Tensor ranks: from asymptotic spectrum duality to quantum functionals and moment polytopes

Jeroen Zuiddam

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- Key insight: complexity characterized by tensor rank and asymptotic behaviour
- Recent years: many tensor rank notions with various applications

(talk by Derksen in previous workshop)

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Christandl-Vrana-Zuiddam (2018)

- We construct such spectral points: quantum functionals

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Christandl - Vrana - Zuiddam (2018)

- We construct such spectral points: quantum functionals
- Using representation theory (Schur-Weyl duality, moment polytopes) and quantum information (talk by Christandl in prev. workshop)

Matrix multiplication $\longrightarrow$ Tensor ranks

$\vdots$

R, Q, SR

(Strassen 1987-1991)

Matrix multiplication $\longrightarrow$ Tensor ranks $\longrightarrow$ Asymptotic behaviour

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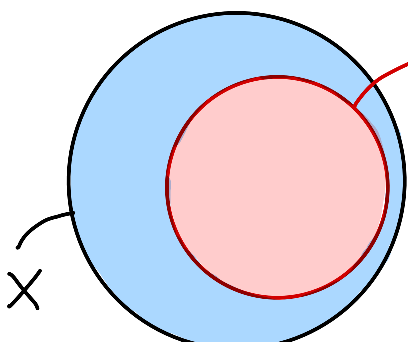
$\longleftarrow$

Christandl–Vrana–Z 2018



Quantum functionals

(Strassen 1987–1991)

Matrix multiplication $\to$ Tensor ranks $\to$ Asymptotic behaviour $\to$ Asymptotic spectrum

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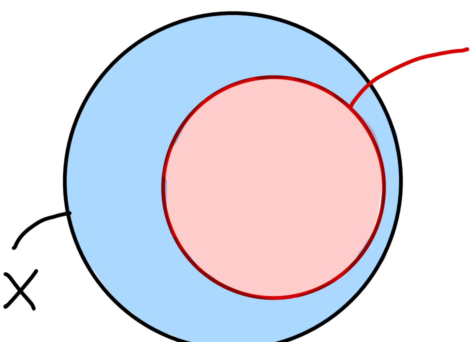
Christandl–Vrana–Z 2018



Quantum functionals $\leftsquigarrow$ Moment polytopes

(Strassen 1987–1991)

Matrix multiplication → Tensor ranks → Asymptotic behaviour → Asymptotic spectrum

⋮

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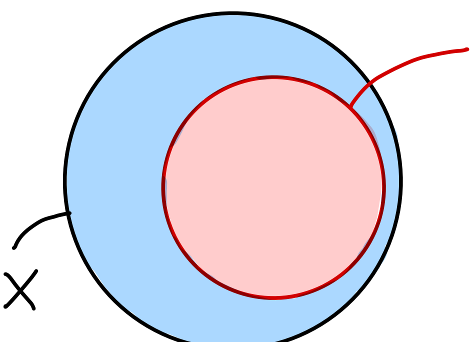
Van den Berg – C – Lysikov – Nieuwboer – Walter – Z 2025

Christandl – Vrana – Z 2018



Quantum functionals ⇜ Moment polytopes ⇜ Algorithms, explicit constructions

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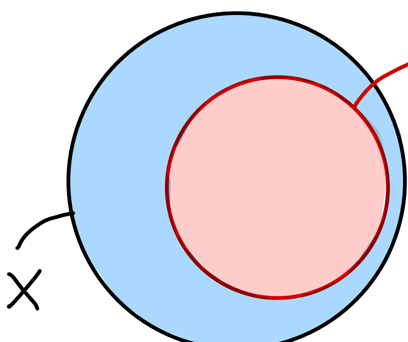

Quantum functionals $\leftsquigarrow$ Moment polytopes $\leftsquigarrow$ Algorithms, explicit constructions

Survey: Wigderson – Z 2025, Trento lectures 2025

1. Tensor ranks

2. Asymptotic spectrum duality

3. Quantum functionals

Intermezzo. Representation theory

4. Moment polytopes

5. Practical algorithm, explicit moment polytopes, explicit quantum functionals, explicit "hard" tensors

1. Tensor ranks

tensor $\quad T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$

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$$T = \sum_{ijk} T_{ijk}\, e_i \otimes e_j \otimes e_k$$

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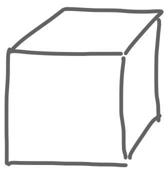

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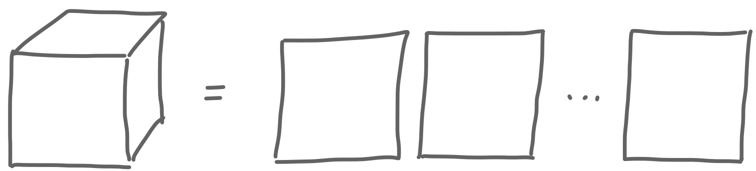

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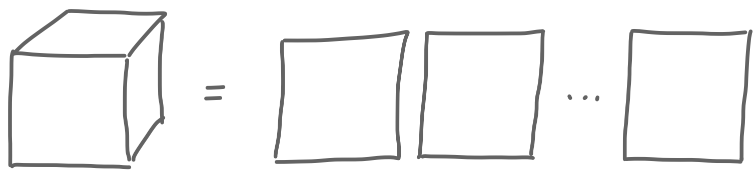

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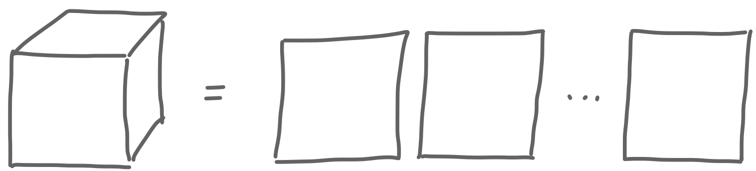

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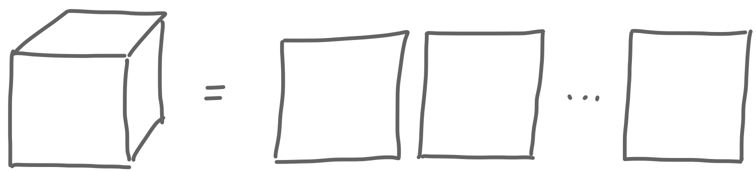

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Generic tensor has rank $\sim n^2$

Best know explicit construction has rank $c \cdot n$

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Lemma $\quad 2^\omega = \underset{\sim}{R}(\langle 2,2,2\rangle)$

Problem $\quad$ Is there <u>any</u> $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ such that $\underset{\sim}{R}(T) > n$

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more ranks:

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slice rank

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slice rank $\quad SR(T) = \min\{ a+b+c : \exists g \in GL,$

$$\operatorname{supp}(g \cdot T) \subseteq [a] \times [n] \times [n] \cup [n] \times [b] \times [n] \cup [n] \times [n] \times [c]. \}$$

