Asymptotic tensor rank is characterized by polynomials

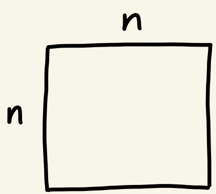
Jeroen Zuiddam

with Christandl, Hoeberechts, Nieuwboer, Vrana

1. Tensors and ranks
2. Algebraic complexity theory
3. Polynomials
4. Consequences

1. Tensors and ranks

matrix



rank

minimal

$$M = \sum_{i=1}^{r} u_i \otimes v_i$$

- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have $\det = 0$

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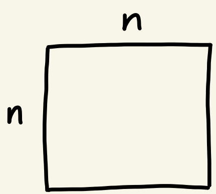

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- $R(A \otimes B) = R(A)R(B)$

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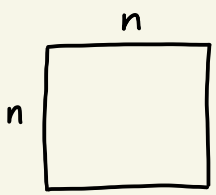

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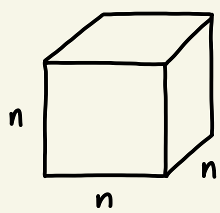

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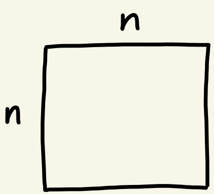

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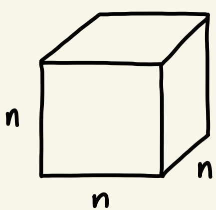

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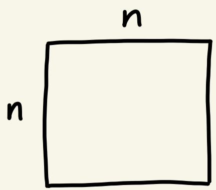

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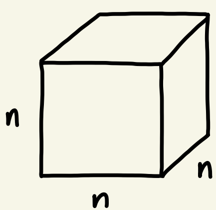

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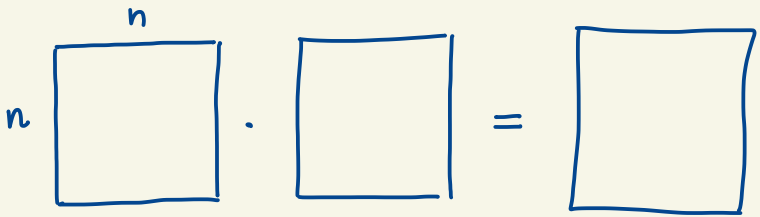
$\Rightarrow$ asymptotic tensor rank: $\underset{\sim}{R}(T) := \lim_{n \to \infty} R(T^{\otimes n})^{1/n}$

2. Algebraic complexity theory



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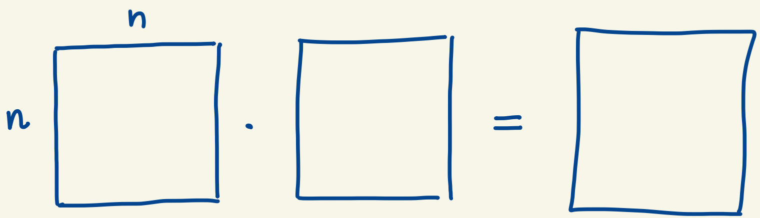

matrix mult. exponent

$O(n^{\omega})$ arithmetic operations

$2 \leq \omega \leq 2.3\ldots$

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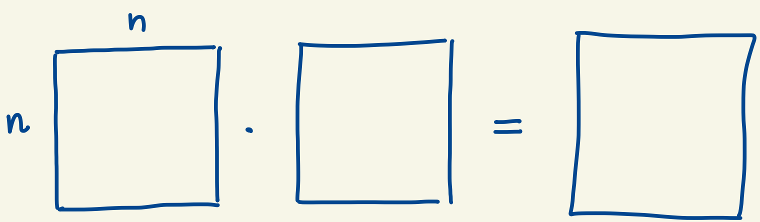

