an eight-dimensional flow field $U = \frac{\phi}{c} + \boldsymbol{A} + j\boldsymbol{W} + j\frac{\psi}{c}$. Using the Euler-Lagrange equations, we find

$$\boldsymbol{a} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \nabla\xi. \tag{31}$$

The simplest case $U = \boldsymbol{A}$, where $\boldsymbol{A} = -\sqrt{\frac{GM}{r}}\hat{\boldsymbol{r}}$.
We have $\xi = -\frac{1}{2}\boldsymbol{A}^2 = -\frac{1}{2}\frac{GM}{r}$. Therefore

$$\boldsymbol{a} = -\nabla\xi = -\frac{GM}{r^2}\hat{\boldsymbol{r}}. \tag{32}$$

Hence, for an inwardly flowing spacetime flow field $A = -\sqrt{\frac{GM}{r}}\hat{\boldsymbol{r}}$, we reproduce the Newtonian acceleration law in gravity.

The Gullstrand–Painlevé coordinates

The Schwarzschild solution around a stationary, non-rotating mass is

$$ds^2 = \left(1-\beta^2\right)c^2dt^2 - \left(1-\beta^2\right)^{-1}dr^2 - r^2d\Omega^2, \qquad (33)$$

where $\beta = \frac{v}{c} = \sqrt{\frac{r_s}{r}}$ is the escape velocity at a distance r, where $r_s = \sqrt{\frac{2GM}{c^2}}$ is the Schwarzschild radius.

Making the coordinate transformation $dt = dT - \frac{\beta}{1-\beta^2}$, the line element becomes

$$ds^2 = c^2dT^2 - (dr + \beta c dT)^2 - r^2d\Omega^2, \qquad (34)$$

which are the Gullstrand–Painlevé coordinates.

$dr + \beta c dT = dr + v_e dT$, describes a flow of spacetime towards the gravitating mass at the escape velocity β. The local time coordinate T is now equal to the proper time of a free-falling observer from infinity and space is flat.

The speed of light

Electromagnetic radiation satisfies $ds^2 = 0$, giving the equation $(cdT + (dr + \beta cdT))(cdT - (dr + \beta cdT)) = 0$, indicating two solutions. Dividing through by dT, we find

$$v_{\mathrm{EM}} = \frac{dr}{dT} = c\,(\pm 1 - \beta) = \pm c - v_{esc} = c\left(\pm 1 - \sqrt{\frac{r_s}{r}}\right). \quad (35)$$

Hence, at the event horizon, with $r = r_s$, the outbound light velocity is zero, as it is balanced by the inflow velocity of space. The Gullstrand–Painlevé coordinates, describes the velocity of inflowing space at a given radius r and thus also corresponds to the free-fall observer from infinity in gravity. With $dT = 0$ the observer views a flat space.

Kerr metric

The Schwarzschild solution can be generalized to the Kerr solution for rotating black holes. Surprising, in terms of the flow model, this involves adding a bivector twist term to the inflowing space[6]. With a velocity/twist field at each point thus implies a multivector field $\boldsymbol{v} + j\boldsymbol{w}$.

Hence, we can view the electromagnetic field as operating in an inertial space, but when this space flows due to the presence of mass, this appears as gravity. The space itself is transformed using the Galilean transforms, whereas within the space we have the Lorentz transformations.

[6] Andrew JS Hamilton and Jason P Lisle. The river model of black holes. American Journal of Physics, 76(6):519–532, 2008.

Conclusions of generalized spacetime

- We show Minkowski spacetime is an emergent property of physical space, when modeled with $C\ell(\Re^3)$.
- We produce a generalised invariant interval, the nature of null particles, generalised Lorentz transformations as well as Maxwell's equations, directly from the algebra.
- We can include a description of gravity as a flow of this generalized spacetime.
- The 8D framework predicts a range of new physical effects.
- Time becomes two-dimensional, and the metric now includes an arrow to time.
- Einstein acknowledged that SR was a theory of principle rather than a preferable constructive theory. With the starting assumption of $C\ell(\Re^3)$ we can construct the Minkowski spacetime arena of SR.

