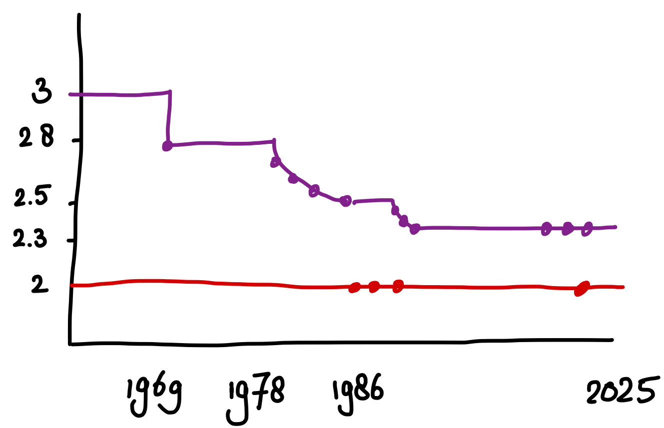
Applying the asymptotic spectrum from the theory of matrix multiplication to other complexity problems

Jeroen Zuiddam

Strassen (1969)

- Multiply $n \times n$ matrices in $\leq O(n^{2.8})$ operations
- Since then, many new ideas, upper bounds, barriers: Schönhage, Coppersmith, Winograd, …, Stothers, Le Gall, DeepMind, V.-Williams, Alman, …
- Currently: $\leq O(n^{2.37})$, best exponent still open



Strassen (1986-1991)

- Developed asymptotic spectrum duality, explaining, simplifying and extending the many ideas and techniques in matrix multiplication research

- While motivated by matrix multiplication, this theory is far more general

- leads to a broader framework that suits many other problems and settings

  - matrix multiplication
  - circuit complexity
  - communication
  - packing
  - cap set problem
  - entanglement ...

What is the cost of a task
if we have to perform it many times?

Direct-sum problems/
Economies of scale/
Savings

$$\lim_{n\to\infty} F(T^n)^{1/n}$$

Calvin Laboratory Experiment

(A) double espresso

(B) 2 x espresso

Results: (A) and (B) take the same time

$\rightsquigarrow$ no economies of scale

Shannon (1956)

Shannon capacity problem

- Motivated by zero-error communication, asks for determining the rate of growth of the independence number under strong graph power
- Wide range of upper and lower bound methods: Shannon (1956), Lovász (1979), Haemers, Alon, Schrijver, ..., DeepMind
- Despite this, even small instances have remained open, e.g. odd cycles of length ≥ 7

fast matrix multiplication $\leftrightsquigarrow$ Shannon capacity of graphs

[Strassen, 1969]

$$\underset{\sim}{R}(T) = \lim_{n\to\infty} R(T^{\otimes n})^{1/n}$$

[Shannon, 1956]

$$\Theta(G) = \lim_{n\to\infty} \alpha(G^{\boxtimes n})^{1/n}$$

[Christandl-Vrana-Z, 2018]

[Z, 2018]

[de Boer, Buys, Polak, Luchnikov, Vrana, ...]

Asymptotic spectrum duality

[Strassen, 1987-1991]

- [Wigderson-Z, 2025]
- [Trento lectures, 2025]

More applications: [Robere-Z, 2021], ..., [Alman-Li, 2025]

Asymptotic spectrum duality

Economies of scale problem	$\rightsquigarrow$ Asymptotic behaviour	$\rightsquigarrow$ Asymptotic relations	$\rightsquigarrow$ Asymptotic spectrum
	$\lim_{n\to\infty} R(T^{\otimes n})^{1/n}$	$S^{\otimes n} \leq T^{\otimes(n+o(n))}$	X

- matrix multiplication
- Shannon capacity
- cap set problem
- entanglement
- communication
- packing

⋮

Asymptotic spectrum duality

More knowledge of X gives stronger methods...

