Geometric Rank of Tensors

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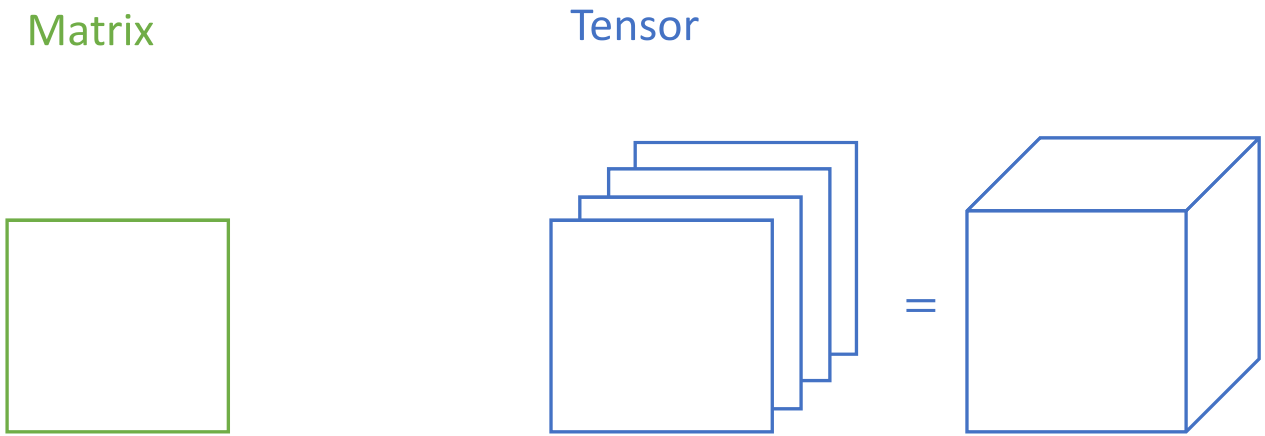
Joint work with Swastik Kopparty and Guy Moshkovitz

Tensors are 3-d arrays



Tensors play a central role in computer science, mathematics and physics

Algebraic Complexity Theory:

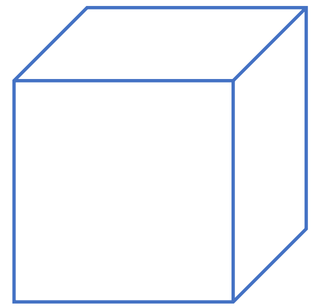
Complexity of Matrix Multiplication

Quantum Information Theory:

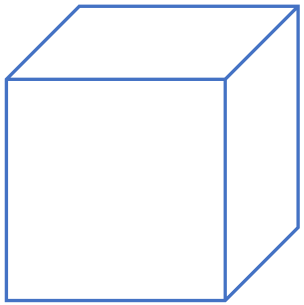
Understanding Entanglement

Extremal Combinatorics:

Cap set problem



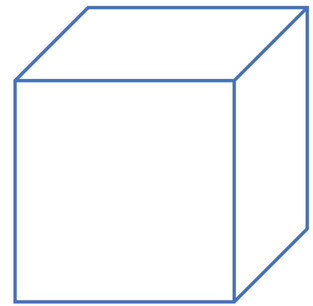
Motivated by these problems we introduce a new tensor parameter



Geometric Rank

Geometric Rank of tensors extends the classical rank of matrices

Matrix Rank



Geometric Rank

Slice Rank

Subrank

Analytic Rank

Tensor Rank

Border Rank

Main results on Geometric Rank

- basic properties and invariances
- develop tools to reason about, and sometimes exactly compute it
- intimate connections to the other important notions for tensors
- answer an old question of Strassen on the (Border) Subrank of matrix multiplication, the "dual" of the more famous Tensor Rank.

Geometric Rank provides new interesting route to upper bound

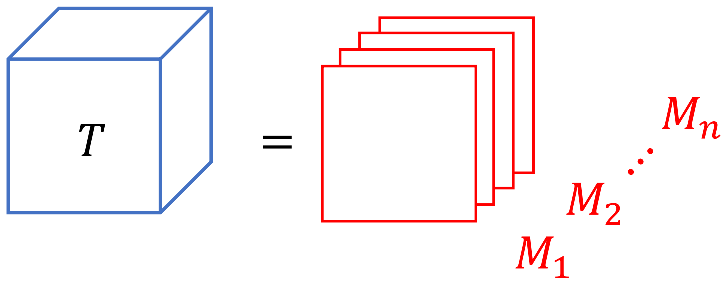
- Subrank of tensors

  important in complexity theory for matrix multiplication and barriers

- Independence number of Hypergraphs

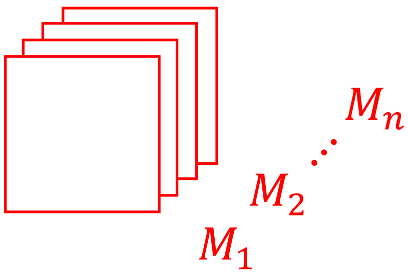
  important in combinatorics in the context of specific natural hypergraphs, as in cap set problem and Erdős–Szemerédi sunflower problem

Geometric Rank



Geometric Rank

system of equations



$\rightsquigarrow$

$$(x_1, \dots, x_n)\, M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$(x_1, \dots, x_n)\, M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$\vdots$$

$$(x_1, \dots, x_n)\, M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

Geometric Rank

$$(x_1, \dots, x_n)\, M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$(x_1, \dots, x_n)\, M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$\vdots$$