Chirality-a fundamental symmetry of nature

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- Objects in the plane can be flipped using 3D, so can the chirality of 3D objects be flipped by rotating using a fourth dimension?

7

[7] Michel Petitjean. Chirality in geometric algebra. Mathematics, 9(13):1521,

A signed unit area $e_1 e_2$ represents chirality in the plane

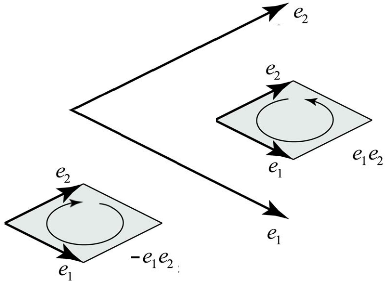


Figure: A mirror image of a unit area about the line e_1 is equivalent to a rotation about the e_1 axis by 180 degrees, using a third dimension. The rotation direction flips from anti-clockwise to clockwise.

The mirror image

A reflection of a geometric object M, about an axis $\hat{\boldsymbol{m}}$ is given by

$$\boldsymbol{v}' = \hat{\boldsymbol{m}}\boldsymbol{v}\hat{\boldsymbol{m}}, \tag{36}$$

where $\hat{\boldsymbol{m}}$ is unit vector.
For our unit area $\hat{A} = e_1 e_2$, if we reflect this about the vector e_1, we find

$$\hat{A}' = e_1 \hat{A} e_1 = e_1(e_1 e_2)e_1 = -e_1 e_2 = e_2 e_1. \tag{37}$$

Hence, the chirality of the unit area flips under a reflection, as expected.
Note that this reflection operation only works to flip the chirality in even dimensions.

Reflections equal to rotation in an extra dimension

In n dimensions, we have a general rotation formula

$$\boldsymbol{v}' = \mathrm{e}^{-B\theta/2}\boldsymbol{v}\,\mathrm{e}^{B\theta/2}, \tag{38}$$

where B is a unit bivector describing the plane of rotation.
So for our shape e_1e_2, rotating in the e_1e_3 plane by 180 degrees, we have

$$\boldsymbol{v}' = \mathrm{e}^{-e_1e_3\pi/2}e_1e_2\mathrm{e}^{e_1e_3\pi/2} = e_1e_2\mathrm{e}^{e_1e_3\pi} = -e_1e_2, \tag{39}$$

using the standard result $\mathrm{e}^{e_1e_3\pi} = -1$, flipping the chirality as expected.
If we image a 1D number line, then flipping the direction of a vector is also equivalent to a rotation in the plane by 180 degrees.

Chirality in 3D

The trivector represents chirality in 3D

$$j = e_1 e_2 e_3, \tag{40}$$

describing a signed volume.
If we exploit a higher dimension e_4, rotating in the $e_1 e_4$ plane by π radians, we find

$$\boldsymbol{v}' = \mathrm{e}^{-e_1 e_4 \pi/2} e_1 e_2 e_3 \mathrm{e}^{e_1 e_4 \pi/2} = e_1 e_2 e_3 \mathrm{e}^{e_1 e_4 \pi} = -e_1 e_2 e_3, \tag{41}$$

Hence, this indeed flips the chirality, as hoped.

Chirality in four dimensions

In four dimensions we have the chiral object, the quadvector

$$k = e_1 e_2 e_3 e_4. \tag{42}$$

Utilizing the higher dimension e_5, we can rotate in the $e_1 e_5$ plane by π radians, giving

$$\mathbf{v}' = \mathrm{e}^{-e_1 e_5 \pi/2} e_1 e_2 e_3 e_4 \mathrm{e}^{e_1 e_5 \pi/2} = e_1 e_2 e_3 e_4 \mathrm{e}^{e_1 e_5 \pi} = -e_1 e_2 e_3 e_4, \tag{43}$$

Hence, this also flips the chirality, as required.

