Representation & Gauge Freedom

in Electromagnetic & Acoustic Field Theory

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AGACSE

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Context

Recent experiments measured spin angular momentum
not predicted by standard electromagnetic and acoustic field theories

Open Access

Magnetic and Electric Transverse Spin Density of Spatially Confined Light

Martin Neugebauer, Jörg S. Eismann, Thomas Bauer, and Peter Banzer

Phys. Rev. X **8**, 021042 – Published 14 May 2018

Observation of acoustic spin

Chengzhi Shi, Rongkuo Zhao, Yang Long, Sui Yang, Yuan Wang, Hong Chen, Jie Ren ✉, Xiang Zhang ✉ Author Notes

National Science Review, Volume 6, Issue 4, July 2019, Pages 707–712,
https://doi.org/10.1093/nsr/nwz059

Our recent work addresses this gap in the theory:

PAPER • OPEN ACCESS

Acoustic versus electromagnetic field theory: scalar, vector, spinor representations and the emergence of acoustic spin

Lucas Burns[5,1,2], Konstantin Y Bliokh[5,3], Franco Nori[3,4] and Justin Dressel[1,2]

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New Journal of Physics, Volume 22, May 2020
Citation Lucas Burns *et al* 2020 *New J. Phys.* **22** 053050
DOI 10.1088/1367-2630/ab7f91

Quantum Stud.: Math. Found. (2024) 11:27–67
https://doi.org/10.1007/s40509-024-00317-8

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REGULAR PAPER

Spacetime geometry of acoustics and electromagnetism

Lucas Burns · Tatsuya Daniel · Stephon Alexander · Justin Dressel

Context

Physics Reports
Volume 589, 8 August 2015, Pages 1-71

Spacetime algebra as a powerful tool for electromagnetism

Justin Dressel[a b], Konstantin Y. Bliokh[b c], Franco Nori[b d]

Adv. Appl. Clifford Algebras (2019) 29:62

Advances in Applied Clifford Algebras

Maxwell's Equations are Universal for Locally Conserved Quantities

Lucas Burns*

Part of a collection:
AGACSE 2018 IMECC – UNICAMP

international journal of aeroacoustics volume 14 · number 7 · 2015 – pages 977 – 1003 977

An acoustic space-time and the Lorentz transformation in aeroacoustics

Alastair L Gregory[1,3,*], Samuel Sinayoko[2,**], Anurag Agarwal[1,†] and Joan Lasenby[1,‡]

PAPER • OPEN ACCESS

Dual electromagnetism: helicity, spin, momentum and angular momentum

Konstantin Y Bliokh[5,1,2], Aleksandr Y Bekshaev[3] and Franco Nori[1,4]

Published 20 March 2013 • © IOP Publishing and Deutsche Physikalische Gesellschaft

New Journal of Physics, Volume 15, March 2013

Citation Konstantin Y Bliokh *et al* 2013 *New J. Phys.* **15** 033026

DOI 10.1088/1367-2630/15/3/033026

Spin and orbital angular momenta of acoustic beams

Konstantin Y. Bliokh and Franco Nori
Phys. Rev. B **99**, 174310 – Published 21 May 2019; Erratum Phys. Rev. B **105**, 219901 (2022)

Transverse spin and surface waves in acoustic metamaterials

Konstantin Y. Bliokh and Franco Nori
Phys. Rev. B **99**, 020301(R) – Published 3 January 2019

Conservation of the spin and orbital angular momenta in electromagnetism

Konstantin Y Bliokh[1,2], Justin Dressel[1,3] and Franco Nori[1,4]

Published 24 September 2014 • © 2014 IOP Publishing Ltd and Deutsche Physikalische Gesellschaft

New Journal of Physics, Volume 16, September 2014

Outline

1. Background on electromagnetic and acoustic field theory

2. Comparison between the theory and experiment

3. A closer look at acoustics

4. Dual symmetric representations

Spacetime Algebra

4	$I = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3$
3	$I\gamma_0 \quad \vert \quad I\gamma_1 \quad I\gamma_2 \quad I\gamma_3$
2	$\vec{\sigma}_1 \quad \vec{\sigma}_2 \quad \vec{\sigma}_3 \quad \vert \quad I\vec{\sigma}_1 \quad I\vec{\sigma}_2 \quad I\vec{\sigma}_3$
1	$\gamma_0 \quad \vert \quad \gamma_1 \quad \gamma_2 \quad \gamma_3$
0	$1 = \sigma_i^2 = \gamma_0^2 = -\gamma_i^2$

$$\boxed{\vec{\sigma}_k \equiv \gamma_k\gamma_0 = \gamma_k \wedge \gamma_0}$$ *Spacetime split*