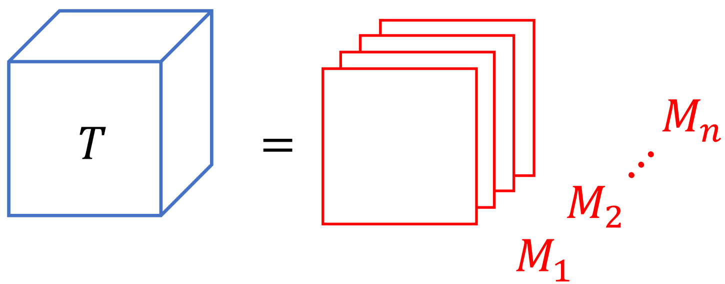
$$(x_1, \dots, x_n)\, M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$\rightsquigarrow$

$\mathrm{GR}(T) = 2n -$ dimension of $V(T)$

$V(T)$ is the set of all solutions

Geometric Rank



$\rightsquigarrow$

$$(x_1, \dots, x_n)\, M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$(x_1, \dots, x_n)\, M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$\vdots$$

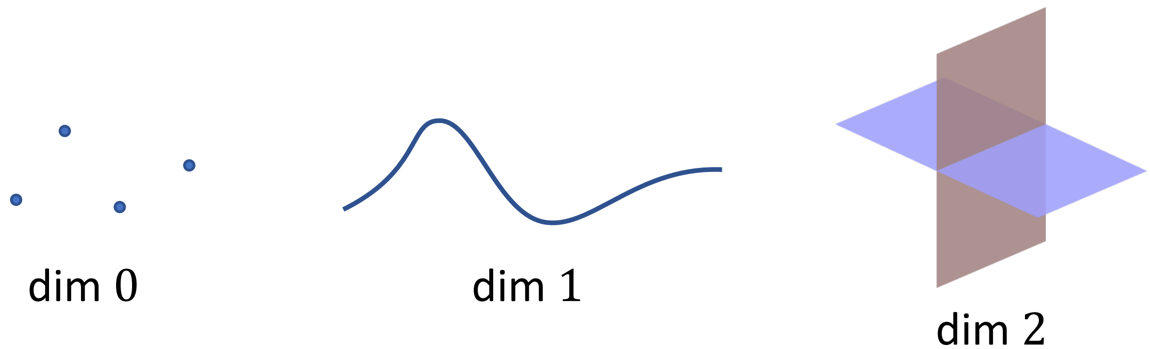
$$(x_1, \dots, x_n)\, M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$\rightsquigarrow$ set of all solutions

$V(T)$

$\mathrm{GR}(T) = 2n -$ dimension of set of solutions $V(T)$

Dimension measures continuous degrees of freedom

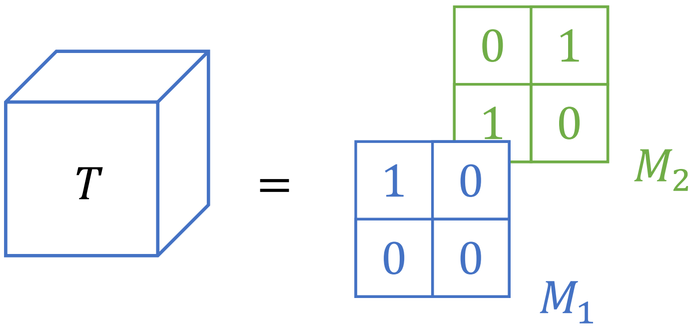


"length of maximal chain of irreducible subvarieties"

Computational intuition:

- If V is a linear space then the dimension equals the one from linear algebra
- If $V = \cup_i W_i$ then $\dim V = \max_i \dim W_i$
- If $V \subseteq W$ then $\dim V \leq \dim W$

Example of Geometric Rank (W-tensor)



$$\rightsquigarrow \quad (x_1, x_2)\, M_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_2 y_1 + x_1 y_2 = 0$$

$$(x_1, x_2)\, M_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 = 0$$

$$V(T) = \{x_1 y_1 = 0,\ x_2 y_1 + x_1 y_2 = 0\}$$

$$= \{x_1 = 0, x_2 = 0\} \ \cup\ \{y_1 = 0, y_2 = 0\} \ \cup\ \{x_1 = 0, y_1 = 0\}$$

$$\mathrm{GR}(T) = 4 - \dim V(T) = 4 - 2 = 2$$

Geometric Rank takes values between 0 and n because the system is bilinear

$$(x_1, \dots, x_n)\, M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$(x_1, \dots, x_n)\, M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$\vdots$$

$$(x_1, \dots, x_n)\, M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

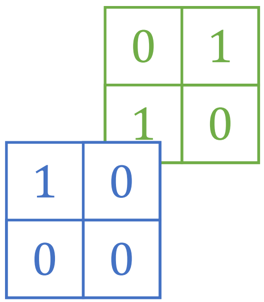
Always:

$$\{x_1 = \cdots = x_n = 0\} \subseteq V(T)$$

$$n \leq \dim V(T)$$

$$\mathrm{GR}(T) = 2n - \dim V(T) \leq n$$

Computing Geometric Rank is easy in practice for small tensors



$\rightsquigarrow$

system of equations:

$$x_2 y_1 + x_1 y_2 = 0$$
$$x_1 y_1 = 0$$

$\rightsquigarrow$

dimension:

2

Macaulay2

```
R = CC[x1,x2,y1,y2];
dim ideal(x1*y1, x2*y1 + x1*y2)
```

Sage

```
A.<x1,x2,y1,y2> = AffineSpace(4, CC);
Ideal([x1*y1, x2*y1 + x1*y2]).dimension()
```

Computational complexity of Geometric Rank is not known

Computing dimension of variety that is:

linear: easy

bilinear: not known to be easy or hard (at least we are not aware)

general: hard

Koiran:

NP-hard $\leq$ dimension of general variety $\leq$ PSPACE

The outline of this talk:

I. Tensors and Applications

II. Fundamental Properties of Geometric rank

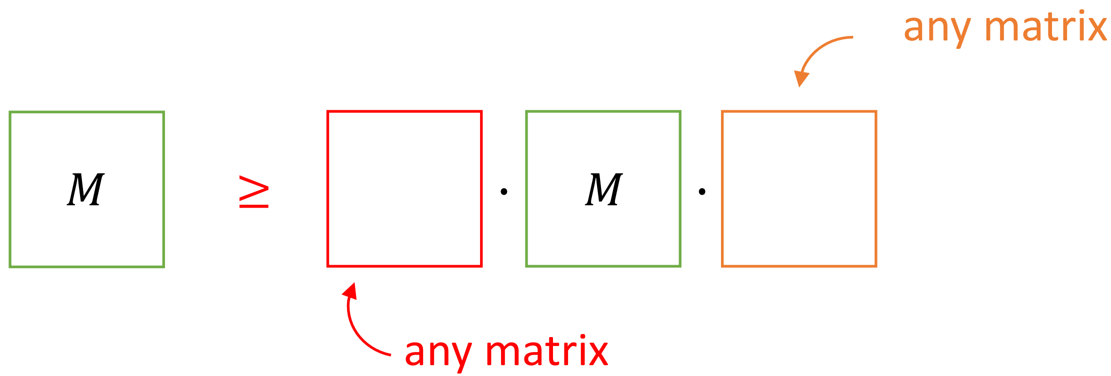
III. As upper bound on Subrank

I. Tensors and Applications

Guassian elimination

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2/3 \\ 0 & -2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

“Guassian order” on matrices



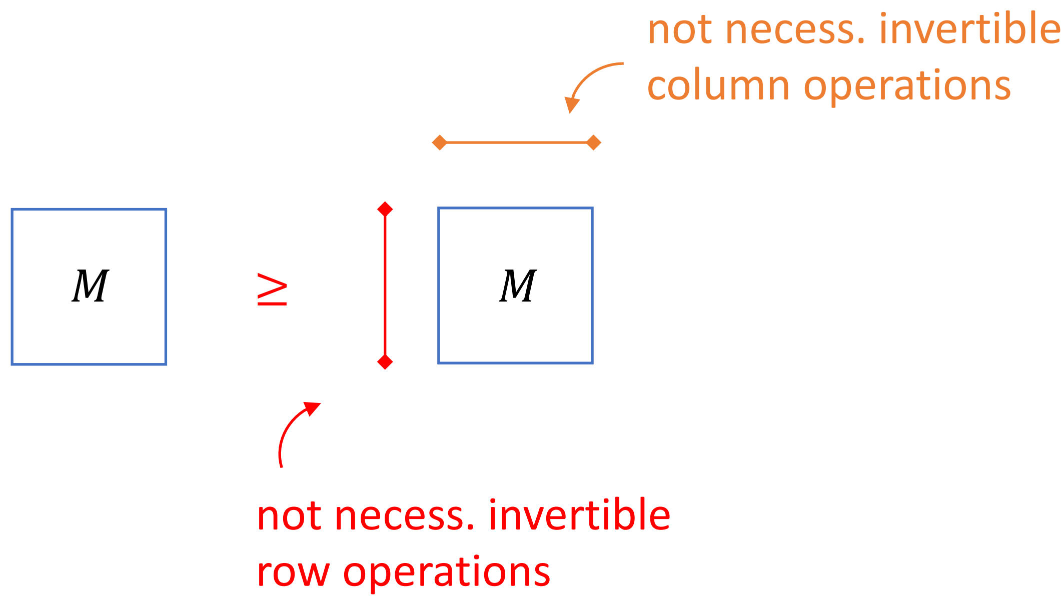
Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \geq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

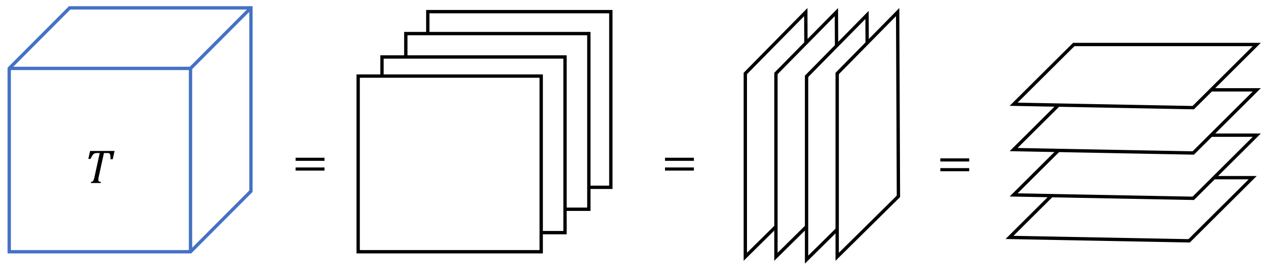
Matrix Rank completely determines the Gaussian order

