Local Space Structure by Geometric Algebra Using the Hurwitz Unit Quaternions

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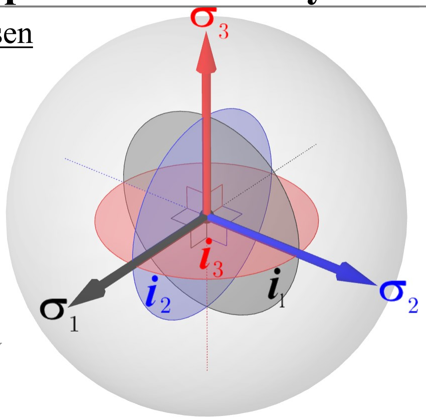


A unit sphere isometry

The quaternion unit bivector basis

Tree geometric perpendicular plane bivectors

$\underbrace{i_3 \perp i_1 \perp i_2 \perp i_3}_{\text{Geometric perpendicular}} \implies \underbrace{i_1\cdot i_2 = i_2\cdot i_3 = i_3\cdot i_1 = 0}_{\text{Algebraic orthogonal}}$.

Their intersection give one central locality; these units are Hamilton ***i,j,k*** *interconnected*:

$i_3 = i_2 i_1,\ \ i_2 = i_1 i_3,\ \ i_1 = i_3 i_2,\ \ i_1 i_2 i_3 = +1,$

negative signature $i_1^2 = i_2^2 = i_3^2 = -1 = i_3 i_2 i_1$.

Dual inherit from the Cartesian unit 1-vector basis

$\underbrace{\sigma_3 \perp \sigma_1 \perp \sigma_2 \perp \sigma_3}_{\text{Geometric perpendicular}} \implies \underbrace{\sigma_1\cdot\sigma_2 = \sigma_2\cdot\sigma_3 = \sigma_3\cdot\sigma_1 = 0}_{\text{Algebraic orthogonal}},$

and units $\sigma_k^2 = 1$, implying orthonormality for $\mathcal{G}_3(\mathbb{R})$.

$\mathcal{G}_3(\mathbb{R}) = \mathcal{G}_3^- + \mathcal{G}_3^+$, have chiral pseudoscalar duality

$i \coloneqq \sigma_1\sigma_2\sigma_3 \implies i_1 = i\sigma_1,\ \ i_2 = i\sigma_2,\ \ i_3 = i\sigma_3.$

Bivectors are rotation invariant in their own plane spinning freely generating 1-spinors, each with active intrinsic Angular Momentum preserved bivector area.

The local orthogonal Quantum Mechanic spin½ components

$S_1 = \frac{1}{2}\hbar i_1,\ \ S_2 = \frac{1}{2}\hbar i_2,\ \ S_3 = \frac{1}{2}\hbar i_3,\ \ \in \mathcal{G}_3^+,\ \ (\hbar = 1)$

and $S_k^\dagger = -S_k$ are the Angular Momenta active area bivectors.

The *interconnected* commutator product (one example of tree)

$[S_2, S_1] = 2[S_2 \times S_1] = S_2 S_1 - S_1 S_2 = \hbar S_3 \implies \underbrace{S_2 S_1 = ½\hbar S_3}_{\text{orthogonal product}}.$

These have dual 1-vector ***direction*** components

$s_1 = \frac{1}{2}\hbar\sigma_1,\ \ s_2 = \frac{1}{2}\hbar\sigma_2,\ \ s_3 = \frac{1}{2}\hbar\sigma_3 \ \in \mathcal{G}_3^- \ \Leftarrow \ s_k = \frac{1}{2}\hbar i i_k,$

with parity inversion $\overline{s_k} = -s_k$, and $s_1 s_2 = \frac{1}{2}\hbar i s_3 = \frac{1}{2}\hbar S_3$,

The eigenvalue: $J^2\left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle \doteq \hbar^2\tfrac{3}{4}\left|\tfrac{3}{4}, \pm\tfrac{1}{2}\right\rangle$ Casimir invariant is

$J^2 = k^2 = s_1^2 + s_2^2 + s_3^2 = s_k s_k \to ¾,\ \ (\hbar = 1)$

Compositions gives rise to ***eight superpositions***

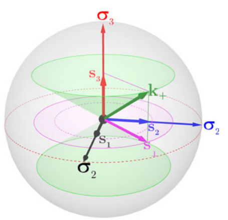
$\{\pm s_1 \pm s_2 \pm s_3\}_8 \to k_\epsilon \in \left\{\tfrac{1}{2}(\pm\sigma_1 \pm\sigma_2 \pm\sigma_3)\right\}_8.$