Source-free EM & Acoustics

Electromagnetism

$$\mathcal{L}_{\text{EM}} = \frac{1}{2}(\epsilon_0 \vec{E}^2 - \mu_0 \vec{H}^2)$$

$$\epsilon_0 \, \partial_t \vec{E} - \vec{\nabla} \times \vec{H} = 0$$

$$\mu_0 \, \partial_t \vec{H} + \vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{H} = 0$$

$\vec{E}$: electric field

$\vec{H}$: magnetic field

ϵ_0 : permittivity

μ_0 : permeability

$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$: speed of light

Acoustics

$$\mathcal{L}_{\text{ac}} = \frac{1}{2}(\rho \vec{v}^2 - \beta P^2)$$

$$\rho \, \partial_t \vec{v} + \vec{\nabla} P = 0$$

$$\beta \, \partial_t P + \vec{\nabla} \cdot \vec{v} = 0$$

$$\vec{\nabla} \times \vec{v} = \vec{0}$$

$\vec{v}$: velocity

P : pressure

ρ : mass density

β : compressibility

$c = \frac{1}{\sqrt{\rho \beta}}$: speed of sound

Spacetime Representations

Electromagnetism

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}(\epsilon_0 \vec{E}^2 - \mu_0 \vec{H}^2)$$

3D

$$\epsilon_0\, \partial_t \vec{E} = \vec{\nabla} \times \vec{H}$$

$$\mu_0\, \partial_t \vec{H} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{H} = 0$$