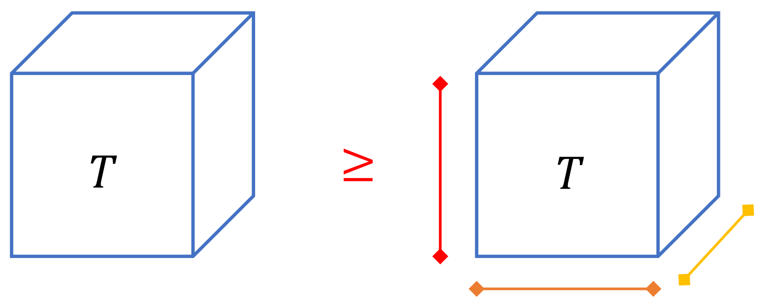
$M \geq N$ if and only if $\mathrm{rank}(M) \geq \mathrm{rank}(N)$

Recall once more:



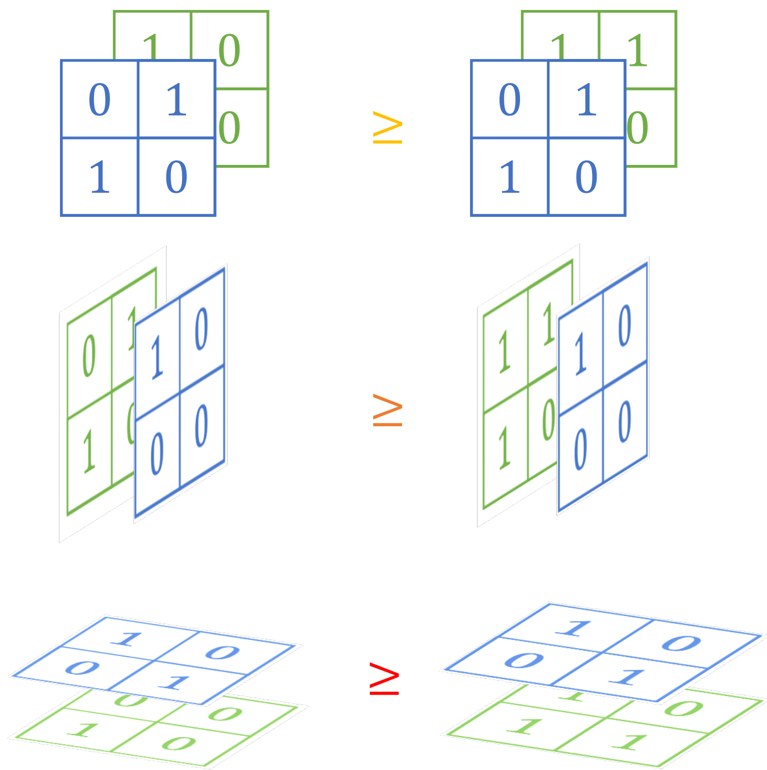
Gaussian order on Tensors generalizes the one on matrices



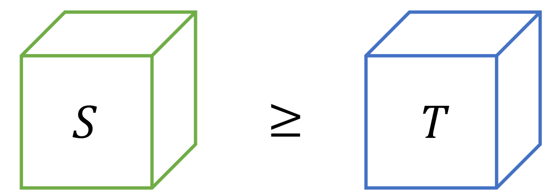


not necess. invertible slice operations in any of the three directions

Examples of slice operations:



Gaussian order in Mathematics, Physics and Computer Science



Complexity of Matrix Multiplication	diagonal tensor	$\geq$	matrix multiplication tensor
Classifying Quantum Entanglement	3-partite pure state	$\geq$ SLOCC	3-partite pure state
Hypergraph Independence Number	tensor supported on hypergraph	$\geq$	diagonal tensor

Matrix Rank completely determines the Gaussian order on matrices

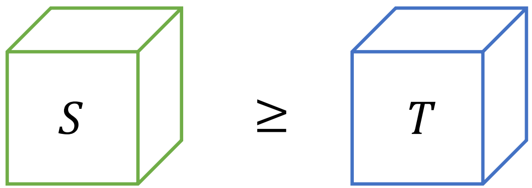
$M \geq N \quad \Leftrightarrow \quad \mathrm{R}(M) \geq \mathrm{R}(N)$

For tensors that level of complete understanding is out of reach

$S \geq T \quad \Leftrightarrow \quad ?$

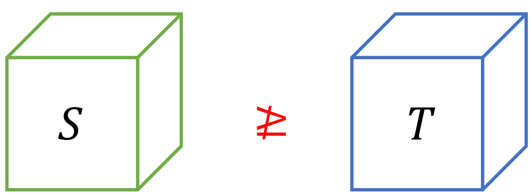
(NP-hard problem)

Our aim is to find monotones for the Gaussian order:



$$\Longrightarrow \quad F(S) \geq F(T)$$

Monotones serve as obstructions:

