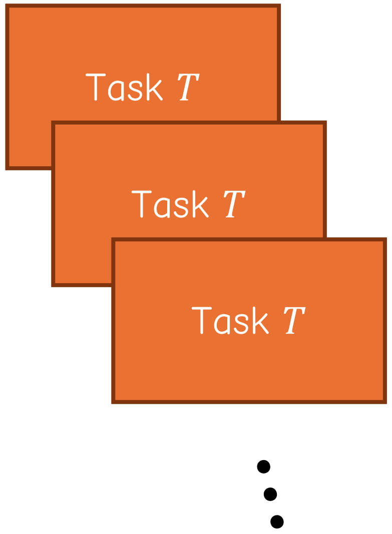
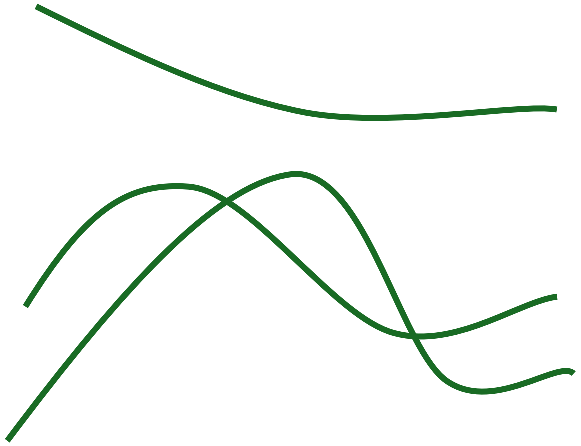
Asymptotic spectrum duality and entanglement polytopes

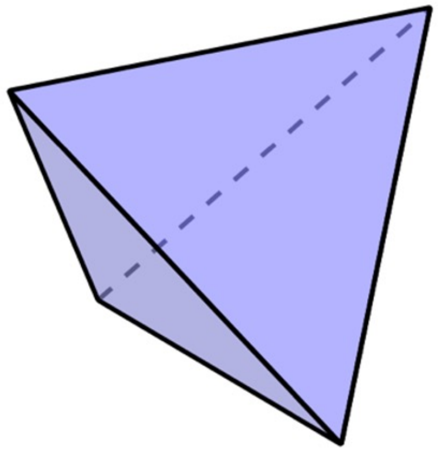
Jeroen Zuiddam



1. Can we achieve enonomies of scale?



2. Asymptotic spectrum duality



3. Entanglement polytopes

1. Economies of scale

Repeated task (computation, communication, ...)