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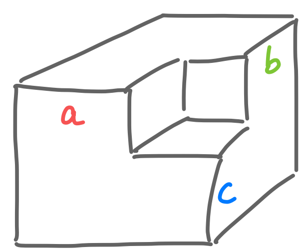

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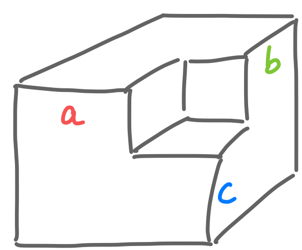

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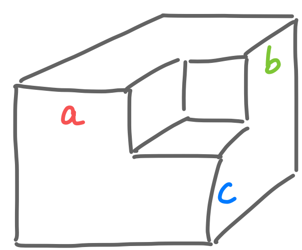
$Q \leq \underline{Q} \leq SR$



Applications:

- combinatorics
- barriers

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Asymptotic restriction $\quad S \lesssim T \quad :\Leftrightarrow \quad S^{\otimes n} \leq T^{\otimes (n+o(n))}$

2. Asymptotic spectrum duality

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Problem What is X? Ex: Flattening ranks $R_1, R_2, R_3 \in X$

Algebraic geometry	"Asymptotic" real geometry
Hilbert	Strassen

Algebraic geometry

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Challenge: Find X !

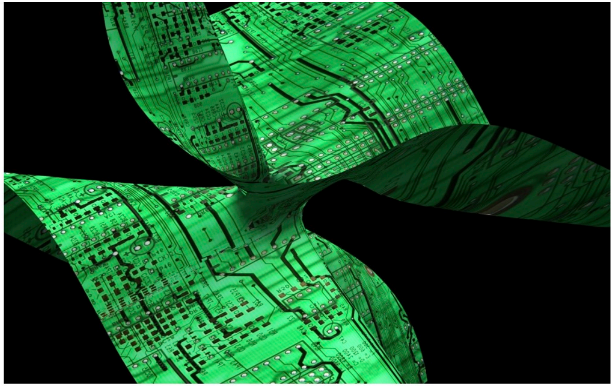
Calvin Laboratory History

Geometric Complexity Theory

Program
Algorithms and Complexity in Algebraic Geometry

Location
Calvin Lab Auditorium

Date
Monday, Sept. 15 – Friday, Sept. 19, 2014



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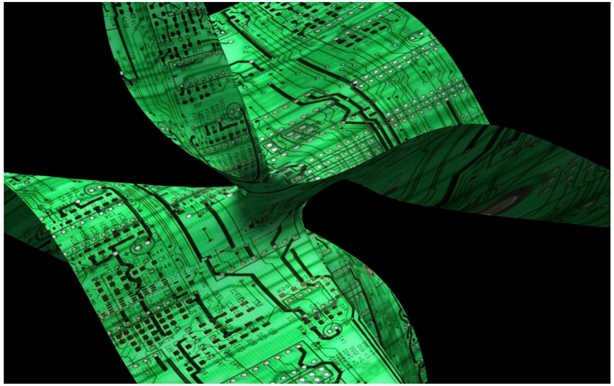

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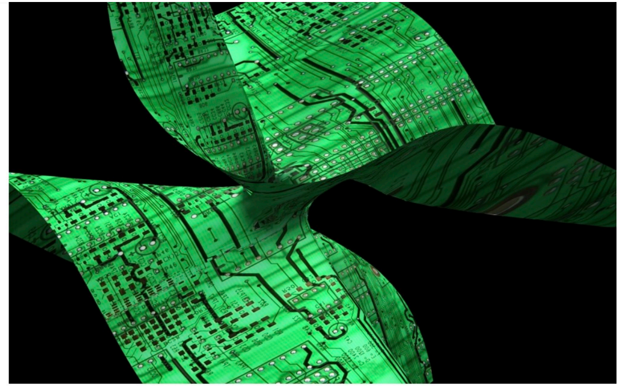

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2 – 2:50 p.m.	Asymptotic Spectra: Old and New Insights Markus Bläser, Universität des Saarlandes	Video

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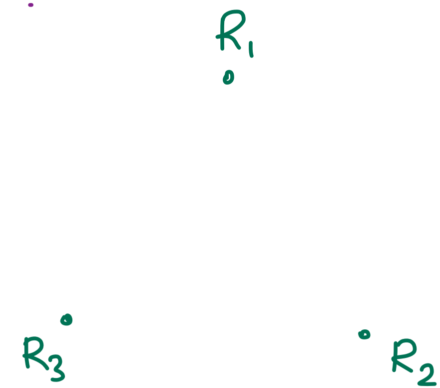

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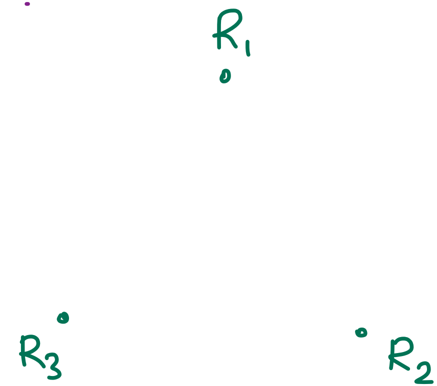

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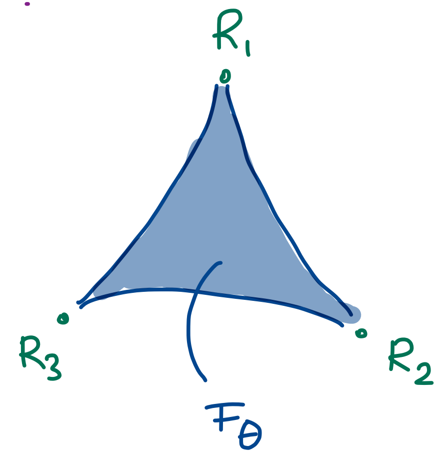

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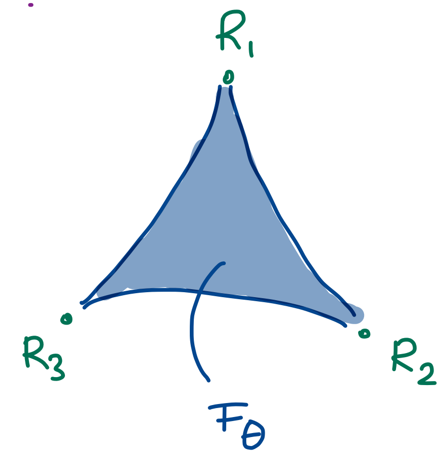

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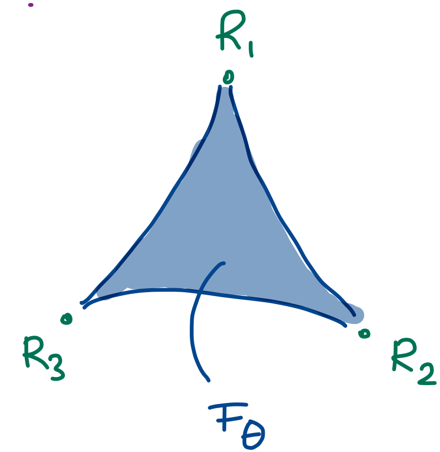