Eight orientations of *four* ***directions*** $k_\kappa \in \mathcal{G}_3^-$

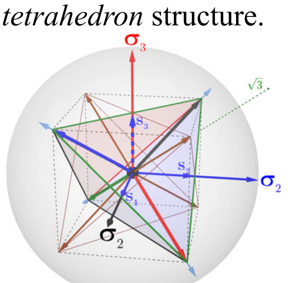
$k_{(\kappa+4)} = \overline{k_\kappa} = -k_\kappa,\ \ \kappa = 0,1,2,3,$ specific:

$$\left\{\begin{aligned} k_0 &= \tfrac{1}{2}(-\sigma_1 - \sigma_2 - \sigma_3),\\ k_1 &= \tfrac{1}{2}(-\sigma_1 + \sigma_2 + \sigma_3),\\ k_2 &= \tfrac{1}{2}(+\sigma_1 - \sigma_2 + \sigma_3),\\ k_3 &= \tfrac{1}{2}(+\sigma_1 + \sigma_2 - \sigma_3)\end{aligned}\right\}_4,\quad \begin{aligned} k_\kappa^2 &= ¾,\\ k_\kappa k_\kappa &= 3.\end{aligned}$$

Symmetric regularity $k_0 + k_1 + k_2 + k_3 = 0$, pointing out a *tetraon* in a *tetrahedron* structure.



Spin½ projection $s_3 = \pm½\sigma_3$, of $k_\pm = s_1 + s_2 \pm s_3$, with the transversal plane oscillation $s_\perp(\phi) = ½(\sigma_1 + \sigma_2)(\phi)$.



Tetrahedron structure in a unit cube central internal in one unit sphere supported of four *tetraon* spokes

Orthogonal Unit Quaternions have *eight* group elements

$U_\perp(\mathbb{H}) \coloneqq \{\pm 1,\ \pm i_1,\ \pm i_2,\ \pm i_3\}_8,$

The ***Hurwitz Unit Quaternions*** have 24 group elements

$U(\mathbb{H}) \coloneqq \left\{\pm 1,\ \pm i_1,\ \pm i_2,\ \pm i_3,\ \left\{\tfrac{1}{2}(\pm 1 \pm i_1 \pm i_2 \pm i_3)\right\}_{16}\right\}_{24}.$

Unit-½-quaternions (½-versors) have *sixteen* group elements

$\varrho_\epsilon \in U_{½}(\mathbb{H}) \coloneqq \left\{\tfrac{1}{2}(\pm 1 \pm i_1 \pm i_2 \pm i_3)\right\}_{16}.$

Each of these *sixteen* unit ½-versors is written $\quad \kappa = 0,1,2,3$

$\varrho_\epsilon = \frac{1}{2}\left(\epsilon_0 1 + \epsilon_1 i_1 + \epsilon_2 i_2 + \epsilon_3 i_3\right) = \varrho_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)},\ \ \boxed{\epsilon_\kappa = \pm 1,}$

these forms the *interconnected* units of a normal invariant subgroup

$\varrho_c = \varrho_a\varrho_b\varrho_a^\dagger = \varrho_a\varrho_b\varrho_a^{-1},$ all in $U_{½}(\mathbb{H})$, and $\varrho_\epsilon\varrho_\epsilon^\dagger = 1,$

inside the full closed Hurwitz Unit Quaternion group

$U_{½}(\mathbb{H})_{16} = U(\mathbb{H})_{24} \backslash U_\perp(\mathbb{H})_8 \subset U(\mathbb{H})_{24} \overleftarrow{\subset} \mathbb{H}.$

Adolf Hurwitz (1859-1919)

The *sixteen* distinct ½-versors of $\mathbb{U}_{½}(\mathbb{H})$ is

$$\left.\begin{aligned} \varrho_0 &= ½(+1 - i_1 - i_2 - i_3), & \varrho_0^\dagger &= ½(+1 + i_1 + i_2 + i_3),\\ \varrho_1 &= ½(+1 - i_1 + i_2 + i_3), & \varrho_1^\dagger &= ½(+1 + i_1 - i_2 - i_3),\\ \varrho_2 &= ½(+1 + i_1 - i_2 + i_3), & \varrho_2^\dagger &= ½(+1 - i_1 + i_2 - i_3),\\ \varrho_3 &= ½(+1 + i_1 + i_2 - i_3), & \varrho_3^\dagger &= ½(+1 - i_1 - i_2 + i_3),\end{aligned}\right\}\ \text{for } \epsilon_0 = +1, \text{ where } \varrho^2 = \varrho - 1 = -\varrho^\dagger \ \Downarrow$$

