

Asymptotic tensor rank
is characterized by polynomials

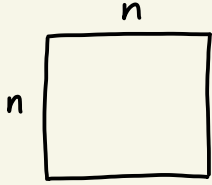
Jeroen Zuiddam

with Christandl, Hoeberechts, Nieuwboer, Vrana

1. Tensors and ranks
2. Algebraic complexity theory
3. Polynomials
4. Consequences

1. Tensors and ranks

matrix



rank

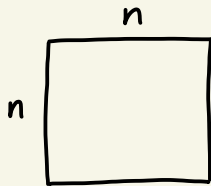
$$M = \sum_{i=1}^r u_i \otimes v_i$$

minimal (with a red arrow pointing to the superscript r)

- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have $\det = 0$

1. Tensors and ranks

matrix



rank

$$M = \sum_{i=1}^r u_i \otimes v_i$$

minimal

- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have $\det = 0$

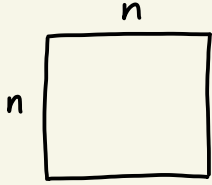
Kronecker product

$$A \otimes B$$

$$\bullet R(A \otimes B) = R(A)R(B)$$

1. Tensors and ranks

matrix



rank ↖ minimal

$$M = \sum_{i=1}^r u_i \otimes v_i$$

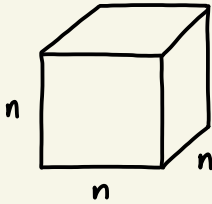
- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have $\det = 0$

Kronecker product

$$A \otimes B$$

$$\bullet R(A \otimes B) = R(A)R(B)$$

tensor



tensor rank

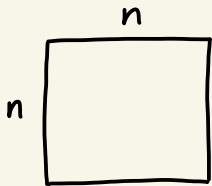
$$T = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$$

↖ minimal

- NP-hard

1. Tensors and ranks

matrix



rank

$$M = \sum_{i=1}^r u_i \otimes v_i$$

↖ minimal

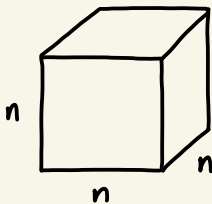
- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have $\det = 0$

Kronecker product

$$A \otimes B$$

$$\bullet R(A \otimes B) = R(A)R(B)$$

tensor



tensor rank

$$T = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$$

↖ minimal

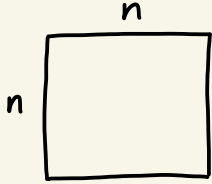
- NP-hard

$$T \otimes S$$

$$\bullet R(T \otimes S) \leq R(T)R(S)$$

1. Tensors and ranks

matrix



rank ↖ minimal

$$M = \sum_{i=1}^r u_i \otimes v_i$$

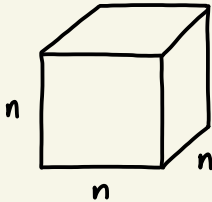
- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have $\det = 0$

Kronecker product

$$A \otimes B$$

$$\bullet R(A \otimes B) = R(A)R(B)$$

tensor



tensor rank

$$T = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$$

↖ minimal

- NP-hard

$$T \otimes S$$

$$\bullet R(T \otimes S) \leq R(T)R(S)$$

$\Rightarrow$ asymptotic tensor rank: $\underset{\sim}{R}(T) := \lim_{n \rightarrow \infty} R(T^{\otimes n})^{1/n}$

2. Algebraic complexity theory

$$\begin{matrix} & n \\ n & \square \end{matrix} \cdot \square = \square$$

2. Algebraic complexity theory

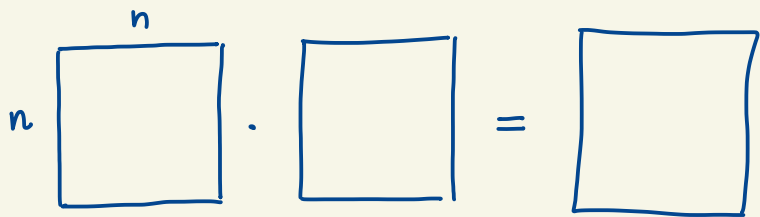
A diagram illustrating the complexity of matrix multiplication. It shows three squares. The first square has a vertical side labeled 'n' and a horizontal side labeled 'n'. To its right is a dot. The second square is empty. To its right is an equals sign. The third square is empty. This represents the equation: an n by n matrix multiplied by an n by n matrix results in an n by n matrix.

$O(n^{\omega})$ arithmetic operations

matrix mult. exponent

$$2 \leq \omega \leq 2.3\dots$$

2. Algebraic complexity theory



A diagram illustrating matrix multiplication. It consists of three squares. The first square has a blue 'n' to its left and a blue 'n' above it. A blue dot is placed to the right of the first square. A blue equals sign is placed to the right of the second square. The third square is empty.