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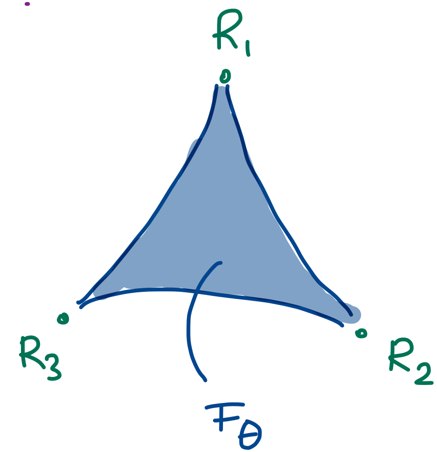

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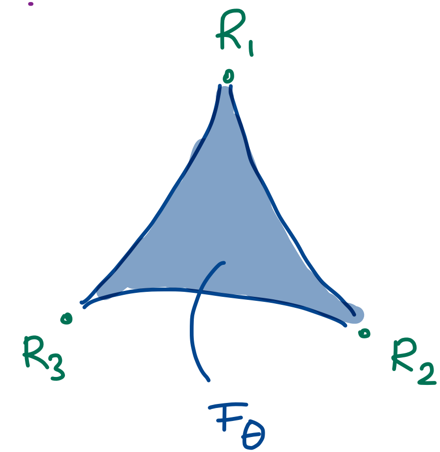

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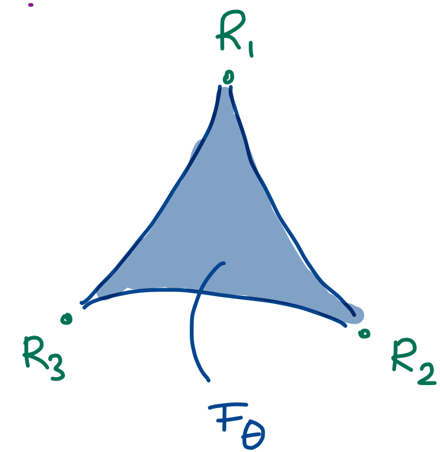

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$\Rightarrow$ asymptotic rank conjecture, $\omega = 2$, ...

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$\rightsquigarrow$ Coarser classification that is meaningful and finite.
Moment polytopes $\Delta(T)$

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Representation theory description

$\Delta(T) \overset{!}{=}$ ... $T^{\otimes n}$ ..

Intermezzo: Representation theory

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Subrepresentation $\quad G \circlearrowright V, \quad W \subseteq V, \quad G\cdot W \subseteq W$

Intermezzo: Representation theory

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Subrepresentation $G \circlearrowright V$, $W \subseteq V$, $G \cdot W \subseteq W$

Irreducible representation (irrep) $G \circlearrowright V$ such that $\{0\}, V \subseteq V$ are the only subrepresentations

Intermezzo: Representation theory

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Isomorphism $G \circlearrowright V, W$, $\phi : V \to W$ bijective, G-equivariant. $V \cong W$.

Symmetric group irreps

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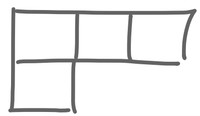
Ex. $\lambda = (3,1) \vdash_2 4$ 

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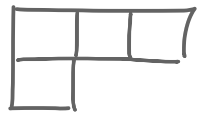
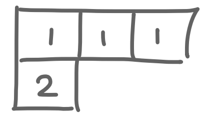
Ex. $\lambda = (3,1) \vdash_2 4$  $\rightsquigarrow$ 

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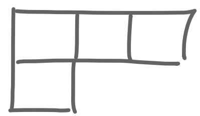
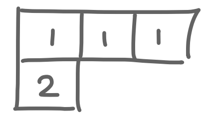
Ex. $\lambda = (3,1) \vdash_2 4$  $\rightsquigarrow$ 

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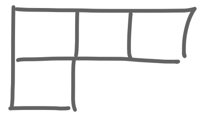
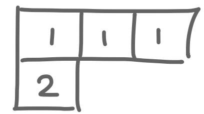
Ex. $\lambda = (3,1) \vdash_2 4$  $\rightsquigarrow$  $v_\lambda = (e_1 \wedge e_2) \otimes e_1 \otimes e_1$

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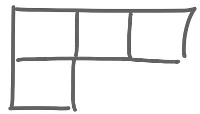
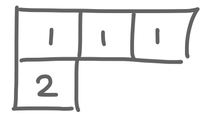
Ex. $\lambda = (3,1) \vdash_2 4$  $\rightsquigarrow$ 

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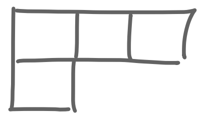
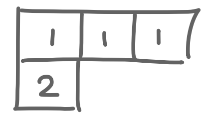
Ex. $\lambda = (3,1) \vdash_2 4$  $\rightsquigarrow$ 

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match!

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Quantum functionals

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Proof of $F_\theta \in X$ uses both descriptions, semigroup property of Kronecker coefficients, entropy inequalities, ..

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Algorithm. Given T, outputs a defining set of inequalities for $\Delta(T)$

Previously known $\Delta(T)$:

- for every $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
- for generic $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ $n = 3, 4, 5$ [Vergne-Walter, Ressayre]

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nurmiev classification of GL-orbits in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

Moment polytopes for $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

We compute all 29 of them: arxiv.org/abs/2510.08336

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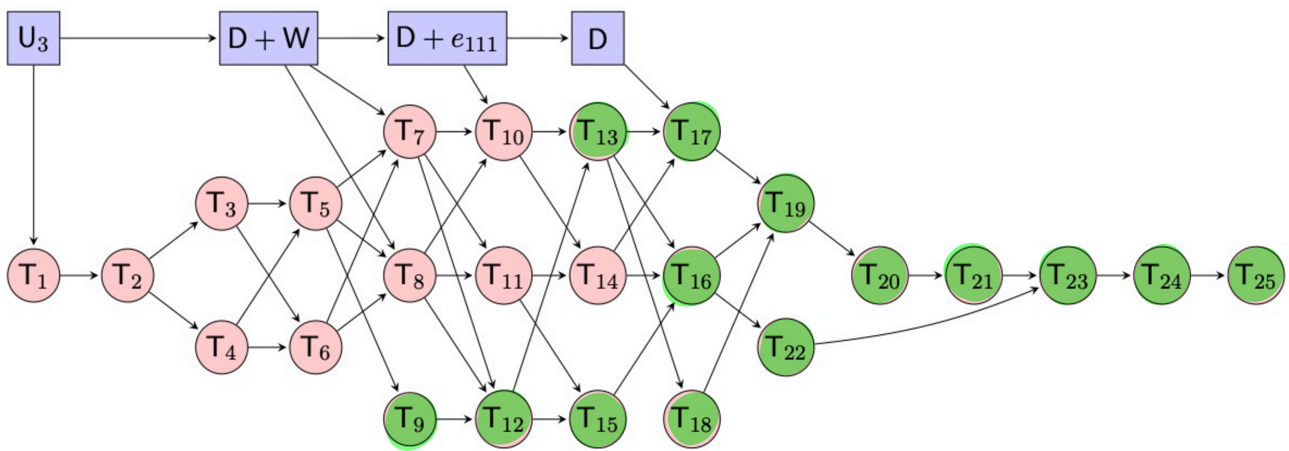