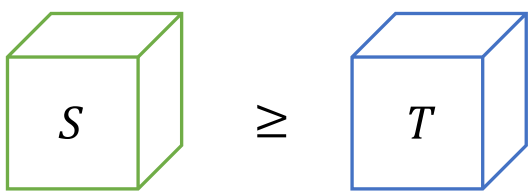


$$\Longleftarrow \quad F(S) < F(T)$$

II. Fundamental Properties of Geometric Rank

Theorem 1

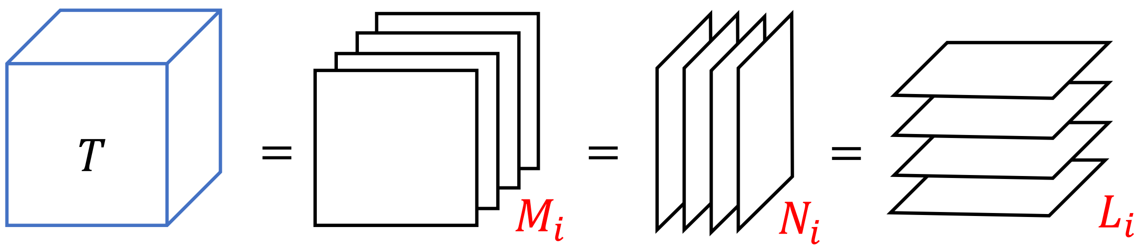
(Geometric Rank is monotone under the Gaussian order on tensors)



$$\Longrightarrow \quad \mathrm{GR}(S) \geq \mathrm{GR}(T)$$

Theorem 2

("Fundamental Theorem of Multilinear Algebra", by analogy)



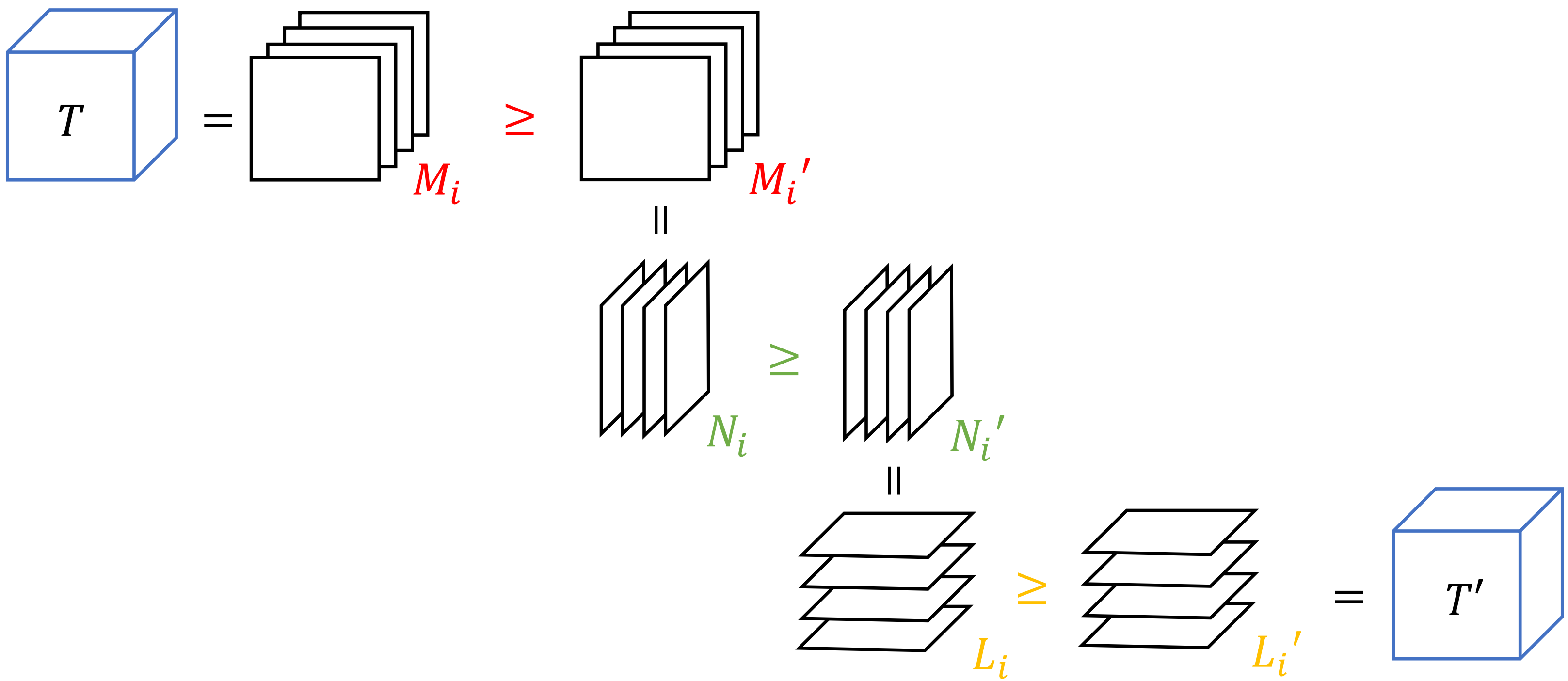
$$\mathrm{GR}(T) = \operatorname{codim}\{(u,v) : \forall i \;\; u^\top M_i v = 0\}$$ (definition)

$$= \operatorname{codim}\{(u,v) : \forall i \;\; u^\top N_i v = 0\}$$

$$= \operatorname{codim}\{(u,v) : \forall i \;\; u^\top L_i v = 0\}$$

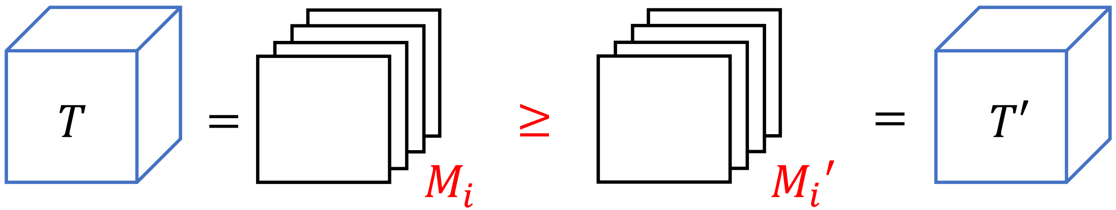
Theorem 1 (Monotonicity) $T \geq T' \Longrightarrow \mathrm{GR}(T) \geq \mathrm{GR}(T')$