$$\downarrow$$

4D

$$F = \vec{E}/c + \mu_0 \vec{H} I$$

Spacetime Bivector

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\langle F^2 \rangle$$

$$\nabla F = 0$$

Acoustics

$$\mathcal{L}_{\mathrm{ac}} = \frac{1}{2}(\rho \vec{v}^2 - \beta P^2)$$

3D

$$\rho\, \partial_t \vec{v} = -\vec{\nabla} P$$

$$\beta\, \partial_t P = -\vec{\nabla} \cdot \vec{v}$$

$$\vec{\nabla} \times \vec{v} = \vec{0}$$

$$\downarrow$$

4D

$$p = (P/c + \rho \vec{v})\gamma_0$$

Spacetime Vector

Energy-momentum density

$$\mathcal{L}_{\mathrm{ac}} = \frac{1}{2}\langle p^2 \rangle$$

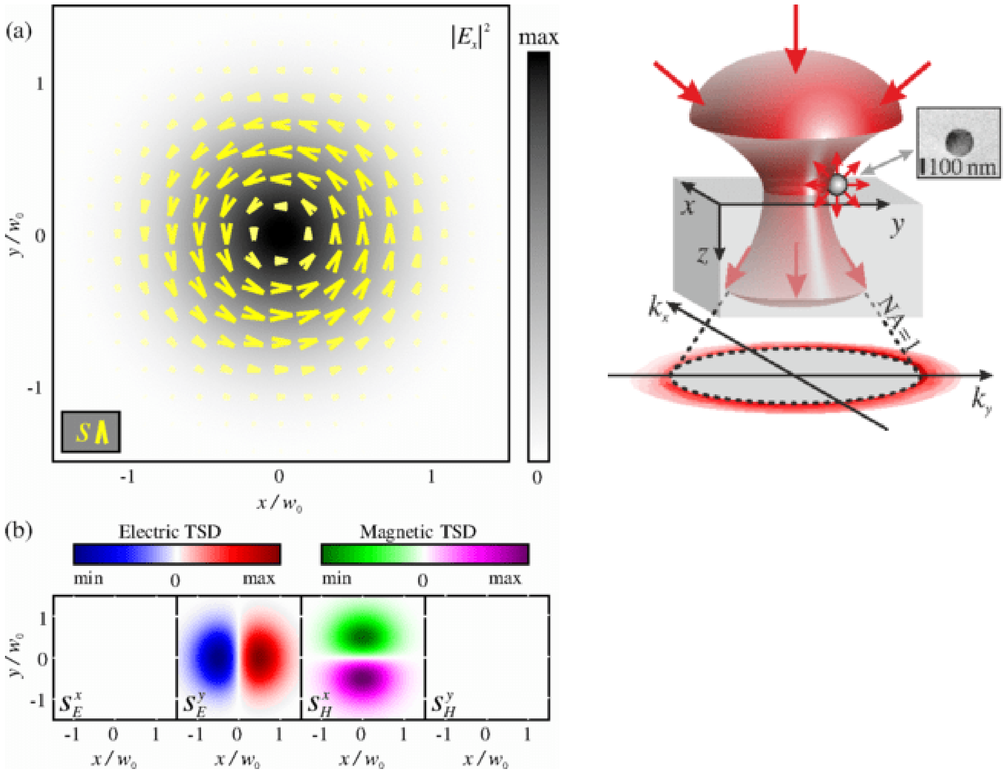
$$\nabla p = 0$$

Gregory et al. (2015)

Spin Experiments

Electromagnetic Spin

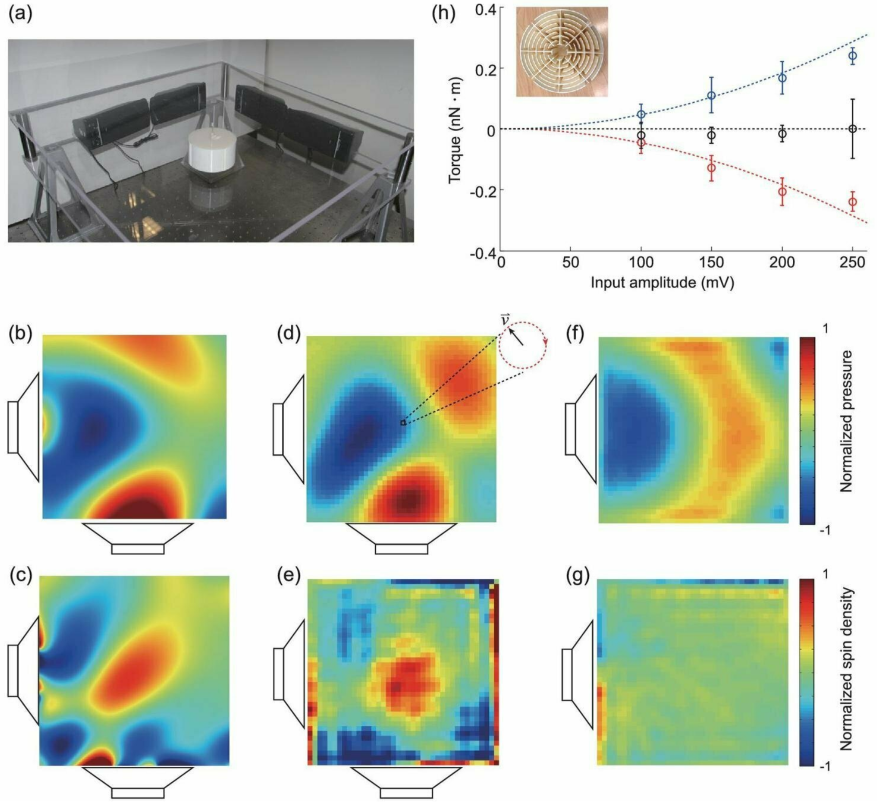
$$\vec{S}_{\text{EM}} = \frac{\rho}{4\omega}\text{Im}(\epsilon_0 \vec{E}^* \times \vec{E} + \mu_0 \vec{H}^* \times \vec{H})$$



Neugebauer et al. (**2018**)

Acoustic Spin

$$\vec{S} = \frac{\rho}{2\omega}\text{Im}(\vec{v}^* \times \vec{v})$$



Shi et al. (**2019**)

What does theory predict?