Chirality in n dimensions

We define the chiral object, the pseudoscalar of n dimensions

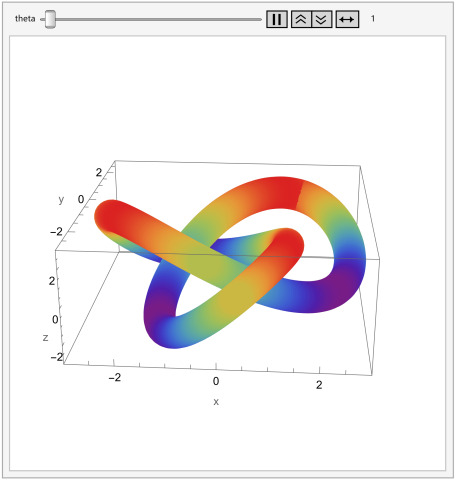
$$z = e_1 e_2 e_3 \ldots e_n. \tag{44}$$

Rotating in the $e_1 e_{n+1}$ plane by π radians, we would have

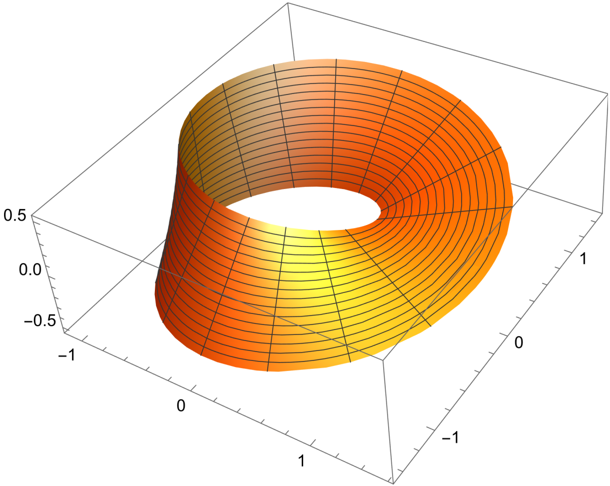
$$\boldsymbol{v}' = \mathrm{e}^{-e_1 e_{n+1}\pi/2} e_1 e_2 e_3 \ldots e_n \mathrm{e}^{e_1 e_{n+1}\pi/2} = e_1 e_2 e_3 \ldots e_n \mathrm{e}^{e_1 e_{n+1}\pi} = -e_1 e_2 e_3 \ldots \tag{45}$$

Hence, this also flips the chirality in n dimensions, as hoped.

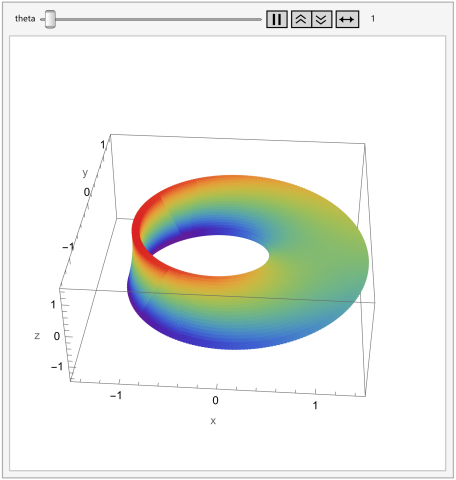
Flipping the chiral trefoil knot in 4D



The Möbius strip: a chiral object

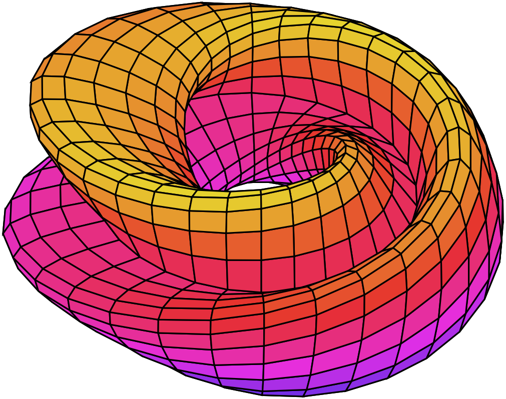


Flipping the chiral Möbius strip using 4D



The Klein bottle

The Klein bottle is an intrinsically 4D object, with two chiral versions and an achiral one. Need 5D to invert chirality.



Conclusion

- Exploring chirality from one to n dimensions, we show how it can be inverted through rotations utilizing one higher dimension.
- GA provides a very simple and general rotation formula applying to any number of dimensions.