1. Matrix multiplication

1.1 Asymptotic rank and restriction

1.2 Asymptotic spectrum duality

1.3 Consequences

2. Shannon capacity of graphs

2.1 Asymptotic spectrum duality and distance

2.2 Converging sequences

2.3 New lower bound for C_{15}

("Laser method for Shannon capacity")

3. Open problems and further directions

1. Matrix multiplication 1.1 Asymptotic rank and restriction

[Strassen 1969]

$$R(\langle 2,2,2\rangle) \leq 7 \quad \Rightarrow \quad 2^{\omega} \leq 7$$

$$\forall n, r \quad R(\langle n,n,n\rangle) \leq r \quad \Rightarrow \quad n^{\omega} \leq r$$

[BCLR]

$$\forall n, r \quad \underline{R}(\langle n,n,n\rangle) \leq r \quad \Rightarrow \quad n^{\omega} \leq r$$

[Schönhage]

$\forall n_i, r$ $\qquad \underline{R}\left(\bigoplus_{i=1}^{k} \langle n_i, n_i, n_i \rangle\right) \leq r \implies \sum_{i=1}^{k} n_i^{\omega} \leq r$

$\forall a_i, b_i, c_i, r$ $\qquad \underline{R}\left(\bigoplus_{i=1}^{k} \langle a_i, b_i, c_i \rangle\right) \leq r \implies \sum_{i=1}^{k} (a_i b_i c_i)^{\omega/3} \leq r$

Survey [Bläser]

$$R(\langle 2,2,2\rangle) \leq 7$$

$$R(\langle 2,2,2\rangle^{\otimes k})^{1/k} \leq 7 = 2^{\log_2 7}$$

$$\underset{\sim}{R}(\langle 2,2,2\rangle) := \lim_{k\to\infty} R(\langle 2,2,2\rangle^{\otimes k})^{1/k} = 2^{\omega}$$

Asymptotic rank $\underset{\sim}{R}(T) := \lim_{k\to\infty} R(T^{\otimes k})^{1/k} \overset{!}{=} \inf_{k} R(T^{\otimes k})^{1/k}$

Conjecture: $\omega = 2$ i.e. $\underset{\sim}{R}(\langle 2,2,2\rangle) = 4$

Asymptotic rank conjecture

$$\forall\, T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d \text{ concise}, \quad \underset{\sim}{R}(T) = d$$

$S \in V_1 \otimes V_2 \otimes V_3$, $T \in W_1 \otimes W_2 \otimes W_3$

Restriction

$$S \leq T \quad :\Leftrightarrow \quad \exists A_i : W_i \to V_i, \; S = (A_1 \otimes A_2 \otimes A_3) \cdot T$$

Asymptotic restriction

$$S \lesssim T \quad :\Leftrightarrow \quad S^{\otimes n} \leq T^{\otimes (n + o(n))}$$

$$\overset{!}{\Rightarrow} \quad \underset{\sim}{R}(S) \leq \underset{\sim}{R}(T)$$

Theorem

- $\underset{\sim}{R}\left(\bigoplus_{i=1}^{k} \langle a_i, b_i, c_i \rangle \right) \leq r \quad \Rightarrow \quad \sum_{i=1}^{k} (a_i b_i c_i)^{\omega/3} \leq r$

- $\bigoplus_{i=1}^{k} \langle a_i, b_i, c_i \rangle \lesssim \langle r \rangle \quad \Rightarrow \quad \sum_{i=1}^{k} (a_i b_i c_i)^{\omega/3} \leq r$

1.2. Asymptotic spectrum duality

Asymptotic behaviour

"Asymptotic spectrum"

$$\underset{\sim}{R}$$

$$\lesssim \qquad \longleftrightarrow \qquad X$$

duality

$$\underset{\sim}{Q}$$

$$\lim_{n\to\infty} Q(T^{\otimes n})^{1/n}$$

Asymptotic spectrum of tensors

$$X = \{ \phi : \{\text{tensors}\} \to \mathbb{R}_{\geq 0} : \quad \forall S, T \quad \forall n \in \mathbb{N}$$