Standard Electromagnetism

$$\nabla \wedge F = 0 \implies F = \nabla \wedge A_e$$

$\downarrow$ Vector potential representation

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\langle \nabla \wedge A_e \nabla \wedge A_e \rangle$$

$\downarrow$ Noether procedure

$$\vec{S}_{\mathrm{EM}} = \epsilon_0 \vec{E} \times \vec{A}_e$$

$\downarrow$ Complexify & cycle average (monochromatic light wave)

$$\vec{S}_{\mathrm{EM}} = \epsilon_0 \mathrm{Im}(\vec{E}^* \times \vec{E})/2\omega$$

Missing term

Standard Acoustics

$$\nabla \wedge p = 0 \implies p = -\nabla \phi$$

$\downarrow$ Scalar potential representation

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\langle \nabla \phi \nabla \phi \rangle$$

$\downarrow$ Noether procedure

$$\vec{S}_{\mathrm{ac}} = 0$$

Wrong

Non-Standard Representations

Electromagnetism

$$\nabla \cdot F = 0 \implies F = \nabla \cdot (A_m I)$$

↓ Trivector potential

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\langle \nabla \cdot (A_m I) \nabla \cdot (A_m I) \rangle$$

↓ Noether procedure

$$\vec{S}_{\mathrm{EM}} = \mu_0 \vec{H} \times \vec{A}_m$$

↓ Complexify & cycle average (monochromatic light wave)

$$\vec{S}_{\mathrm{EM}} = \mu_0 \mathrm{Im}(\vec{H}^* \times \vec{H})/2\omega$$

Other half!

Acoustics

$$X = \vec{x} + I\vec{y}$$

$$\nabla \cdot p = 0 \implies p = \nabla \cdot (\vec{x} + I\vec{y})$$

↓ Bivector potential

$$\mathcal{L}_{\mathrm{ac}} = \frac{1}{2}\langle \nabla \cdot X \nabla \cdot X \rangle$$

↓ Noether procedure

$$\vec{S}_{\mathrm{ac}} = \vec{x} \times (\rho \vec{v})$$

↓ Complexify & cycle average (monochromatic sound wave)

$$\vec{S}_{\mathrm{EM}} = \rho \mathrm{Im}(\vec{v}^* \times \vec{v})/2\omega$$

Correct!*

Burns et al. (2020)

Dual-Symmetrized Representation

Dual Electromagnetism

$$F = \nabla A = \nabla(A_e + A_m I)/2$$

$\downarrow$ Odd multivector potential

$$\mathcal{L}_{\text{EM}}^{\text{dual}} = \frac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle$$

$\downarrow$ $F = \langle F \rangle_2, \nabla \widetilde{A} = 0$

$$\vec{S}_{\text{EM}} = \frac{1}{2}\left(\epsilon_0 \vec{E} \times \vec{A}_e + \mu_0 \vec{H} \times \vec{A}_m\right)$$

$\downarrow$

$$\vec{S}_{\text{EM}} = \text{Im}(\epsilon_0 \vec{E}^* \times \vec{E} + \mu_0 \vec{H}^* \times \vec{H})/4\omega$$

Correct!

Bliokh et al. (2013)

Dual Acoustics

$$p = -\nabla\psi = -\nabla(\phi + X + I\phi_w)/2$$

$\downarrow$ Even multivector potential

$$\mathcal{L}_{\text{ac}}^{\text{dual}} = \frac{1}{2}\langle \nabla\psi \nabla \widetilde{\psi} \rangle$$

$\downarrow$ $p = \langle p \rangle_1, \nabla \widetilde{\psi} = 0$

$$\vec{S}_{\text{ac}} = \frac{\mathbf{1}}{\mathbf{2}} \vec{x} \times (\rho \vec{v})$$

$\downarrow$

$$\vec{S}_{\text{EM}} = \rho \text{Im}(\vec{v}^* \times \vec{v})/\mathbf{4}\omega$$

Correct!

