On rank and subrank of the matrix multiplication tensors

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- characterize <u>computational complexity</u> of multiplying matrices

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  helps

- these tensors are <u>highly structured</u>

  <u>closed under powering</u>

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number of scalar $+/\cdot$ needed?

tensor rank

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Exponent

$$\boxed{\lim_{m \to \infty} R\left(MM_2^{\otimes m}\right)^{1/m} = 2^{\omega}}$$

Tensor rank R

Create tensor from small diagonal tensor

Subrank Q

Create large diagonal tensor from tensor

Tensor rank R

Create tensor from small diagonal tensor

smallest

$$T = \underbrace{U \otimes V \otimes W}_{\text{find}} \cdot \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

$$\left(T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \right)$$

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Create large diagonal tensor from tensor

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$$\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$$

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Applications:

- matrix multiplication
- circuit complexity [Raz]

- Matrix multiplication (also!)
- Additive combinatorics

Subrank application: Matrix multiplication <u>barriers</u>

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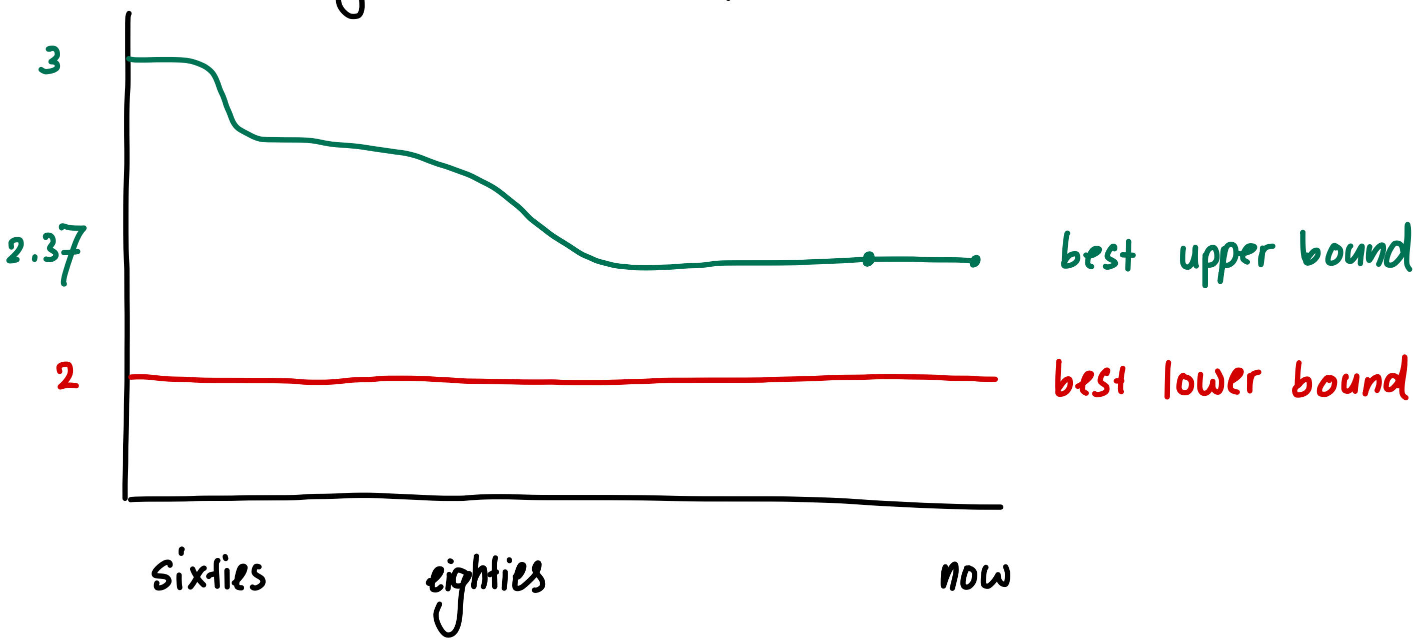
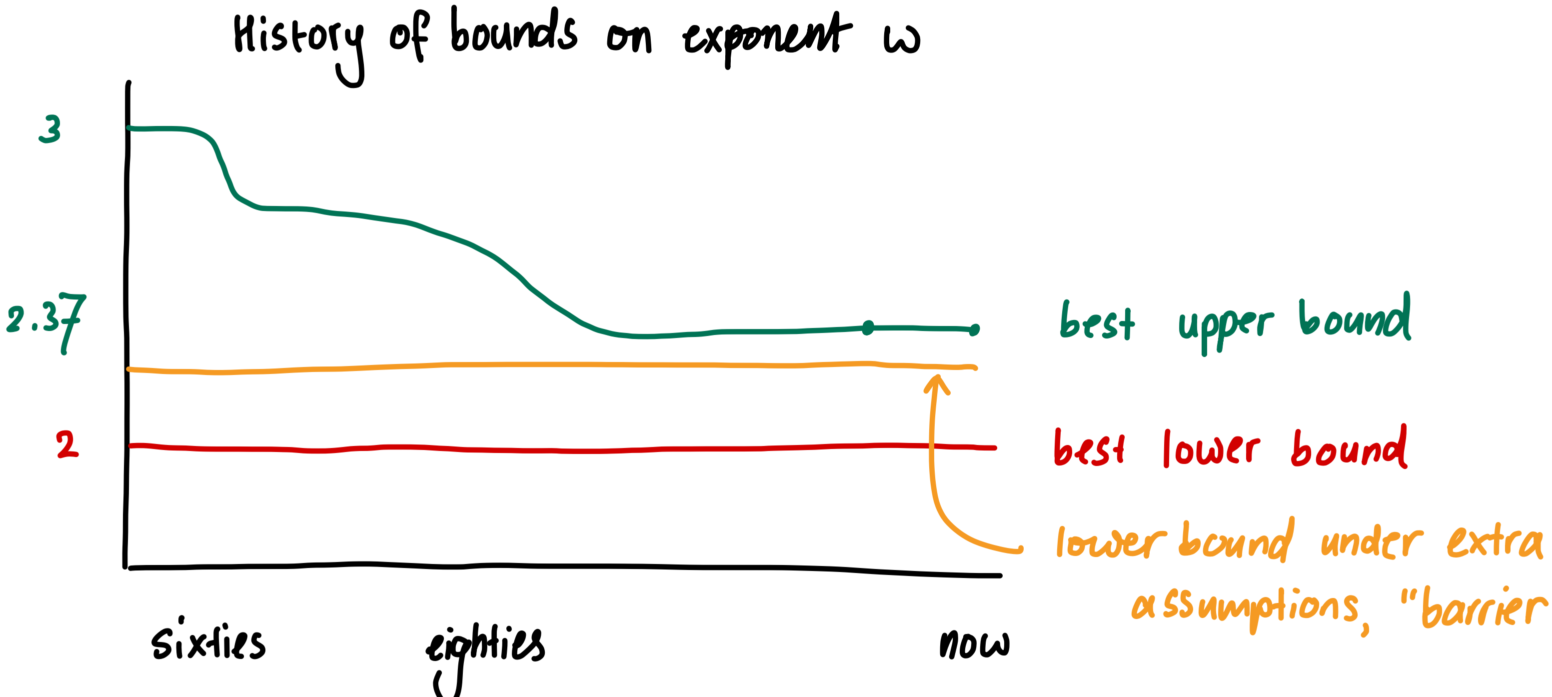
History of bounds on exponent ω