(1) $\phi(S \otimes T) = \phi(S) \cdot \phi(T)$,

(2) $\phi(S \oplus T) = \phi(S) + \phi(T)$,

(3) $\phi(\langle n \rangle) = n$,

(4) $S \leq T \Rightarrow \phi(S) \leq \phi(T) \quad \}$

Lemma
- $S \underset{\sim}{\leq} T \Rightarrow \phi(S) \leq \phi(T)$
- $\underset{\sim}{Q}(T) \leq \phi(T) \leq \underset{\sim}{R}(T)$

Proof.

$$\underset{\sim}{R}(T) \leq r \Rightarrow R(T^{\otimes n}) \leq r^{n+o(n)} \Rightarrow T^{\otimes n} \leq \langle r^{n+o(n)} \rangle \Rightarrow \phi(T)^n \leq r^{n+o(n)}$$

$\square$

Lemma • $S \underset{\sim}{\leq} T \implies \phi(S) \leq \phi(T)$

• $\underset{\sim}{Q}(T) \leq \phi(T) \leq \underset{\sim}{R}(T)$

Theorem (Strassen, 1988)

• $S \underset{\sim}{\leq} T \iff \forall \phi \in X,\ \phi(S) \leq \phi(T)$

• $\underset{\sim}{R}(T) = \max\limits_{\phi \in X} \phi(T)$

• $\underset{\sim}{Q}(T) = \min\limits_{\phi \in X} \phi(T)$

Question: Are $\underset{\sim}{R}, \underset{\sim}{Q} \in X$? No!

Algebraic geometry

Hilbert

- $I \subseteq \sqrt{I} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$

proper maximal ideals

$(x_1 - a_1, x_2 - a_2, \ldots)$

evaluation in $\mathbb{C}^n$

- $I(Z(I)) = \sqrt{I}$

- $q = 0$ on $Z(I)$

$\Rightarrow \exists \ell,\ q^{\ell} \in I$

"Asymptotic" real geometry

Strassen

- $\leq \ \subseteq \ \lesssim \ = \bigcap_{\widehat{\leq} \supseteq \leq} \widehat{\leq}$

"maximal Strassen preorders"

$\phi \in X : T_1 \mapsto a_1, T_2 \mapsto a_2, \ldots$

evaluation in $\mathbb{R}^n_{\geq 0}$

- $\leq(X(\leq)) = \lesssim$

- $q_1 \leq q_2$ on $X(\leq)$

$\Rightarrow q_1^{\ell} \leq q_2^{\ell + o(\ell)}$

1.3 Consequences

Duality theorem

- $S \underset{\sim}{\leq} T \iff \forall \phi \in X,\ \phi(S) \leq \phi(T)$
- $\underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$

Consequences:

Theorem $\underset{\sim}{R}$ is subadditive and submultiplicative

Theorem $\forall S, T,\ \underset{\sim}{R}(S \oplus T) = \underset{\sim}{R}(S) + \underset{\sim}{R}(T) \iff \underset{\sim}{R}(S \otimes T) = \underset{\sim}{R}(S)\underset{\sim}{R}(T)$

Theorem

$$\bigoplus_{i=1}^{\ell} \langle a_i, b_i, c_i \rangle \underset{\sim}{\leq} \langle r \rangle \implies \sum_{i=1}^{\ell} (a_i b_i c_i)^{\omega/3} \leq r$$

Duality theorem $\underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$

Theorem $\underset{\sim}{R}(S \otimes T) \leq \underset{\sim}{R}(S)\underset{\sim}{R}(T)$.

Proof $\underset{\sim}{R}(S \otimes T) = \phi(S \otimes T) = \phi(S)\phi(T) \leq \underset{\sim}{R}(S)\underset{\sim}{R}(T)$. $\square$

Theorem. $\forall S, T$, $\underset{\sim}{R}(S \oplus T) = \underset{\sim}{R}(S) + \underset{\sim}{R}(T) \iff \underset{\sim}{R}(S \otimes T) = \underset{\sim}{R}(S)\underset{\sim}{R}(T)$.