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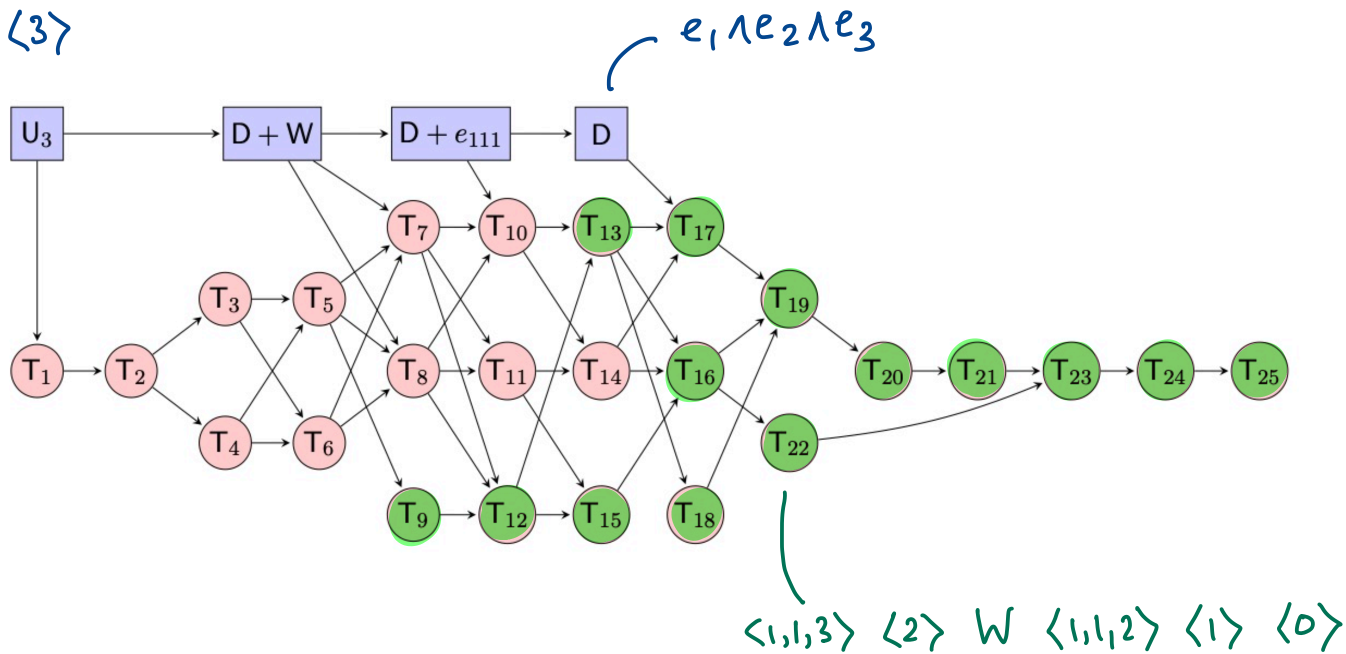