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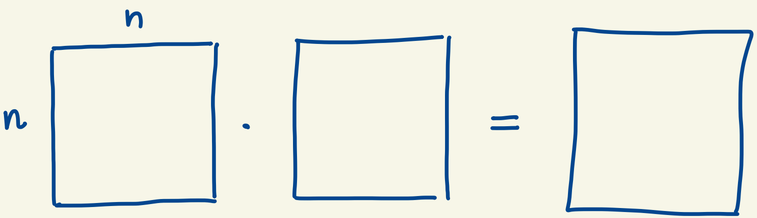

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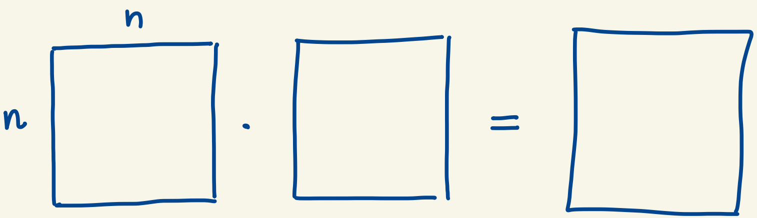

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For any concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $\underset{\sim}{R}(T) = m$

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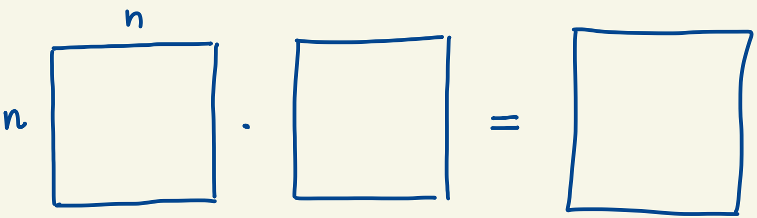

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What properties does $\underset{\sim}{R}$ have? Computable? Semicontinuous? Integer-valued?

3. Polynomials

$V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, $A \subseteq V$, $r \in \mathbb{R}$

<u>Theorem 1</u> $\{T \in V : \underset{\sim}{R}(T) \leq r\}$ is Zariski-closed.

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Def. $\underset{\sim}{R}[A] := \sup\{\underset{\sim}{R}(T) : T \in A\}$

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Theorem 2 $\Rightarrow$ Theorem 1.

- Let $A = \{T \in V : \underset{\sim}{R}(T) \leq r\}$. Then $\underset{\sim}{R}[A] \leq r$.
- By Theorem 2, $\underset{\sim}{R}[\bar{A}] \leq r$.
- So for all $T \in \bar{A}$, $\underset{\sim}{R}(T) \leq r$, so $T \in A$.
- Then $\bar{A} \subseteq A$ $\square$

Theorem 2 $\underset{\sim}{R}[\overline{A}] = \underset{\sim}{R}[A]$

important ingredient

Lemma 3 $\{T^{\otimes n} : T \in \overline{A}\} \subseteq \text{span}\{T^{\otimes n} : T \in A\}$

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$\uparrow$ important ingredient

Lemma 3 $\{T^{\otimes n} : T \in \overline{A}\} \subseteq \text{span}\,\{T^{\otimes n} : T \in A\}$

Proof

- Let ℓ be a linear form vanishing on the RHS
- $f : T \mapsto \ell(T^{\otimes n})$ is a polynomial vanishing on A
- So f vanishes on $\overline{A}$
- Then ℓ vanishes on the LHS $\square$

Theorem 2 $\underset{\sim}{R}[\bar{A}] = \underset{\sim}{R}[A]$

Proof sketch:

To prove: $\leq$

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- $\underset{\sim}{R}(T) \leq \underset{\sim}{R}[A]$ $\square$

4. Consequences and more

$$\mathcal{R} := \{ \underset{\sim}{R}(T) . T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}, d_1, \ldots, d_k \in \mathbb{N} \}$$

<u>Theorem</u> $\mathcal{R}$ is well-ordered. (Every non-increasing sequence stabilizes.)

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Open problems:

1. Is $\mathcal{R}$ discrete from below?

2. Is $\{T \in V : \underset{\sim}{R}(T) \leq r\}$ an irreducible variety? $\Uparrow$

3. Use lower-semicontinuity of $\underset{\sim}{R}$ on concrete sequence of tensors.

$\Uparrow$

Strassen's asymptotic rank conjecture