Conjecture: $\omega = 2$

$\mathcal{O}(n^{\omega})$ arithmetic operations

matrix mult. exponent

$$2 \leq \omega \leq 2.3\dots$$

2. Algebraic complexity theory

$$\begin{matrix} n \\ \square \end{matrix} \cdot \square = \square$$

Conjecture: $\omega = 2$

Th: $R(T) = 2^\omega$

$\uparrow$

matrix mult. tensor $\in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$

$\swarrow$ matrix mult. exponent
 $O(n^\omega)$ arithmetic operations

$$2 \leq \omega \leq 2.3 \dots$$

2. Algebraic complexity theory

$$\begin{array}{c} n \\ \square \end{array} \cdot \square = \square$$

n

$\mathcal{O}(n^{\omega})$ arithmetic operations

matrix mult. exponent

$$2 \leq \omega \leq 2.3 \dots$$

Conjecture: $\omega = 2$

Th: $\tilde{R}(T) = 2^{\omega}$

↑

matrix mult. tensor $\in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$

Strassen's asymptotic rank conjecture:

For any concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $\tilde{R}(T) = m$

2. Algebraic complexity theory

$$\begin{matrix} n \\ \square \end{matrix} \cdot \square = \square$$

$\mathcal{O}(n^{\omega})$ arithmetic operations
matrix mult. exponent

$$2 \leq \omega \leq 2.3\dots$$

Conjecture: $\omega = 2$

Th: $\mathcal{R}(T) = 2^{\omega}$
 $\uparrow$
matrix mult. tensor $\in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$

Strassen's asymptotic rank conjecture:

For any concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $\mathcal{R}_{\infty}(T) = m$

What properties does $\mathcal{R}$ have? Computable? Semicontinuous? Integer-valued?

3. Polynomials $V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, $A \in V$, $r \in \mathbb{R}$

Theorem 1 $\{T \in V : \rho(T) \leq r\}$ is Zariski-closed.

3. Polynomials $V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, $A \in V$, $r \in \mathbb{R}$

Theorem 1 $\{T \in V : R_{\sim}(T) \leq r\}$ is Zariski-closed.

Def. $R_{\sim}[A] := \sup \{R_{\sim}(T) : T \in A\}$

Theorem 2 $R_{\sim}[\overline{A}] = R_{\sim}[A]$

3. Polynomials $V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, $A \subseteq V$, $r \in \mathbb{R}$

Theorem 1 $\{T \in V : \underline{R}(T) \leq r\}$ is Zariski-closed.

Def. $\underline{R}[A] := \sup \{\underline{R}(T) : T \in A\}$

Theorem 2 $\underline{R}[\bar{A}] = \underline{R}[A]$

Theorem 2 $\Rightarrow$ Theorem 1.

- Let $A = \{T \in V : \underline{R}(T) \leq r\}$. Then $\underline{R}[A] \leq r$.
- By Theorem 2, $\underline{R}[\bar{A}] \leq r$.
- So for all $T \in \bar{A}$, $\underline{R}(T) \leq r$, so $T \in A$.
- Then $\bar{A} \subseteq A$ $\square$

Theorem 2 $\mathcal{R}[\overline{A}] = \mathcal{R}[A]$

↑ important ingredient

Lemma 3 $\{T^{\otimes n} : T \in \overline{A}\} \subseteq \text{span} \{T^{\otimes n} : T \in A\}$

Theorem 2 $\mathcal{R}[\overline{A}] = \mathcal{R}[A]$

↑ important ingredient

Lemma 3 $\{T^{\otimes n} : T \in \overline{A}\} \subseteq \text{span} \{T^{\otimes n} : T \in A\}$

Proof

- Let ℓ be a linear form vanishing on the RHS
- $f \cdot T \mapsto \ell(T^{\otimes n})$ is a polynomial vanishing on A
- So f vanishes on $\overline{A}$
- Then ℓ vanishes on the LHS $\square$

Theorem 2 $\mathcal{R}[\overline{A}] = \mathcal{R}[A]$

Proof sketch:

To prove: $\leq$

Theorem 2 $\tilde{R}[\bar{A}] = \tilde{R}[A]$

Proof sketch:

To prove: $\leq$

- Let $T \in \bar{A}$. Then $T^{\otimes n} = \sum_{i=1}^{p(n)} \alpha_i S_i^{\otimes n}$ with $S_i \in A$ (Lemma 3)

$\leq \dim \text{Sym}^n(V)$ grows at most polynomially

Theorem 2 $\tilde{R}[\bar{A}] = \tilde{R}[A]$

Proof sketch:

To prove: $\leq$

• Let $T \in \bar{A}$. Then $T^{\otimes n} = \sum_{i=1}^{p(n)} \alpha_i S_i^{\otimes n}$ with $S_i \in A$ (Lemma 3)

$$\bullet \quad T^{\otimes nm} = \sum_{i_1, \dots, i_m = 1}^{p(n)} \alpha_{i_1} \dots \alpha_{i_m} S_{i_1}^{\otimes n} \otimes \dots \otimes S_{i_m}^{\otimes n}$$

$\leq \dim \text{Sym}^n(V)$ grows at most polynomially

Theorem 2 $\tilde{R}[\bar{A}] = \tilde{R}[A]$

Proof sketch:

To prove: $\leq$

• Let $T \in \bar{A}$. Then $T^{\otimes n} = \sum_{i=1}^{p(n)} \alpha_i S_i^{\otimes n}$ with $S_i \in A$ (Lemma 3)

• $T^{\otimes nm} = \sum_{i_1, \dots, i_m=1}^{p(n)} \alpha_{i_1} \dots \alpha_{i_m} S_{i_1}^{\otimes n} \otimes \dots \otimes S_{i_m}^{\otimes n}$

• $R(T^{\otimes nm}) \leq p(n)^m \max_{i_1, \dots, i_m} R(S_{i_1}^{\otimes n}) \dots R(S_{i_m}^{\otimes n})$

$\leq \dim \text{Sym}^n(V)$ grows at most polynomially

Theorem 2 $\tilde{R}[\bar{A}] = \tilde{R}[A]$

Proof sketch:

To prove: $\leq$

• Let $T \in \bar{A}$. Then $T^{\otimes n} = \sum_{i=1}^{p(n)} \alpha_i S_i^{\otimes n}$ with $S_i \in A$ (Lemma 3)

$$\bullet \quad T^{\otimes nm} = \sum_{i_1, \dots, i_m = 1}^{p(n)} \alpha_{i_1} \dots \alpha_{i_m} S_{i_1}^{\otimes n} \otimes \dots \otimes S_{i_m}^{\otimes n}$$

$$\bullet \quad R(T^{\otimes nm}) \leq p(n)^m \max_{i_1, \dots, i_m} R(S_{i_1}^{\otimes n}) \dots R(S_{i_m}^{\otimes n})$$

$$\leq p(n)^m \tilde{R}[A]^{mn} + \text{small part and skipping details}$$

$\leq \dim \text{Sym}^n(V)$ grows at most polynomially

Theorem 2 $\tilde{R}[\bar{A}] = \tilde{R}[A]$

Proof sketch:

To prove: $\leq$

• Let $T \in \bar{A}$. Then $T^{\otimes n} = \sum_{i=1}^{p(n)} \alpha_i S_i^{\otimes n}$ with $S_i \in A$ (Lemma 3)

$$\bullet \quad T^{\otimes nm} = \sum_{i_1, \dots, i_m=1}^{p(n)} \alpha_{i_1} \dots \alpha_{i_m} S_{i_1}^{\otimes n} \otimes \dots \otimes S_{i_m}^{\otimes n}$$

$$\bullet \quad R(T^{\otimes nm}) \leq p(n)^m \max_{i_1, \dots, i_m} R(S_{i_1}^{\otimes n}) \dots R(S_{i_m}^{\otimes n})$$

$$\leq p(n)^m \tilde{R}[A]^{mn} + \text{small part and skipping details}$$

$$\bullet \quad \tilde{R}(T) \leq \tilde{R}[A] \quad \square$$

4. Consequences and more

$$\mathcal{R} := \left\{ \tilde{R}(T) \cdot T \in \mathbb{F}^{d_1} \otimes \dots \otimes \mathbb{F}^{d_k}, d_1, \dots, d_k \in \mathbb{N} \right\}$$

Theorem $\mathcal{R}$ is well-ordered. (Every non-increasing sequence stabilizes.)

Noetherianity of $\mathbb{F}^{d_1} \otimes \dots \otimes \mathbb{F}^{d_k}$ and $\tilde{R}(T) \geq \max_i d_i$ for concise T

4. Consequences and more

$$\mathcal{R} := \left\{ \tilde{R}(T) \cdot T \in \mathbb{F}^{d_1} \otimes \dots \otimes \mathbb{F}^{d_k}, d_1, \dots, d_k \in \mathbb{N} \right\}$$

Theorem $\mathcal{R}$ is well-ordered. (Every non-increasing sequence stabilizes.)

Noetherianity of $\mathbb{F}^{d_1} \otimes \dots \otimes \mathbb{F}^{d_k}$ and $\tilde{R}(T) \geq \max_i d_i$ for concise T

Theorem $\mathcal{R}$ is complete (over $\mathbb{C}$).

Baire property for affine varieties over $\mathbb{C}$.

4. Consequences and more

$$\mathcal{R} := \left\{ \tilde{R}(T) \cdot T \in \mathbb{F}^{d_1} \otimes \dots \otimes \mathbb{F}^{d_k}, d_1, \dots, d_k \in \mathbb{N} \right\}$$

Theorem $\mathcal{R}$ is well-ordered. (Every non-increasing sequence stabilizes.)

Noetherianity of $\mathbb{F}^{d_1} \otimes \dots \otimes \mathbb{F}^{d_k}$ and $\tilde{R}(T) \geq \max_i d_i$ for concise T

Theorem $\mathcal{R}$ is complete (over $\mathbb{C}$).

Baire property for affine varieties over $\mathbb{C}$.

Open problems:

1. Is $\mathcal{R}$ discrete from below?

2. Is $\{T \in V : \tilde{R}(T) \leq r\}$ an irreducible variety? $\Uparrow$

3. Use lower-semicontinuity of $\tilde{R}$ on concrete sequence of tensors.

Strassen's asymptotic rank conjecture $\Uparrow$