Proof $\Rightarrow$ • $\exists \phi \in X$, $\phi(S \oplus T) = \underset{\sim}{R}(S \oplus T)$.

- $\phi(S) + \phi(T) = \phi(S \oplus T) \overset{!}{=} \underset{\sim}{R}(S \oplus T) = \underset{\sim}{R}(S) + \underset{\sim}{R}(T)$
- $\phi(S) = \underset{\sim}{R}(S)$ and $\phi(T) = \underset{\sim}{R}(T)$.
- $\underset{\sim}{R}(S)\underset{\sim}{R}(T) = \phi(S)\phi(T) = \phi(S \otimes T) \leq \underset{\sim}{R}(S \otimes T) \leq \underset{\sim}{R}(S)\underset{\sim}{R}(T)$

$\Leftarrow$ Similar. $\square$

Theorem

$$\bigoplus_{i=1}^{\ell} \langle a_i, b_i, c_i \rangle \lesssim \langle r \rangle \Rightarrow \sum_{i=1}^{\ell} (a_i b_i c_i)^{\omega/3} \leq r$$

Proof

- $2^{\omega} \stackrel{!}{=} \phi(\langle 2,2,2 \rangle) = \underbrace{\phi(\langle 2,1,1 \rangle)}_{2^{x_1}} \cdot \underbrace{\phi(\langle 1,2,1 \rangle)}_{2^{x_2}} \cdot \underbrace{\phi(\langle 1,1,2 \rangle)}_{2^{x_3}}$

- Claim: $\phi(\langle a_i, b_i, c_i \rangle) = a_i^{x_1} b_i^{x_2} c_i^{x_3}$ (exercise)

- Then $\sum_{i=1}^{\ell} a_i^{x_1} b_i^{x_2} c_i^{x_3} \leq r$.

- The asymptotic spectrum is S_3-invariant: $(123)\phi, (123)^2\phi \in X$

- so also $\sum_{i=1}^{\ell} a_i^{x_3} b_i^{x_1} c_i^{x_2} \leq r$ and $\sum_{i=1}^{\ell} a_i^{x_2} b_i^{x_3} c_i^{x_1} \leq r$.

- Convexity: $\sum_{i=1}^{\ell} (a_i b_i c_i)^{\frac{x_1+x_2+x_3}{3}} \leq r$ and $x_1 + x_2 + x_3 = \omega$ $\square$

Asymptotic behaviour

"Asymptotic spectrum"

$\underset{\sim}{R}$

$\underset{\sim}{Q}$

$\underset{\sim}{\leq} \quad \xleftrightarrow[\text{duality}]{} \quad X$

$\rightsquigarrow$ Constructing elements of X

$\rightsquigarrow$ Structure of X

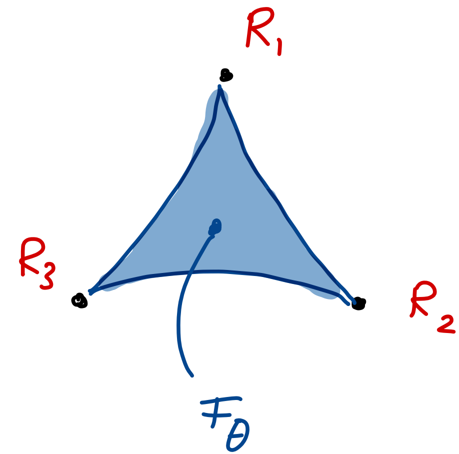
What is in the asymptotic spectrum?

Flattening ranks $R_1, R_2, R_3 \in X$

Quantum functionals $F_\theta \in X$, $\theta_1, \theta_2, \theta_3 \geq 0$,
$\sum_i \theta_i = 1$

- quantum information
- representation theory

} moment polytopes



If $\{F_\theta : \theta\} = X$, then the asymptotic rank conjecture holds (and $\omega = 2$).