Proof:



By Fundamental Theorem we may focus on the first step.

Focus on one step:



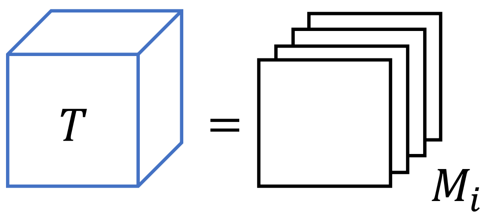
$V(T) = \{(u, v) : \forall i \;\; u^\top M_i v = 0\}$

$V(T') = \{(u, v) : \forall i \;\; u^\top M_i' v = 0\}$

- By assumption: M_i' are in the span of the M_i
- $V(T) \subseteq V(T')$
- $\dim V(T) \leq \dim V(T')$.
- $\mathrm{GR}(T) = \mathrm{codim}\, V(T) \geq \mathrm{codim}\, V(T') = \mathrm{GR}(T')$. ■

Fundamental Theorem follows from:

Theorem 3
(Method for computing Geometric Rank)



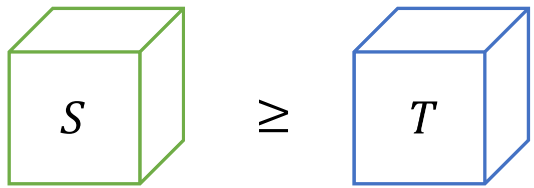
$$T(u) := u_1 M_1 + \cdots + u_n M_n$$

$$\mathrm{GR}(T) = \min_j \operatorname{codim} \{u : \operatorname{rank} T(u) = j\} + j$$

Proof: relies on a fiber dimension theorem applied to the projection $(u, v) \longmapsto u$

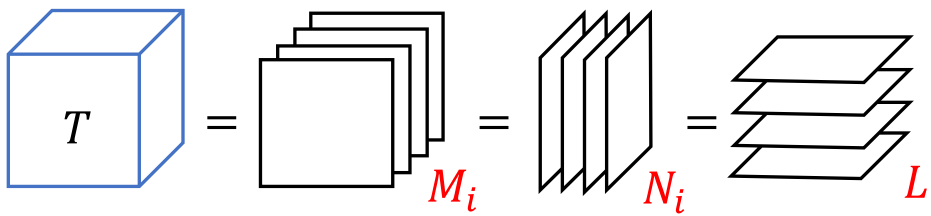
Summarizing

Theorem 1



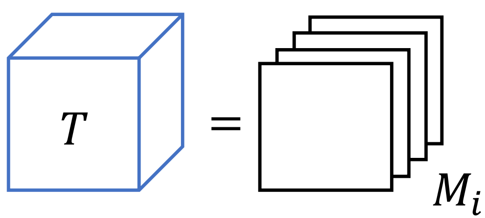
$\Longrightarrow$ $\mathrm{GR}(S) \geq \mathrm{GR}(T)$