Burns et al. (2020)

Observations

- In vacuum, physical fields have a degeneracy of potential representations.
- The dynamical fields varied in a Lagrangian are the *potential* fields, *not* the physical fields.
- Canonical Noether currents are representation and gauge dependent, with gauge symmetries determined by representation.
- Transferrable part of spin is gauge invariant, analogous to the gauge invariance of voltage differences.
- **Experiments can differentiate between representations with identical equations of motion through canonical spin measurements.**

A Closer Look

Dual Electromagnetism

$$F = \langle \nabla (A_e + A_m I)/2 \rangle_2$$

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}_e - \vec{\nabla} \times \vec{A}_m$$

$$\vec{H} = -\vec{\nabla}\phi_m - \partial_t \vec{A}_m + \vec{\nabla} \times \vec{A}_e$$

$$\vec{S}_{\mathrm{EM}} = \frac{1}{2}(\vec{E} \times \vec{A}_e + \vec{H} \times \vec{A}_m)$$

Well studied

Dressel et al. (2014)

Dual Acoustics

$$p = -\langle \nabla (\phi + \vec{x} + I\vec{y} + I\phi_w)/2 \rangle_1$$

$$P = -\partial_t \phi - \vec{\nabla} \cdot \vec{x}$$

$$\rho\vec{v} = \vec{\nabla}\phi + \partial_t \vec{x} - \vec{\nabla} \times \vec{y}$$

$$\vec{S}_{\mathrm{ac}} = \frac{1}{2}\vec{x} \times \vec{(\rho\vec{v})}$$

New!

Burns et al. (2020)

A Closer Look at Acoustics

$$p = -\langle \nabla(\phi + \vec{x} + I\vec{y} + I\phi_w)/2 \rangle_1$$

Velocity potential

(couples to scalar sources)

$$P = -\partial_t \phi - \vec{\nabla} \cdot \vec{x}$$

No contribution to p

$$\vec{v} = \partial_t \vec{x} + \vec{\nabla}\phi - \vec{\nabla} \times \vec{y} c$$

Rotational displacement

(couples to vorticity sources)

$$\vec{S}_{\mathrm{ac}} = \frac{1}{2}\vec{x} \times \vec{(}\rho\vec{v})$$

Linear Displacement

(couples to vector sources)

Microscopic interpretations discussed in Burns et al. (2024)

A Closer Look at Acoustics

$$p = (P/c + \rho\vec{v})\gamma_0 = -\langle\nabla(\phi + \vec{x} + I\vec{y} + I\phi_w)/2\rangle_1$$

Gauge freedoms

$$\phi(r) \mapsto \phi(r) + c$$

$$\vec{x} \mapsto \vec{x} + \vec{\nabla} \times \vec{a}$$

$$\vec{y} \mapsto \vec{y} + \vec{\nabla}\phi + \partial_t\vec{a}$$

$$\phi_w$$

Maxwell gauge

$$p = \nabla X = \nabla \cdot (\vec{x} + I\vec{y})$$

Analogy to electromagnetism:

Pressure P ~ Charge Density ρ

Velocity **v** ~ Charge Current **J**

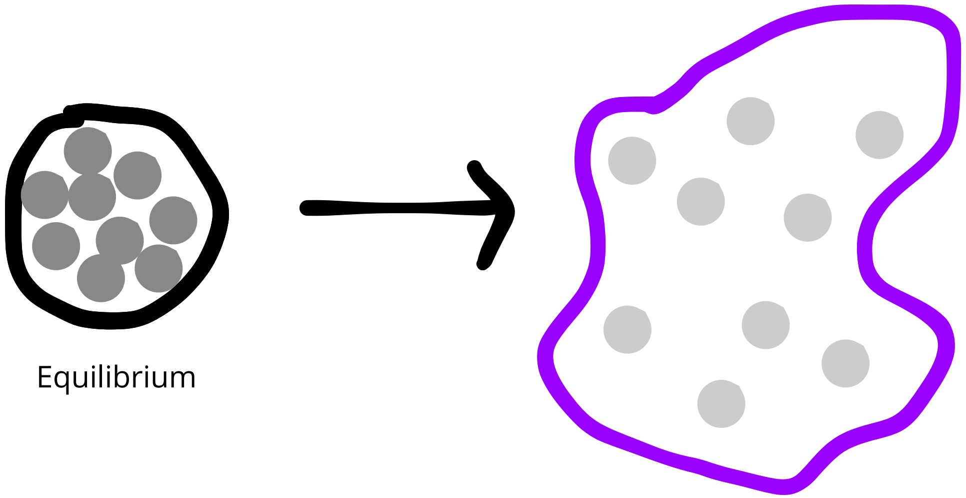
A Closer Look at Acoustics

Example: Linear displacement potential in Maxwell gauge

$$P_V = \int P dV = -\rho c^2 \int \vec{\nabla} \cdot \vec{x} dV = -\rho c^2 \oint \vec{x} \cdot d\vec{a}$$