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Vertices	Generic	D	D + W	$D + e_{111}$	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}	T_{24}	T_{25}
(1, 0, 0 | 1, 0, 0 | 1, 0, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	
(1, 0, 0 | 1/2, 1/2, 0 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●		
(1/2, 1/2, 0 | 1, 0, 0 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				
(1/2, 1/2, 0 | 1/2, 1/2, 0 | 1, 0, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				
(1/2, 1/2, 0 | 1/2, 1/2, 0 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●					
(1, 0, 0 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●	●	●	●						●			
(1/3, 1/3, 1/3 | 1, 0, 0 | 1/3, 1/3, 1/3)	●		●	●	●	●	●	●	●	●	●	●		●	●			●											
(1/3, 1/3, 1/3 | 1/3, 1/3, 1/3 | 1, 0, 0)	●		●	●	●	●	●	●	●	●	●	●		●	●														
(3/4, 1/4, 0 | 1/2, 1/2, 0 | 1/2, 1/4, 1/4)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●						
(3/4, 1/4, 0 | 1/2, 1/4, 1/4 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●								
(1/2, 1/4, 1/4 | 3/4, 1/4, 0 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●		●	●			●											
(1/2, 1/4, 1/4 | 1/2, 1/2, 0 | 3/4, 1/4, 0)	●	●	●	●	●	●	●	●	●	●	●	●		●	●			●											
(1/2, 1/2, 0 | 3/4, 1/4, 0 | 1/2, 1/4, 1/4)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●						
(1/2, 1/2, 0 | 1/2, 1/4, 1/4 | 3/4, 1/4, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●								
(2/3, 1/3, 0 | 2/3, 1/3, 0 | 1/3, 1/3, 1/3)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●						
(2/3, 1/3, 0 | 1/3, 1/3, 1/3 | 2/3, 1/3, 0)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●								
(1/3, 1/3, 1/3 | 2/3, 1/3, 0 | 2/3, 1/3, 0)	●	●	●	●	●	●	●	●	●	●	●	●		●	●			●											
(1/2, 1/2, 0 | 1/2, 1/2, 0 | 1/3, 1/3, 1/3)	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●					●							
(1/2, 1/2, 0 | 1/3, 1/3, 1/3 | 1/2, 1/2, 0)	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●												
(1/3, 1/3, 1/3 | 1/2, 1/2, 0 | 1/2, 1/2, 0)	●		●	●	●	●	●	●	●	●	●	●		●															
(2/3, 1/6, 1/6 | 2/3, 1/6, 1/6 | 1/2, 1/2, 0)	●		●	●	●	●	●	●	●	●	●	●		●	●														
(2/3, 1/6, 1/6 | 1/2, 1/2, 0 | 2/3, 1/6, 1/6)	●		●	●	●	●	●	●	●	●	●	●		●	●			●											
(1/2, 1/2, 0 | 2/3, 1/6, 1/6 | 2/3, 1/6, 1/6)	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●	●	●	●									
(2/3, 1/6, 1/6 | 1/2, 1/2, 0 | 1/3, 1/3, 1/3)	●	●	●	●	●	●	●	●	●	●	●	●		●															
(2/3, 1/6, 1/6 | 1/3, 1/3, 1/3 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●		●				●											
(1/2, 1/2, 0 | 2/3, 1/6, 1/6 | 1/3, 1/3, 1/3)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				●								
(1/2, 1/2, 0 | 1/3, 1/3, 1/3 | 2/3, 1/6, 1/6)	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				●								
(1/3, 1/3, 1/3 | 2/3, 1/6, 1/6 | 1/2, 1/2, 0)	●	●	●	●	●	●	●	●	●	●	●	●		●				●											
(1/3, 1/3, 1/3 | 1/2, 1/2, 0 | 2/3, 1/6, 1/6)	●	●	●	●	●	●	●	●	●	●	●	●		●															
(1/2, 1/4, 1/4 | 1/2, 1/2, 0 | 1/2, 1/4, 1/4)	○	○	○	○	○	○	○	○	○	○	○	○		○	●														
(1/2, 1/4, 1/4 | 1/2, 1/4, 1/4 | 1/2, 1/2, 0)	○	○	○	○	○	○	○	○	○	○	○	○		○	●			○											
(1/2, 1/2, 0 | 1/2, 1/4, 1/4 | 1/2, 1/4, 1/4)	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○		●		○								
(2/3, 1/3, 0 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	○		●		○	○	○	●	●	●		●	○			●			●										
(1/3, 1/3, 1/3 | 2/3, 1/3, 0 | 1/3, 1/3, 1/3)	○		●		○	○	○	●	●	●	●	●			●														
(1/3, 1/3, 1/3 | 1/3, 1/3, 1/3 | 2/3, 1/3, 0)	○		●		○	○	●	○	○	●	●	●			●														
(1/2, 1/2, 0 | 1/2, 1/3, 1/6 | 1/3, 1/3, 1/3)	○		●		○	○	○	●	●	●		●	○			●													
(1/2, 1/2, 0 | 1/3, 1/3, 1/3 | 1/2, 1/3, 1/6)	○		●		○	○	○	●	●	●		●	○			●													
(1/2, 1/3, 1/6 | 1/2, 1/2, 0 | 1/3, 1/3, 1/3)	○		●		○	○	○	●	●	●	●	●																	
(1/2, 1/3, 1/6 | 1/3, 1/3, 1/3 | 1/2, 1/2, 0)	○		●		○	○	●	○	○	●	●	●																	
(1/3, 1/3, 1/3 | 1/2, 1/2, 0 | 1/2, 1/3, 1/6)	○		●		○	○	○	●	●	●	●	●																	
(1/3, 1/3, 1/3 | 1/2, 1/3, 1/6 | 1/2, 1/2, 0)	○		●		○	○	●	○	○	●	●	●																	
(1/2, 1/2, 0 | 2/5, 2/5, 1/5 | 2/5, 3/10, 3/10)	○		●		○	○	○	●	●	●		●	○			●													
(1/2, 1/2, 0 | 2/5, 3/10, 3/10 | 2/5, 2/5, 1/5)	○		●		○	○	○	●	●	●		●	○			●													
(2/5, 2/5, 1/5 | 1/2, 1/2, 0 | 2/5, 3/10, 3/10)	○		●		○	○	○	●	●	●	●	●																	
(2/5, 2/5, 1/5 | 2/5, 3/10, 3/10 | 1/2, 1/2, 0)	○		●		○	○	●	○	○	●	●	●																	
(2/5, 3/10, 3/10 | 1/2, 1/2, 0 | 2/5, 2/5, 1/5)	○		●		○	○	○	●	●	●	●	●																	
(2/5, 3/10, 3/10 | 2/5, 2/5, 1/5 | 1/2, 1/2, 0)	○		●		○	○	●	○	○	●	●	●																	
(3/8, 3/8, 1/4 | 1/2, 1/2, 0 | 3/8, 3/8, 1/4)	○		●		○	○	○	●	●	●	●	●																	
(3/8, 3/8, 1/4 | 3/8, 3/8, 1/4 | 1/2, 1/2, 0)	○		●		○	○	●	○	○	●	●	●																	
(1/2, 1/2, 0 | 3/8, 3/8, 1/4 | 3/8, 3/8, 1/4)	○		●		○	○	○	●	●	●		●	○			●													