What is the structure of the asymptotic spectrum

- **Theorem**

  $X|_{\langle n,n,n\rangle,\, n\in\mathbb{N}}$ is "log-convex" $\rightsquigarrow$ potentially very powerful unused algorithmic methods, "error correction"

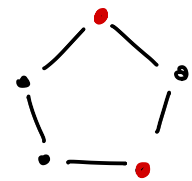
- **Theorem** $\rightsquigarrow$ $T_1, T_2, \ldots \to \langle n,n,n\rangle$

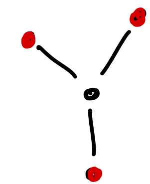
  - $\forall F \in X$, F is semicontinuous
  - $\underset{\sim}{R}$ is semicontinuous

  $\underset{\sim}{R}(T_i) \leq r \Rightarrow \underset{\sim}{R}(\langle n,n,n\rangle) \leq r$

2. Shannon capacity of graphs

Independent set :





Independence number $\alpha(G)$: $\quad 2 \quad\quad 3$

Strong product $\quad G \boxtimes H$: graph whose adjacency matrix is the tensor product of those of G and H

Cohomomorphism $\quad G \leq H$: if there is a map $V(G) \to V(H)$ preserving independent sets

Shannon capacity $\quad \Theta(G) := \lim_{n\to\infty} \alpha(G^{\boxtimes n})^{1/n} \overset{!}{=} \sup_{n} \alpha(G^{\boxtimes n})^{1/n}$

2.1 Asymptotic spectrum duality and distance

$$\Theta(G) = \lim_{n\to\infty} \alpha(G^{\boxtimes n})^{1/n} = \sup_{n} \alpha(G^{\boxtimes n})^{1/n}$$

Duality theorem (Strassen 1988, Zuiddam 2018) $\Theta(G) = \min_{F \in X} F(G)$

Def. Asymptotic spectrum X := set of all functions F: graphs $\to \mathbb{R}_{\geq 0}$ that are $\boxtimes$-mult., $\sqcup$-add., K_1-norm. and monotone under cohomomorphism

Ex. X contains: Lovász theta, fractional Haemers bound (Bukh-Cox), fractional clique covering number, ...

Def. Asymptotic spectrum distance: $d(G,H) = \max_{F \in X} |F(G) - F(H)|$

Def. Asymptotic spectrum distance: $d(G,H) = \max_{F \in X} |F(G) - F(H)|$

Lemma $G_i \to H \Rightarrow \Theta(G_i) \to \Theta(H)$

Proof Let $\varepsilon > 0$

There is N s.t for all $i > N$ and all $F \in X$, $|F(G_i) - F(H)| < \varepsilon$

Let $F_{G_i}, F_H \in X$ s.t. $F_{G_i}(G_i) = \Theta(G_i)$ and $F_H(H) = \Theta(H)$ (duality)

$\Theta(H) = F_H(H) > F_H(G_i) - \varepsilon \geq \Theta(G_i) - \varepsilon$ (same with H and G_i swapped) □

Lemma The following are equivalent: (1) $d(G,H) \leq \frac{a}{b}$

(2) $(\underbrace{G \sqcup \cdots \sqcup G}_{b})^{\boxtimes n} \leq (\underbrace{H \sqcup \cdots \sqcup H}_{b} \sqcup \underbrace{\bullet \cdots \bullet}_{a})^{\boxtimes (n + o(n))}$ and G, H swapped

(3) $(E_b \boxtimes G)^{\boxtimes n} \leq ((E_b \boxtimes H) \sqcup E_a)^{\boxtimes (n + o(n))}$ "