Net pressure (particle density) of a region

Displacement of boundary from equilibrium



Note invariance under
$\vec{x} \mapsto \vec{x} + \vec{\nabla} \times \vec{a}$

Microscopic interpretations discussed in Burns et al. (2024)

Acoustics with Sources

$$\mathcal{L}_{\mathrm{ac}} = \langle \tfrac{1}{2}\nabla\psi\nabla\widetilde{\psi} + \psi\widetilde{\boldsymbol{\Lambda}}\rangle$$

$$p = -\langle\nabla(\phi + \vec{x} + I\vec{y} + I\boldsymbol{\phi_w})/2\rangle_1$$

$$\boldsymbol{\Lambda} = \nu + \vec{F} - I\vec{\Omega} + I\boldsymbol{\nu_w}$$

$$\nabla\widetilde{\psi}\nabla = \nabla p = \boldsymbol{\Lambda}$$

$$\downarrow$$

Vector constraint

$$p = \nabla\psi = \widetilde{\psi}\nabla$$

$$\vec{\nabla}\cdot\vec{y} = 0$$

$$\nabla\phi_w = 0 = \nu_w$$

$$\partial_t P + \vec{\nabla}\cdot\vec{v} = \boldsymbol{\nu}$$

$$\partial_t\vec{v} + \vec{\nabla}P = \vec{F}$$

$$\vec{\nabla}\wedge\vec{v} = I\vec{\Omega}$$

Example sources

Hole in boundary of system

Directed speaker

Spinning propeller

Microscopic interpretations discussed in Burns et al. (2024)

Why this form of Lagrangian?

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{dual}} = \tfrac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle ?$$

Evolution is generated by *duality transformations*

$$F = (\vec{E}_0 + I\vec{B}_0)e^{Ik\cdot x} \qquad\qquad A = (A_e^0 + A_m^0 I)e^{Ik\cdot x}$$

Helicity is conserved in vacuum by virtue of symmetry under duality

$$J_\chi I = \tfrac{1}{2}(\vec{A}_e \cdot \vec{H} - \vec{A}_m \cdot \vec{E} + \vec{E} \times \vec{A}_e + \vec{H} \times \vec{A}_m)\gamma_0 I$$

The traditional electromagnetic Lagrangian

fails to predict helicity conservation

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{e}} = \tfrac{1}{2}\langle \nabla \wedge A_e \nabla \wedge A_e \rangle$$ *Does not vanish on shell*

Why this form of Lagrangian?

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{dual}} = \frac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle$$

$$\mathcal{L}_{\mathrm{ac}}^{\mathrm{dual}} = \frac{1}{2}\langle \nabla \psi \nabla \widetilde{\psi} \rangle$$

$$\begin{aligned} \nabla(Me^{I\beta})\nabla\widetilde{(Me^{I\beta})} &= \nabla M e^{I\beta} \dot{\nabla} e^{I\beta} \dot{\widetilde{M}} \\ &= \nabla M e^{I\beta} e^{-I\beta} \nabla \widetilde{M} \\ &= \nabla M \nabla \widetilde{M} \end{aligned}$$

Dual symmetric for all multivector fields M.

Why this form of Lagrangian?

$$\mathcal{L}_{\text{EM}}^{\text{dual}} = \tfrac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle$$

$$\nabla \widetilde{A} = 0$$

$$\Longleftrightarrow$$

$$\nabla A_e = \nabla A_m I$$

Bliokh et al. (2013)

Dressel et al. (2014)

$$\mathcal{L}_{\text{ac}}^{\text{dual}} = \tfrac{1}{2}\langle \nabla \psi \nabla \widetilde{\psi} \rangle$$

$$\nabla \widetilde{\psi} = 0$$

$$\Longleftrightarrow$$

$$\nabla(\vec{x} + I\vec{y}) = \nabla(\phi + I\phi_w)$$

Duality (massless) constraint does heavy lifting:

Forces Lagrangian to vanish on shell, and restricts # of DOFs, $\nabla A = \langle \nabla A \rangle_2$ in EM.