Theorem 2



$\rightsquigarrow$ same notion of GR

Theorem 3



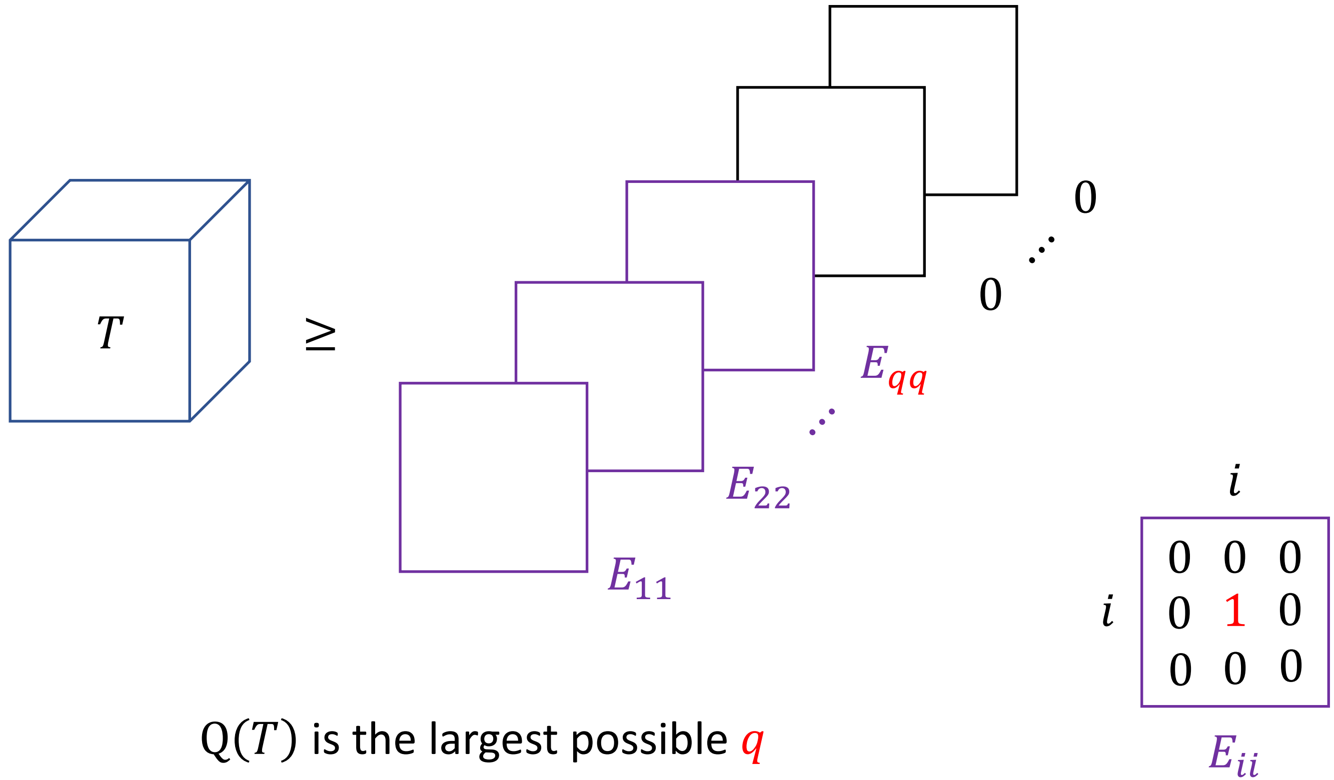
$$T(u) := u_1 M_1 + \cdots + u_n M_n$$

$$\mathrm{GR}(T) = \min_j \operatorname{codim} \{u : \operatorname{rank} T(u) = j\} + j$$

III. As upper bound on Subrank

The Subrank of T is the size of the largest diagonal tensor smaller than T

Strassen 1987



$Q(T)$ is the largest possible q

Subrank of tensors

Complexity theory
matrix multiplication and barriers

Combinatorics
Hypergraph independence number, cap set problem,
and Erdős–Szemerédi sunflower problem

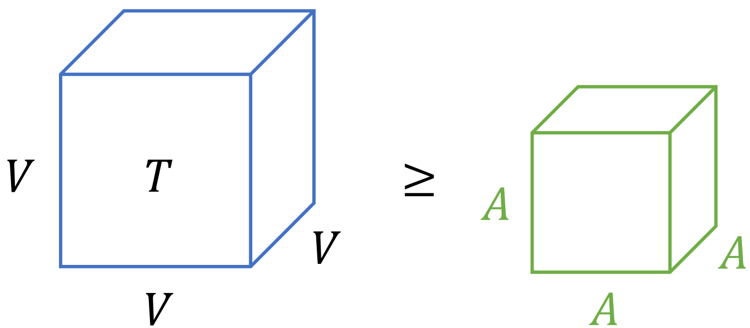
Quantum Information
distilling GHZ states by SLOCC

Subrank upper bounds hypergraph independence number

Hypergraph: symmetric subset $E \subseteq V \times V \times V$

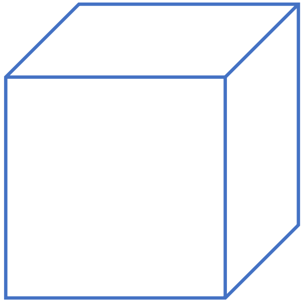
Independent set: $A \subseteq V$ such that $E \cap (A \times A \times A) = \emptyset$

Tensor T supported on $E \cup \{(i,i,i) : i \in V\}$.



$\mathrm{Q}(T) \geq |A|$

Upper bounds on Subrank



Slice Rank

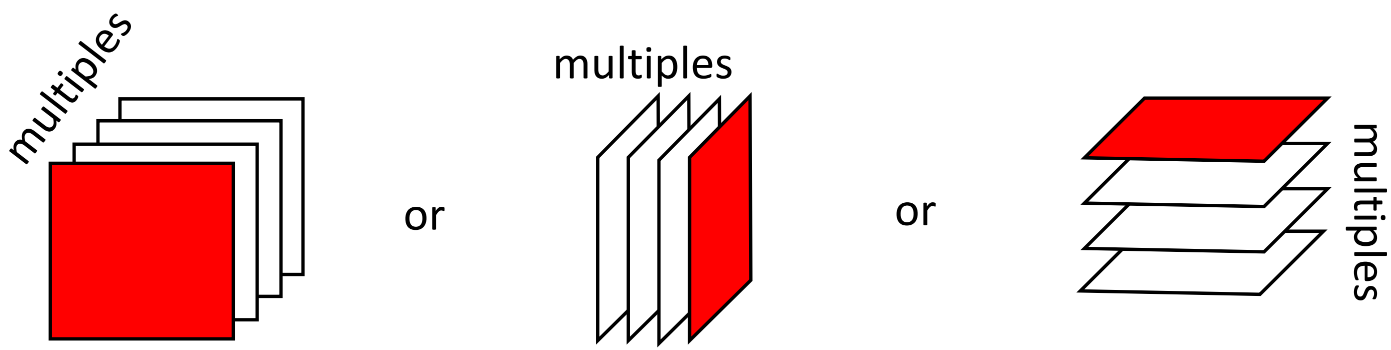
Analytic Rank

Geometric Rank

Slice Rank is the smallest number of slice rank one tensors summing to T

Tao

Slice rank one tensor has slices that are multiples of one slice



Slide Rank upper bounds Subrank

SR

VI

Q

Tao

Application to cap set problem

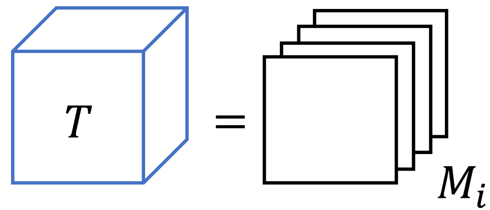
(Gijswijt–Ellenberg)