$$\left.\begin{aligned} \varrho_0^2 &= -\varrho_0^\dagger = ½(-1 - i_1 - i_2 - i_3), & -\varrho_0 &= ½(-1 + i_1 + i_2 + i_3) = (\varrho_0^\dagger)^2,\\ \varrho_1^2 &= -\varrho_1^\dagger = ½(-1 - i_1 + i_2 + i_3), & -\varrho_1 &= ½(-1 + i_1 - i_2 - i_3) = (\varrho_1^\dagger)^2,\\ \varrho_2^2 &= -\varrho_2^\dagger = ½(-1 + i_1 - i_2 + i_3), & -\varrho_2 &= ½(-1 - i_1 + i_2 - i_3) = (\varrho_2^\dagger)^2,\\ \varrho_3^2 &= -\varrho_3^\dagger = ½(-1 + i_1 + i_2 - i_3), & -\varrho_3 &= ½(-1 - i_1 - i_2 + i_3) = (\varrho_3^\dagger)^2.\end{aligned}\right\}\ \epsilon_0 = -1.$$

Unit-½-quaternions as 1-spinors; a scalar and a ***direction*** bivector,

$\varrho_\kappa = \frac{1}{2} + A_\kappa,\quad \varrho_\kappa^\dagger = \frac{1}{2} + A_\kappa^\dagger,\quad \varrho_\epsilon\varrho_\epsilon^\dagger = 1.\quad$ (8 for $\epsilon_0 = +1$)

Eight orientations of *four bivectors* ***direction*** planes of these ½-versors

$$\begin{aligned} A_0 &= \tfrac{1}{2}(-i_1 - i_2 - i_3), & A_0^\dagger &= \tfrac{1}{2}(+i_1 + i_2 + i_3),\\ A_1 &= \tfrac{1}{2}(-i_1 + i_2 + i_3), & A_1^\dagger &= \tfrac{1}{2}(+i_1 - i_2 - i_3),\\ A_2 &= \tfrac{1}{2}(+i_1 - i_2 + i_3), & A_2^\dagger &= \tfrac{1}{2}(-i_1 + i_2 - i_3),\\ A_3 &= \tfrac{1}{2}(+i_1 + i_2 - i_3), & A_3^\dagger &= \tfrac{1}{2}(-i_1 - i_2 + i_3).\end{aligned}$$

Note:

$A_\kappa^\dagger = -A_\kappa$, for each $\kappa = 0,1,2,3$: $A_\kappa^\dagger A_\kappa = ¾$, and $A_\kappa A_\kappa = -¾$,

and the regular *tetrahedron symmetry* balance addition rule

$A_0 + A_1 + A_2 + A_3 = 0 \implies A_0 = -(A_1 + A_2 + A_3)$

The bivectors are rotation invariant in their own plane, spinning freely.

$\psi_{\kappa\pm}^{½} \sim \rho_\kappa U_{\phi_\kappa \pm} = \rho_\kappa e^{\pm\widehat{A}_\kappa ½\phi_\kappa} = \rho_\kappa(\cos ½\phi_\kappa \pm \widehat{A}_\kappa \sin ½\phi_\kappa),$

are oscillating wave function components for the angular momenta 1-spinors.

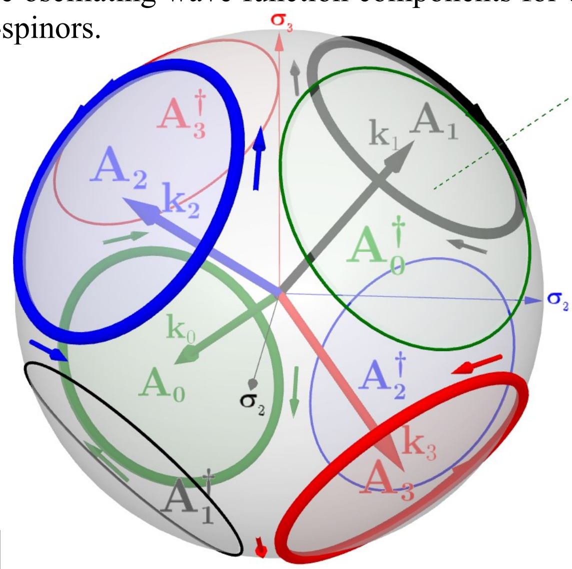


Figure: Display idea of one Spin½ Fermion with four cyclic oscillations in the *tetrahedron* ***direction*** symmetry in a local sphere. The *four* cyclic bivectors A_0, A_1, A_2, A_3, for Angular Momenta outwards *dextral* orientated, and their reversed outwards *sinistral* $A_\kappa^\dagger = -A_\kappa$, generating fluctuating oscillations which on the spherical surface does not exceed *retarded speed of information*. Hence *eight* independent bivector orientation possibilities to generate four 1-spinors, which superposes by synchronisation to one resulting 2-spinor quaternion wavefunction.