$$T^{\otimes n} \qquad (n \to \infty)$$

$$\lim_{n\to\infty} f\left(T^{\otimes n}\right)^{1/n}$$

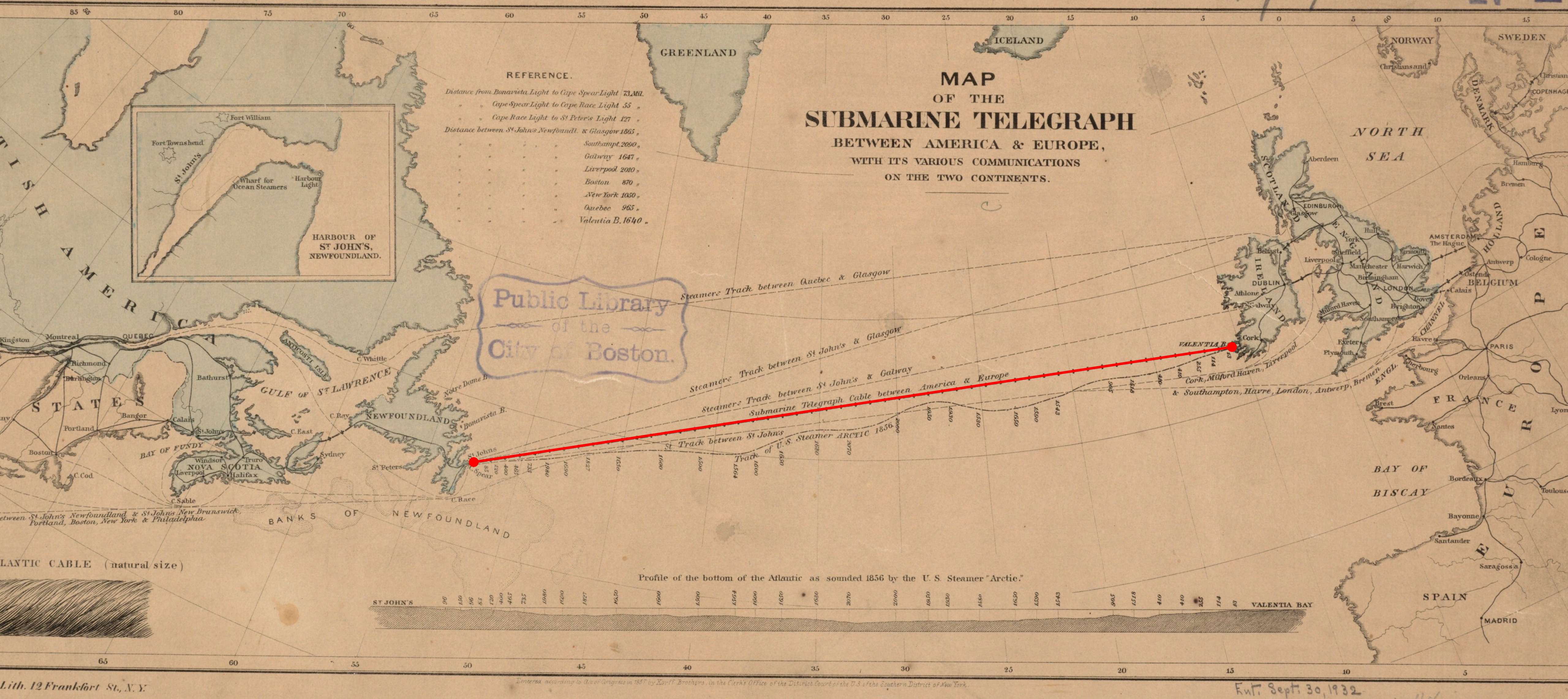
$$T^{\otimes (n+o(n))} \geq S^{\otimes n}$$

Sometimes have a simple characterisation!

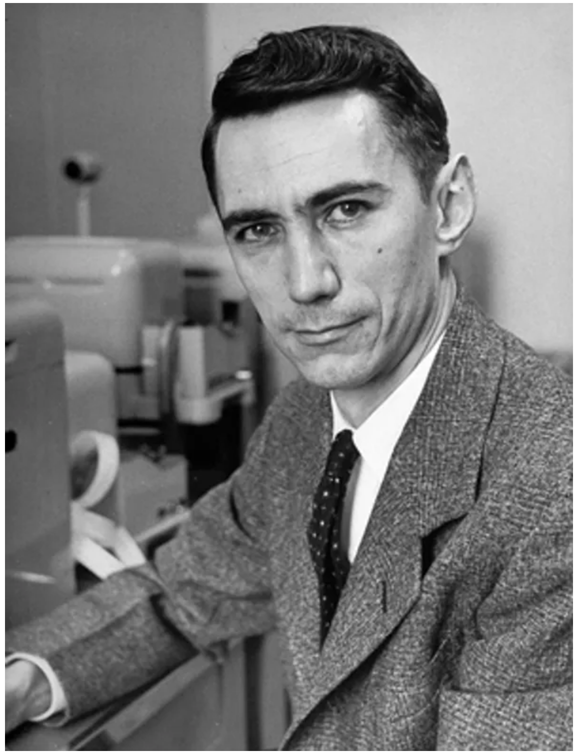
Computer science: amortization/direct-sum problem