Vertices	Generic	D	D + W	$D + e_{111}$	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}	T_{24}	T_{25}
(1/2, 1/2, 0 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	●				●	●	●						●																
(1/3, 1/3, 1/3 | 1/2, 1/2, 0 | 1/3, 1/3, 1/3)	●				●	●	●																						
(1/3, 1/3, 1/3 | 1/3, 1/3, 1/3 | 1/2, 1/2, 0)	●				●	●		●	●																				
(3/5, 1/5, 1/5 | 3/5, 1/5, 1/5 | 2/5, 2/5, 1/5)	○		○	○	○	○	○	○	○	○	○	○		○	●			○											
(3/5, 1/5, 1/5 | 2/5, 2/5, 1/5 | 3/5, 1/5, 1/5)	○		○	○	○	○	○	○	○	○	○	○		○	●														
(1/2, 1/3, 1/6 | 2/3, 1/6, 1/6 | 1/3, 1/3, 1/3)	○		○	○	○	○	○	○	○	○	●	○			●														
(1/2, 1/3, 1/6 | 1/3, 1/3, 1/3 | 2/3, 1/6, 1/6)	○		○	○	○	○	○	○	○	○	●	○			●														
(2/5, 2/5, 1/5 | 3/5, 1/5, 1/5 | 2/5, 2/5, 1/5)	○	○	○	○	○	○	○	○	○	○	●	○			●														
(2/5, 2/5, 1/5 | 2/5, 2/5, 1/5 | 3/5, 1/5, 1/5)	○	○	○	○	○	○	○	○	○	○	●	○			●														
(5/9, 2/9, 2/9 | 5/9, 2/9, 2/9 | 1/3, 1/3, 1/3)	○		○	○	○	○		○	●																				
(4/9, 4/9, 1/9 | 1/3, 1/3, 1/3 | 4/9, 4/9, 1/9)	○		○	○	○	○	●	○	●	●																			
(1/3, 1/3, 1/3 | 4/9, 4/9, 1/9 | 4/9, 4/9, 1/9)	○		○	○	○	○	●	○	●	●	●																		
(5/9, 2/9, 2/9 | 4/9, 4/9, 1/9 | 1/3, 1/3, 1/3)	○	○	○	○	○	○	○	●		●																			
(4/9, 4/9, 1/9 | 5/9, 2/9, 2/9 | 1/3, 1/3, 1/3)	○	○	○	○	○	○	○	●		●																			
(3/5, 1/5, 1/5 | 2/5, 2/5, 1/5 | 1/3, 1/3, 1/3)	○		○	○	○	○		●																					
(2/5, 2/5, 1/5 | 3/5, 1/5, 1/5 | 1/3, 1/3, 1/3)	○		○	○	○	○		●																					
(7/15, 4/15, 4/15 | 1/3, 1/3, 1/3 | 3/5, 1/5, 1/5)	○		○	○	○	○	●	○		●																			
(1/3, 1/3, 1/3 | 7/15, 4/15, 4/15 | 3/5, 1/5, 1/5)	○		○	○	○	○	●	○		●																			
(2/3, 1/3, 1/3 | 7/15, 7/15, 1/15 | 2/5, 2/5, 1/5)	○		●		○	○	○																						
(2/3, 1/3, 1/3 | 2/5, 2/5, 1/5 | 7/15, 7/15, 1/15)	○		●		○	○	○	●		●																			
(7/15, 7/15, 1/15 | 2/3, 1/3, 1/3 | 2/5, 2/5, 1/5)	○		●		○	○	○																						
(7/15, 7/15, 1/15 | 2/5, 2/5, 1/5 | 2/3, 1/3, 1/3)	○		●		○	○		○	○																				
(2/5, 2/5, 1/5 | 2/3, 1/3, 1/3 | 7/15, 7/15, 1/15)	○		●		○	○	○	●		●																			
(2/5, 2/5, 1/5 | 7/15, 7/15, 1/15 | 2/3, 1/3, 1/3)	○		●		○	○		○	○																				
(1/2, 1/4, 1/4 | 1/2, 1/4, 1/4 | 1/2, 1/4, 1/4)	○	○	○	○	○	○	○	○	○	○	●	●			●														
(3/7, 3/7, 1/7 | 3/7, 3/7, 1/7 | 3/7, 2/7, 2/7)	○		○	○	○	○	○	○	●	○		●																	
(3/7, 3/7, 1/7 | 3/7, 2/7, 2/7 | 3/7, 3/7, 1/7)	○		○	○	○	○	○	○	○	○		●																	
(3/7, 2/7, 2/7 | 3/7, 3/7, 1/7 | 3/7, 3/7, 1/7)	○		○	○	○	○	○	○	○	○	○	●																	
(5/12, 5/12, 1/6 | 1/3, 1/3, 1/3 | 1/2, 1/4, 1/4)	○	○	○	○	○	○	●	●		●																			
(1/3, 1/3, 1/3 | 5/12, 5/12, 1/6 | 1/2, 1/4, 1/4)	○	○	○	○	○	○	●	●		●																			
(1/2, 1/4, 1/4 | 1/2, 1/4, 1/4 | 1/3, 1/3, 1/3)	○		○	○	○	○		●																					
(2/5, 2/5, 1/5 | 2/5, 2/5, 1/5 | 2/5, 2/5, 1/5)	○	○	○	○	○	○	○	○	●	○		●																	
(5/9, 2/9, 2/9 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	○		○	○	○	●																							
(1/3, 1/3, 1/3 | 5/9, 2/9, 2/9 | 1/3, 1/3, 1/3)	○		○	○	●	●																							
(1/3, 1/3, 1/3 | 1/3, 1/3, 1/3 | 5/9, 2/9, 2/9)	○		○	○	●			●																					
(4/9, 4/9, 1/9 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	○				○	●																							
(1/3, 1/3, 1/3 | 4/9, 4/9, 1/9 | 1/3, 1/3, 1/3)	○				●	●																							
(1/3, 1/3, 1/3 | 1/3, 1/3, 1/3 | 4/9, 4/9, 1/9)	○				●			●																					
(1/2, 1/4, 1/4 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	○		○	○	●																								
(2/5, 2/5, 1/5 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	○				●																								
(1/3, 1/3, 1/3 | 1/3, 1/3, 1/3 | 1/3, 1/3, 1/3)	●	●	●	●																									

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$

$$= \mathbb{C}[x]/(x^3)$$

$$T_{11} = 113 + 131 + 212 + 321$$

$$= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$T_{14} = 113 + 121 + 132 + 211 + 312$$

$$= \mathbb{C}[x,y]/(x^2, y^2, xy)$$

nurmiev

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$
$$= \mathbb{C}[x]/(x^3)$$

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$$T_{14} = 113 + 121 + 132 + 211 + 312$$
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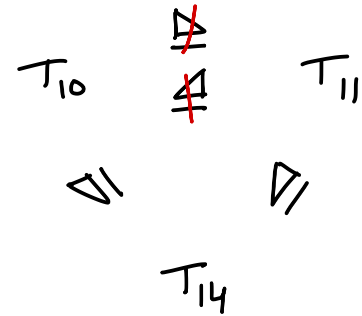
Degeneration Hasse diagram:



nurmiev

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$
$$= \mathbb{C}[x]/(x^3)$$

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$$= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$T_{14} = 113 + 121 + 132 + 211 + 312$$
$$= \mathbb{C}[x,y]/(x^2, y^2, xy)$$

Degeneration Hasse diagram:

$$\begin{tikzcd}
T_{10} \arrow[rr, phantom, "\not\trianglerighteq" above, "\not\trianglelefteq" below] \arrow[dr, phantom, "\trianglerighteq"] & & T_{11} \arrow[dl, phantom, "\trianglerighteq"] \\
& T_{14} &
\end{tikzcd}$$

Asymptotic restriction Hasse diagram:

$$\begin{tikzcd}
T_{10} \arrow[rr, phantom, "?"] \arrow[dr, phantom, "\geq_2"] & & T_{11} \arrow[dl, phantom, "\geq_2"] \\
& T_{14} &
\end{tikzcd}$$

nurmiev

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$

$$= \mathbb{C}[x]/(x^3)$$

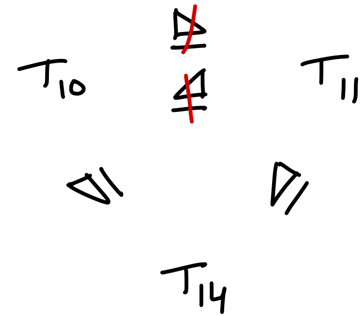
$$T_{11} = 113 + 131 + 212 + 321$$

$$= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

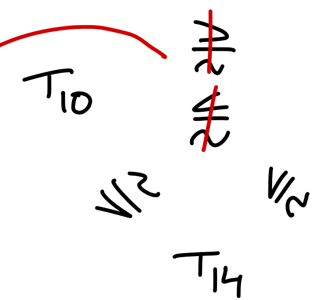
$$T_{14} = 113 + 121 + 132 + 211 + 312$$

$$= \mathbb{C}[x,y]/(x^2, y^2, xy)$$

Degeneration Hasse diagram:



Asymptotic restriction Hasse diagram:



quantum functionals

$\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

Vertices												Polytopes							
												Generic	U_4	$(2v_1+v_2+v_3)\oplus e_{111}$	$(\mathsf{D}+e_{111})\oplus e_{111}$	M_2	$\mathsf{W}\otimes\mathsf{U}_2$	$\mathsf{W}\otimes\mathsf{W}$	$\mathsf{W}\oplus\mathsf{W}$
1	0	0	0	1	0	0	0	1	0	0	0	●	●	●	●	●	●	●	●
1/2	1/2	0	0	1/2	1/2	0	0	1	0	0	0	●	●	●	●	●	●	●	●
1/2	1/2	0	0	1/2	1/2	0	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/2	1/4	1/4	0	1/2	1/2	0	0	3/4	1/4	0	0	●	●	●	●	●	●	●	●
1/3	1/3	1/3	0	2/3	1/3	0	0	2/3	1/3	0	0	●	●	●	●	●	●	●	●
1/3	1/3	1/3	0	1/2	1/2	0	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/3	1/3	1/3	0	1/3	1/3	1/3	0	1	0	0	0	●	●	●	●	●	●	●	●
2/3	1/6	1/6	0	2/3	1/6	1/6	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/3	1/3	1/3	0	2/3	1/6	1/6	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/3	1/3	1/3	0	1/3	1/3	1/3	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/4	1/4	1/4	1/4	1/2	1/2	0	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/3	1/3	1/3	0	1/3	1/3	1/3	0	1/3	1/3	1/3	0	●	●	●	●	●	●	●	●
1/3	1/3	1/6	1/6	1/3	1/3	1/3	0	5/6	1/6	0	0	●	●	●	●	●	●	●	●
1/2	1/6	1/6	1/6	2/3	1/6	1/6	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
2/5	1/5	1/5	1/5	2/5	2/5	1/5	0	4/5	1/5	0	0	●	●	●	●	●	●	●	●
5/8	1/8	1/8	1/8	1/2	1/4	1/4	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/4	1/4	1/4	1/4	1/2	1/4	1/4	0	3/4	1/4	0	0	●	●	●	●	●	●	●	●
7/12	1/4	1/12	1/12	1/3	1/3	1/3	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/2	1/6	1/6	1/6	1/3	1/3	1/3	0	2/3	1/3	0	0	●	●	●	●	●	●	●	●
1/2	1/6	1/6	1/6	1/3	1/3	1/3	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/3	2/9	2/9	2/9	1/3	1/3	1/3	0	2/3	1/3	0	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	3/8	3/8	1/4	0	5/8	3/8	0	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	1/2	1/4	1/4	0	1/2	1/2	0	0	●	●	●	●	●	●	●	●
1/3	2/9	2/9	2/9	7/18	7/18	2/9	0	1/2	1/2	0	0	○	○	○	○		○	●	○
1/3	5/18	5/18	1/9	1/3	1/3	1/3	0	1/2	1/2	0	0	○	○	○	○		○	●	○
1/3	1/3	1/6	1/6	1/3	1/3	1/3	0	1/2	1/2	0	0	○	○	○	○	●	○	●	○
2/7	2/7	2/7	1/7	5/14	5/14	2/7	0	1/2	1/2	0	0	○	○	○	○		○	●	○
1/4	1/4	1/4	1/4	1/3	1/3	1/3	0	1/2	1/2	0	0	●	●	●	●		●		●
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1	0	0	0	●	●	●	●	●	●	●	●
2/5	2/5	1/10	1/10	2/5	3/10	3/10	0	4/5	1/10	1/10	0	●	●	●	●	●	●	●	●
4/7	1/7	1/7	1/7	3/7	3/7	1/7	0	5/7	1/7	1/7	0	●	●	●	●	●	●	●	●
4/9	1/3	1/9	1/9	1/3	1/3	1/3	0	7/9	1/9	1/9	0	●	●	●	●	●	●	●	●
1/3	1/3	1/6	1/6	2/3	1/6	1/6	0	2/3	1/6	1/6	0	●	●	●	●	●	●	●	●
3/10	3/10	1/5	1/5	2/5	3/10	3/10	0	4/5	1/10	1/10	0	●	●	●	●	●	●	●	●
1/3	1/3	1/6	1/6	1/3	1/3	1/3	0	7/9	1/9	1/9	0	●	●	●	●		●	●	●
1/3	2/9	2/9	2/9	1/3	1/3	1/3	0	7/9	1/9	1/9	0	●	●	●	●	●	●	●	●
1/4	1/4	1/4	1/4	3/8	3/8	1/4	0	3/4	1/8	1/8	0	●	●	●	●	●	●	●	●
7/24	7/24	5/24	5/24	1/3	1/3	1/3	0	3/4	1/8	1/8	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	2/5	3/10	3/10	0	7/10	3/20	3/20	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	1/2	1/4	1/4	0	2/3	1/6	1/6	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	1/3	1/3	1/3	0	2/3	1/4	1/12	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	1/3	1/3	1/3	0	2/3	1/6	1/6	0	●	●	●	●	●	●	●	●
8/27	8/27	8/27	1/9	1/3	1/3	1/3	0	13/27	13/27	1/27	0	○	○	○	○		○	●	○
5/12	5/12	1/12	1/12	1/3	1/3	1/3	0	1/3	1/3	1/3	0	○	○	○	○	○	○	●	○
3/7	4/21	4/21	4/21	1/3	1/3	1/3	0	3/7	3/7	1/7	0	○	○	○	○	○	○	●	○
1/2	1/6	1/6	1/6	1/3	1/3	1/3	0	1/3	1/3	1/3	0	○	○	○	○	○	●	●	●
1/4	1/4	1/4	1/4	1/3	1/3	1/3	0	5/12	5/12	1/6	0	○	○	○	○	●	●		●
1/3	1/3	1/6	1/6	1/3	1/3	1/3	0	1/3	1/3	1/3	0	○	○	○	○	○	●		●
1/4	1/4	1/4	1/4	1/3	1/3	1/3	0	1/3	1/3	1/3	0	●	●	●	●	●			
5/8	1/8	1/8	1/8	5/8	1/8	1/8	1/8	1/2	1/2	0	0	●	●	●	●	●	●	●	●
3/8	1/4	1/4	1/8	5/8	1/8	1/8	1/8	1/2	1/2	0	0	●	●	●	●		●	●	●
3/10	3/10	3/10	1/10	3/5	1/5	1/10	1/10	1/2	1/2	0	0	●	●	●	●		●	●	●
2/5	1/5	1/5	1/5	3/5	1/5	1/10	1/10	1/2	1/2	0	0	●	●	●	●		●	●	●
1/2	1/6	1/6	1/6	1/2	1/6	1/6	1/6	1/2	1/2	0	0	○	○	○	○	○	○	●	○
2/7	2/7	2/7	1/7	4/7	1/7	1/7	1/7	4/7	3/7	0	0	●	●	●	●		●	●	●
3/10	3/10	3/10	1/10	11/20	3/20	3/20	3/20	1/2	1/2	0	0	●	●	●	●		●		●
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	3/4	1/4	0	0	○	○	○	●	○	○		○
1/4	1/4	1/4	1/4	1/2	1/4	1/8	1/8	5/8	3/8	0	0	●	●	●	●		●	●	●
1/3	1/3	1/6	1/6	1/2	1/6	1/6	1/6	1/2	1/2	0	0	○	○	○	○	○	○	●	○
5/16	5/16	5/16	1/16	5/16	5/16	5/16	1/16	1/2	1/2	0	0	○	○	○	○	○	○	●	○
1/4	1/4	1/4	1/4	1/2	1/6	1/6	1/6	1/2	1/2	0	0	●	●	●	●		●		●
2/5	1/5	1/5	1/5	2/5	1/5	1/5	1/5	3/5	2/5	0	0	○	○	○	○	○	○	●	○
1/4	1/4	1/4	1/4	3/8	3/8	1/8	1/8	1/2	1/2	0	0	○	○	○	●	○	○		○
1/4	1/4	1/4	1/4	1/3	1/3	1/4	1/12	1/2	1/2	0	0	○	○	○	●		○		○
1/4	1/4	1/4	1/4	5/12	1/4	1/6	1/6	1/2	1/2	0	0	○	○	○	●		○		○
3/10	7/30	7/30	7/30	3/10	3/10	3/10	1/10	1/2	1/2	0	0	○	○	○	●		○		○
1/4	1/4	1/4	1/4	3/8	1/4	1/4	1/8	1/2	1/2	0	0	○	○	○	●	○	○		○
5/14	3/14	3/14	3/14	5/14	3/14	3/14	3/14	1/2	1/2	0	0	○	○	○	●	○	○		○
2/7	2/7	3/14	3/14	2/7	2/7	2/7	1/7	1/2	1/2	0	0	○	○	○	●	○	○		○
3/10	3/10	1/5	1/5	3/10	3/10	1/5	1/5	1/2	1/2	0	0	○	○	○	●	○	○		○
5/18	5/18	5/18	1/6	5/18	5/18	5/18	1/6	1/2	1/2	0	0	○	○	○	●	○	○		○
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/2	1/2	0	0	●	●	●		●	●		●
5/12	5/12	1/12	1/12	3/4	1/12	1/12	1/12	1/2	1/4	1/4	0	●	●	●	●		●	●	●
2/5	2/5	1/10	1/10	7/10	1/10	1/10	1/10	3/5	1/5	1/5	0	●	●	●	●		●	●	●
5/12	5/12	1/12	1/12	3/4	1/12	1/12	1/12	1/3	1/3	1/3	0	●	●	●	●	●	●	●	●
5/14	5/14	1/7	1/7	11/14	1/14	1/14	1/14	3/7	2/7	2/7	0	●	●	●	●		●	●	●
1/3	1/3	1/6	1/6	3/4	1/12	1/12	1/12	1/2	1/4	1/4	0	●	●	●	●		●	●	●
5/8	1/8	1/8	1/8	5/8	1/8	1/8	1/8	3/8	3/8	1/4	0	●	●	●	●		●	●	●
5/12	1/4	1/6	1/6	3/4	1/12	1/12	1/12	1/3	1/3	1/3	0	●	●	●	●		●	●	●
5/11	2/11	2/11	2/11	8/11	1/11	1/11	1/11	4/11	4/11	3/11	0	●	●	●	●		●	●	●
5/9	2/9	1/9	1/9	2/3	1/9	1/9	1/9	1/3	1/3	1/3	0	●	●	●	●		●	●	●
2/5	1/5	1/5	1/5	11/15	2/15	1/15	1/15	1/3	1/3	1/3	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	3/4	1/12	1/12	1/12	1/3	1/3	1/3	0	●	●	●	●	●	●	●	●
1/3	2/9	2/9	2/9	2/3	1/9	1/9	1/9	4/9	4/9	1/9	0	●	●	●	●		●	●	●
1/2	1/6	1/6	1/6	2/3	1/9	1/9	1/9	1/3	1/3	1/3	0	●	●	●	●		●		●
1/4	1/4	1/4	1/4	2/3	1/6	1/12	1/12	5/12	5/12	1/6	0	●	●	●	●		●	●	●
1/4	1/4	1/4	1/4	5/8	1/8	1/8	1/8	1/2	3/8	1/8	0	●	●	●	●		●	●	●
11/36	11/36	11/36	1/12	17/36	17/36	1/36	1/36	1/3	1/3	1/3	0	○	○	○	○	○	○	●	○
2/7	2/7	2/7	1/7	4/7	1/7	1/7	1/7	3/7	3/7	1/7	0	○	○	○	○	○	○	●	○
1/4	1/4	1/4	1/4	5/12	5/12	1/12	1/12	1/3	1/3	1/3	0	○	○	○	○	●	●		●
1/4	1/4	1/4	1/4	1/2	1/6	1/6	1/6	1/3	1/3	1/3	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	3/8	3/8	1/8	1/8	1/3	1/3	1/3	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	5/12	5/12	1/6	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/2	1/4	1/4	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	1/3	1/3	1/4	1/12	1/3	1/3	1/3	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	5/12	1/4	1/6	1/6	1/3	1/3	1/3	0	○	○	○	●	○			
4/15	4/15	4/15	1/5	2/5	1/5	1/5	1/5	1/3	1/3	1/3	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	3/8	1/4	1/4	1/8	1/3	1/3	1/3	0	○	○	○	●	○			
2/7	2/7	2/7	1/7	2/7	2/7	2/7	1/7	1/3	1/3	1/3	0	○	○	○	●	○			
2/7	2/7	3/14	3/14	5/14	3/14	3/14	3/14	1/3	1/3	1/3	0	○	○	○	●	○			
3/10	3/10	1/5	1/5	3/10	3/10	1/5	1/5	1/3	1/3	1/3	0	○	○	○	●	○			
1/3	2/9	2/9	2/9	1/3	2/9	2/9	2/9	1/3	1/3	1/3	0	○	○	○	●	○			
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/3	1/3	1/3	0	●	●	●		●			
1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	●	●	●	●	●			