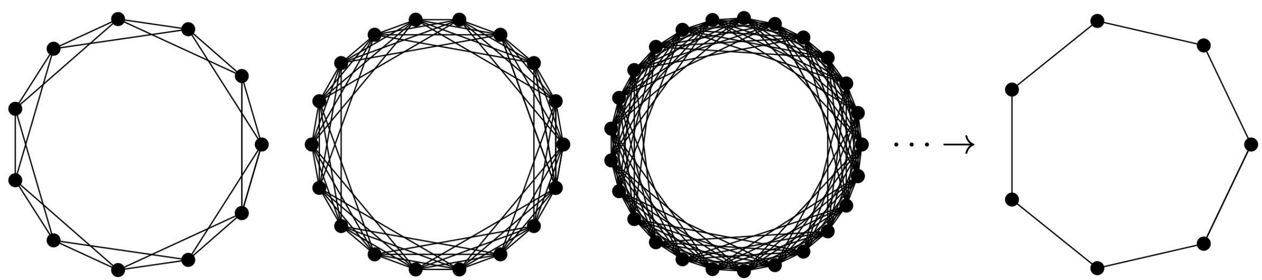
Graph limit approach:

$$G_1, G_2, \ldots \longrightarrow C_7 \quad \Rightarrow \quad \Theta(G_1), \Theta(G_2), \ldots \longrightarrow \Theta(C_7)$$

Questions:

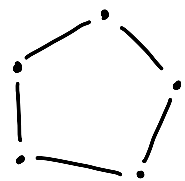
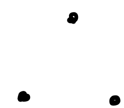
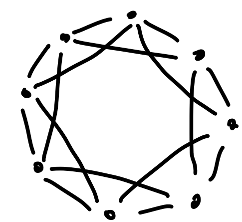
- How to construct converging sequences?
- Where to look for graphs that are "easier to analyse"?

We answer both! [de Boer, Buys, Polak]



2.2. Converging sequences

<u>Def</u> [Zhu, Hell–Nešetřil] Fraction graph $E_{a/b}$ has vertex set $\mathbb{Z}/a\mathbb{Z}$ and an edge $u \sim v$ iff $|u-v| < b \pmod{a}$

$E_{\frac{5}{2}} =$  $E_{\frac{3}{1}} =$  $E_{\frac{8}{3}} =$ 

<u>Lemma</u> [Hell–Nešetřil] $E_{a/b} \leq E_{c/d}$ iff $\frac{a}{b} \leq \frac{c}{d}$ (in $\mathbb{Q}$)

<u>Theorem A</u> For any $a/b \geq 2$, if $c_n/d_n \to a/b$ from above, then $E_{c_n/d_n} \longrightarrow E_{a/b}$.

<u>Theorem B</u> For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy.

Ingredients

Lemma 1. [Vrana] G vertex transitive, $S \subseteq V(G)$, $F \in X$, then

$$F(G[S]) \leq F(G) \leq \frac{|G|}{|S|} \cdot F(G[S]).$$

Lemma 2 [Hell–Nešetřil] Any fraction graph $E_{p/q}$ minus a vertex is equivalent to some fraction graph $E_{p'/q'}$ for $p' < p$, $q' < q$ with $p \cdot q' - q \cdot p' = 1$.

Consequence: $F(E_{p'/q'}) \leq F(E_{p/q}) \leq \frac{p}{p-1} F(E_{p'/q'})$

Theorem A For any $a/b \geq 2$, if $c_n/d_n \to a/b$ from above, then $E_{c_n/d_n} \longrightarrow E_{a/b}$.

Proof sketch: Let a, b coprime.

There are x, y with $x \cdot b - y \cdot a = 1$.

Then $c_n \cdot b - d_n\, a = 1$ for $c_n = x + a \cdot n$ and $d_n = y + b \cdot n$

So:

$$F(E_{a/b}) \leq F(E_{c_n/d_n}) \leq \frac{c_n}{c_n - 1} F(E_{a/b})$$

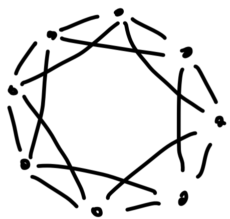
Let $n \to \infty$, then $c_n \to \infty$. $\square$

Theorem B For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy.