Subrank application: Matrix multiplication barriers



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Barrier Theorem (CVZ) small subrank of $T^{\otimes n}$ implies barrier

Two directions in the rest of the talk:

1. How to upper bound subrank?
   for barriers and other applications

2. How to circumvent these barriers
   and improve upper bound on ω?

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- G-stable rank [Derksen] one-parameter subgroups, invariant theory

related

Quantum functionals [Christandl-Vrana-Z] moment polytopes

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Theorem [Moshkovitz-Cohen] GR equals SR up to constant!

Theorem [Derksen-Makam-Z] Huge gap between Q and GR.

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Traditional

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- Schönhage's tau theorem

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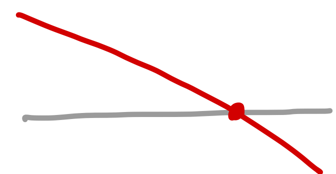

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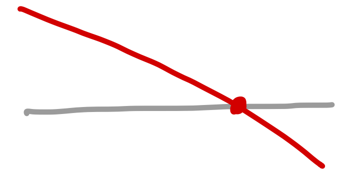

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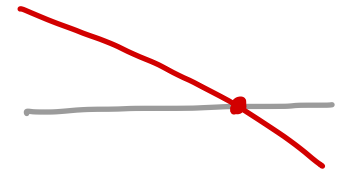

Non-traditional

$\boxed{\text{Strassen's theory of asymptotic spectra}}$

Positivstellensatz

Tensor inequalities $\longleftrightarrow$ Real geometry

$X \subseteq [1,\infty)$

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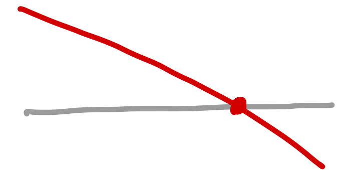

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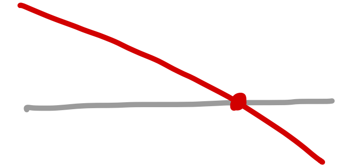

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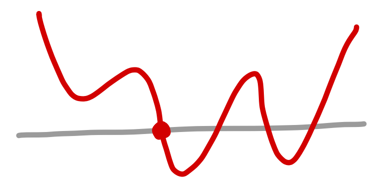

Upcoming paper with Wigderson: survey, exposition, extensions