$\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

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Theorem $\forall n$, $\Delta(\langle n,n,n \rangle)$ is not maximal:

$$\exists p, \quad \Delta(\langle n,n,n \rangle) \not\ni p \qquad \Delta(\langle n^2 \rangle) \ni p$$

arxiv.org/abs/2503.22633

Problem: determine $\Delta(\langle n \rangle)$ and $\Delta(\langle m,m,m \rangle)$ [Bürgisser-Ikenmeyer]

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Problem: Is $\Delta(\langle n\rangle)$ always the largest (generic) moment polytope?

True for $n = 2,3,4$

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From slight modification of above:

Theorem $\underline{Q}(\langle n,n,n \rangle) = \lceil \frac{3}{4} n^2 \rceil$ reproving [Kopparty-Moshkovitz-Z]

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Lemma

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Theorem

$$\left.\begin{matrix} 1&0&0\\0&0&0\\0&0&1 \end{matrix}\;\right|\;\begin{matrix} 0&0&0\\0&1&0\\0&0&1 \end{matrix}\;\left|\;\begin{matrix} 0&1&0\\0&0&1\\0&0&0 \end{matrix}\right. \qquad \left.\begin{matrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{matrix}\;\right|\;\begin{matrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{matrix}\;\left|\;\begin{matrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{matrix}\;\right|\;\begin{matrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{matrix} \qquad \cdots$$

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Proof idea: gradient flow along moment map reduces problem to $U \subseteq GL$.

Open problems

(1) Are the quantum functionals all of X ?

(2) What other (asymptotic) information does $\Delta(T)$ give?

(3) Is there an efficient algorithm to test freeness ?