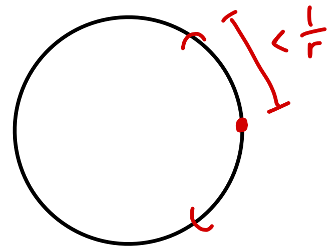
Proof sketch: Continued fraction convergents: $\frac{p_0}{q_0} < \frac{p_2}{q_2} \cdots < r < \cdots \frac{p_3}{q_3} < \frac{p_1}{q_1}$

Satisfy: $q_n \cdot p_{n-1} - p_n \cdot q_{n-1} = (-1)^n$. $\square$

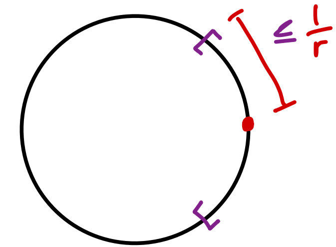
Infinite graphs as limit points



$E_{8/3}$



E_r^{o}



E_r^{c}

Theorem

For any irrational $r \geq 2$, $d(E_r^{o}, E_r^{c}) = 0$ and if $a_n/b_n \longrightarrow r$, then $E_{a_n/b_n} \longrightarrow E_r^{o}$

Theorem

$d(E_{p/q}^{c}, E_{p/q}^{o}) = 0$ iff $a_n/b_n \to p/q$ from below $\Rightarrow$ $E_{a_n/b_n} \longrightarrow E_{p/q}$.

2.3 New lower bound for C_{15}

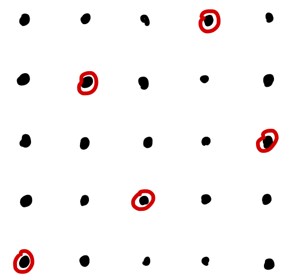
- "Finite version" of graph limit approach:

auxiliary graph H close to target graph

- Orbit independent sets:

$\Theta(C_5) = \sqrt{5}$

$\alpha(C_5^{\boxtimes 2}) = 5$



$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$

(Shannon 1956)

G	H	orbit independent set in $H^{\boxtimes k}$	reduction	$\leq \Theta(G)$
$E_{5/2}$	$E_{5/2}$	$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$	$H = G$	2.23 [Sha56]
$E_{7/2}$	$E_{382/108}$	$\{t \cdot (1,7,7^2,7^3,7^4) : t \in \mathbb{Z}_{382}\}$	$G \leq H$	3.25 [PS19]
$E_{9/2}$	$E_{9/2}$	$\{s \cdot (1,0,2) + t \cdot (0,1,4) : s,t \in \mathbb{Z}_9\}$	$H = G$	4.32 [BMR+71]
$E_{11/2}$	$E_{148/27}$	$\{t \cdot (1,11,11^2) : t \in \mathbb{Z}_{148}\}$	$H \leq G$	5.28 [BMR+71]
$E_{13/2}$	$E_{247/38}$	$\{t \cdot (1,19,117) : t \in \mathbb{Z}_{247}\}$	$H \leq G$	6.27 [BMR+71][18]
$E_{15/2}$	$E_{2873/381}$	$\{t \cdot (1,15,1073,1125) : t \in \mathbb{Z}_{2873}\}$	$G \leq H$	7.30 (Section 6.2)

Th Orbits are optimal for $\Theta(E_{p/q})$ when $p/q \to \infty$

Open problems

- Approximation of matrix multiplication in asymptotic spectrum distance? Role of symmetry?
- Complexity of $\lesssim$? For graphs: universal for all countable orders (Luchnikov)
- Construct spectral points
- Structure (connected, convex, ...) of asymptotic spectra
- Links with graphons, sphere packing, tropical geometry, effective Positivstellensatz ...

Further directions

• tensors	Christandl, Vrana, Holtrup, Hoeberechts, ...
• graphs	Buys, Polak, Luchnikov, Vrana, ...
• amortized circuits	Robere - Zuiddam
• nonnegative rank	Chee-Le-Ta
• depth-2 ciruits	Alman-Li
• LOCC entanglement transformations	Bugar - Vrana

...

Survey [Wigderson - Z 2025]