Proof: Monotone $(T \geq S \Rightarrow \mathrm{SR}(T) \geq \mathrm{SR}(S))$

+ Normalized $(\mathrm{SR}(I_n) = n)$

Analytic Rank for tensors over finite fields $\mathbb{F}_p$ (say $\mathbb{F}_2$)

Gowers and Wolf



$$T(u) := u_1 M_1 + \cdots + u_n M_n$$

$$\text{bias}(T) := \mathbb{E}_{u,v,w}\left[(-1)^{v^\top T(u) w}\right] \in (0, \infty)$$

$$\text{AR}(T) := -\log_2 \text{bias}(T)$$

Analytic Rank upper bounds Subrank

$\mathrm{AR}/\mathrm{AR}(I_1)$

↓

Q

Lovett

Briët: application in combinatorics

Proof: Monotone + Normalized

Geometric Rank "extends" Analytic Rank to characteristic 0

Theorem

$$\liminf_{p\to\infty} \mathrm{AR}(T \bmod p) = \mathrm{GR}(T)$$

Proof ingredients:

- $\mathrm{AR}(T \bmod p) = 2n - \log_p |V(T \bmod p)(\mathbb{F}_p)|$
- Generalized Schwartz–Zippel lemma (Dvir–Kollár–Lovett)
- Lang–Weil Theorem

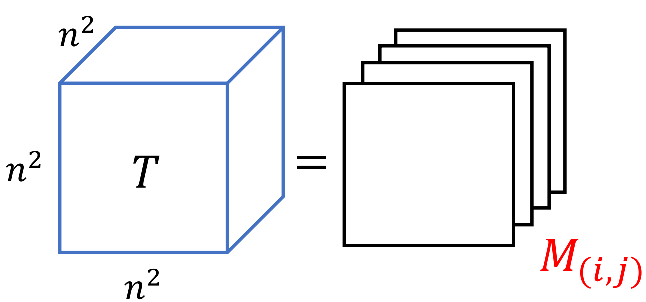
  (the two items above:) $|V(\mathbb{F}_p)| \leftrightsquigarrow \dim V$
- Bertini–Noether Theorem: $V(T) \rightsquigarrow V(T \bmod p)$

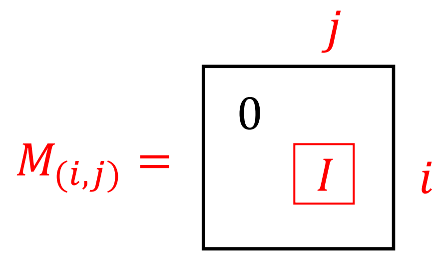
Geometric rank upper bounds Subrank
and is at most Slice Rank

Proof: Monotone + Normalized

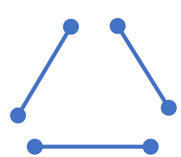
Example (matrix multiplication)

Matrix multiplication tensor





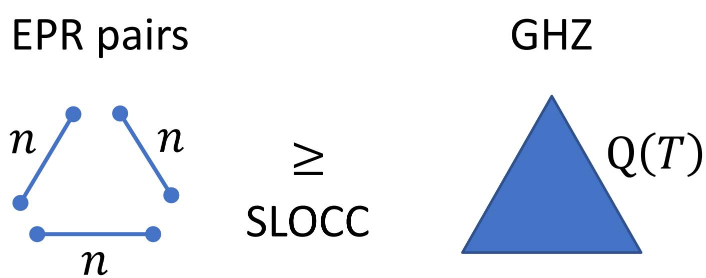
As quantum state: triangle of level-n EPR pairs



Example (matrix multiplication)

Previously (Christandl, Lucia, Vrana and Werner)

$$Q(T) \leq n^2 - n + 1$$



Example (matrix multiplication)

$\mathrm{SR}(T) = n^2$

↓

$\mathrm{GR}(T) = \lceil \frac{3}{4} n^2 \rceil$

Improves:
$\mathrm{Q}(T) \leq n^2 - n + 1$

↓

$\underline{\mathrm{Q}}(T) \geq \lceil \frac{3}{4} n^2 \rceil$ Strassen 1987

↓

$\mathrm{Q}(T) \geq n^{2-o(1)}$ Strassen 1987

Proof uses Theorem 3:

$$\dim V(T) = \max_r \dim\{M \in \mathbb{F}^{n \times n} : \mathrm{rank}\, M = r\} + (n-r)n$$

Question 1

Computational complexity of GR?

(SR is NP-hard.)

Question 2

How much smaller than SR can GR be?

(Big open problem for SR and AR.)

Question 3

Is $\mathrm{GR}(T)$ the limit of analytic ranks?