Dual Lagrangians

$$\mathcal{L} = \langle \nabla M \nabla \widetilde{M} \rangle$$

Equations of motion

$$\nabla M \nabla = 0 = \nabla \widetilde{M} \nabla$$

Grade restrictions yield standard wave equation

$$\nabla M = \langle \nabla M \rangle_n \implies \nabla M = \pm \widetilde{M} \nabla \implies \nabla^2 M = \nabla^2 \widetilde{M} = 0$$

More generally, coupling between grades.

Grade-Complementary Theories

Electromagnetism

$$F_+ = \nabla A = \boldsymbol{W_e} + F + I\boldsymbol{W_m}$$

$$\mathcal{L}_{\text{EM}}^{\text{dual}} = \frac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle$$

Acoustics

$$p_- = -\nabla\psi = p + \boldsymbol{w}I$$

$$\mathcal{L}_{\text{ac}}^{\text{dual}} = \frac{1}{2}\langle \nabla\psi \nabla \widetilde{\psi} \rangle$$

Odd potential A ↔ Odd field $p_- = -\nabla\psi$

Even field $F_+ = \nabla A$ ↔ Even potential ψ

Grade-Complementary Theories

Electromagnetism

$$F_+ = \nabla A = W_e + F + IW_m$$

$$\mathcal{L}_{\text{EM}}^{\text{dual}} = \frac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle$$

Acoustics

$$p_- = -\nabla \psi = p + wI$$

$$\mathcal{L}_{\text{ac}}^{\text{dual}} = \frac{1}{2}\langle \nabla \psi \nabla \widetilde{\psi} \rangle$$

Odd potential A

Even potential ψ

Even field $F_+ = \nabla A$

Odd field $p_- = -\nabla \psi$

Note: Duality/massless condition
$\nabla \widetilde{A} = 0 \implies W_e + IW_m = 0$

Otherwise, W_e contributes a new term proportional to velocity in Lorentz force

Conclusions

- Acoustic fields can carry intrinsic spin, and we can now represent it theoretically.
- Geometric algebra highlights fruitful analogies between electromagnetism and acoustics.
- Geometric algebra simplifies reasoning about representation degeneracy.
- Experimental measurements of Noether currents can differentiate between representations.
- Dual symmetric representations are necessary and understudied.
- Future lightcone: investigate geometrically admitted extensions
  (e.g. pseudoscalar acoustic sources, scalar EM fields)

Dressel, Bliokh, Nori, Physics Reports Phys. Rep. 589, 1–71 (2015).

Burns, Bliokh, Nori, Dressel, New J Physics 22 053050 (2020)

Burns, Daniel, Alexander, Dressel, *Quantum Stud.: Math. Found.* **11**, 27–67 (2024).

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Helicity

Added fields from geometric completion

$$F_+ = W_e + F + IW_m = \nabla A$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{EM}}^{\mathrm{dual}} = \tfrac{1}{2}\langle \nabla A \nabla \widetilde{A} \rangle$$

$$A \mapsto A e^{I\beta}$$

$$\downarrow$$

$$J_\chi = \frac{1}{2}\langle I\widetilde{A}(\nabla A)\rangle_1$$

$$= \frac{1}{4}(W_m A_e - W_e A_m + (IF)\cdot A_e - F\cdot A_m)$$

longitudinal

$$p_- = p + wI = -\nabla\psi$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{ac}}^{\mathrm{dual}} = \tfrac{1}{2}\langle \nabla\psi \nabla\widetilde{\psi} \rangle$$

$$\psi \mapsto \psi e^{I\beta}$$

$$\downarrow$$

$$J_\chi = \frac{1}{2}\langle I\widetilde{\psi}(\nabla\psi)\rangle_1$$

$$= \frac{1}{2}\big(\phi w - \phi_w p + \vec{y}\cdot p - \vec{x}\cdot w - I(\vec{x}\wedge p + \vec{y}\wedge w)\big)$$

longitudinal

Helicity is conserved in vacuum where dual symmetry is present.

Burns et al. (2024)

Belinfante Momentum

$$T(n) = \frac{1}{2}\langle \nabla A n \widetilde{A} \nabla + \nabla \widetilde{A} n A \nabla \rangle$$

$$T(n) = \frac{1}{2}\langle \nabla \psi n \widetilde{\psi} \nabla + \nabla \widetilde{\psi} n \psi \nabla \rangle$$