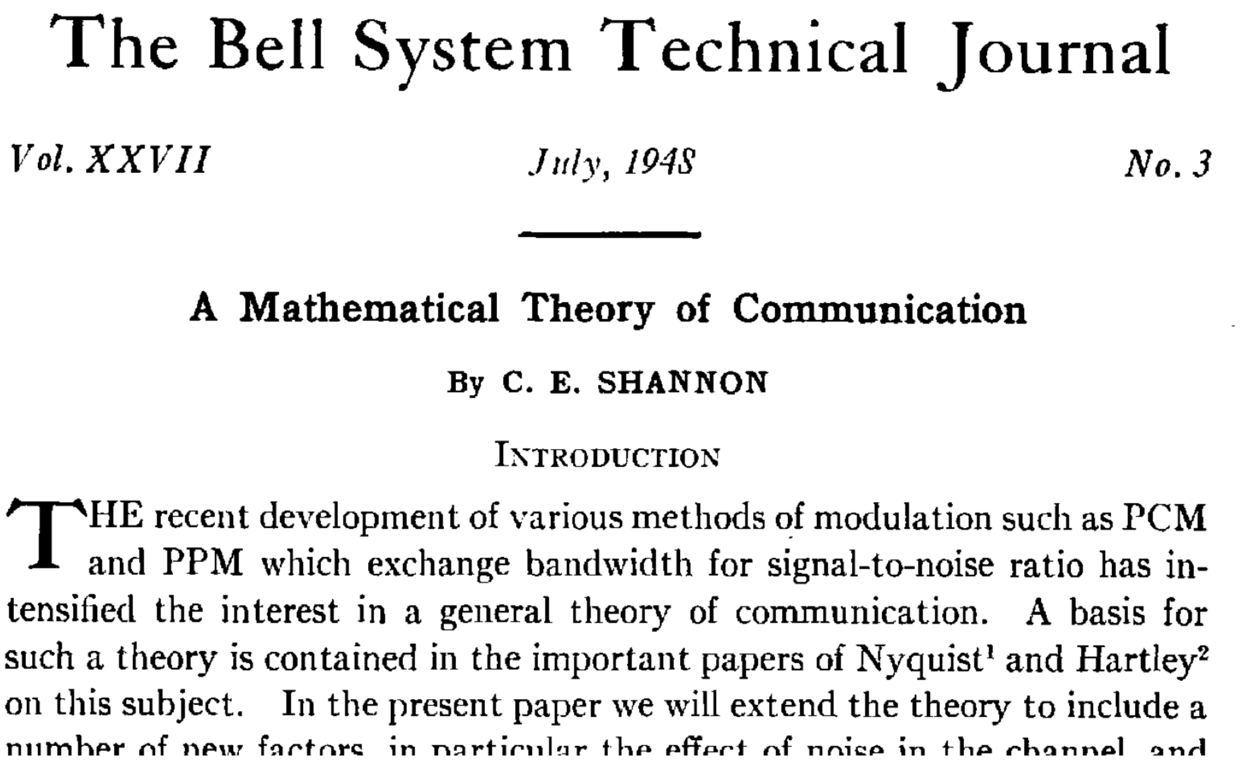
$$f\left(T^{\otimes n}\right) = f(T)^n \text{ ?}$$

H.M.S. Agamemnon
Laying the Atlantic
Telegraph Cable, 1858


MAP
OF THE
SUBMARINE TELEGRAPH
BETWEEN AMERICA & EUROPE,
WITH ITS VARIOUS COMMUNICATIONS
ON THE TWO CONTINENTS.
REFERENCE.
Distance from Bonavista Light to Cape Spear Light 73 Mil.
" " Cape Spear Light to Cape Race Light 55 "
" " Cape Race Light to St. Peter's Light 127 "
Distance between St. John's Newfoundl. & Glasgow 1865 "
" " " " Southampt. 2090 "
" " " " Galway 1647 "
" " " " Liverpool 2010 "
" " " " Boston 870 "
" " " " New York 1050 "
" " " " Quebec 965 "
" " " " Valentia B. 1640 "
HARBOUR OF ST. JOHN'S, NEWFOUNDLAND.
Fort William
Fort Townshend
St. John's
Wharf for Ocean Steamers
Harbour Light
GREENLAND
ICELAND
NORWAY
SWEDEN
DENMARK
NORTH SEA
SCOTLAND
IRELAND
ENGLAND
HOLLAND
BELGIUM
FRANCE
SPAIN
EUROPE
BAY OF BISCAY
NEWFOUNDLAND
NOVA SCOTIA
GULF OF ST LAWRENCE
BAY OF FUNDY
BANKS OF NEWFOUNDLAND
VALENTIA B.
Steamers Track between Quebec & Glasgow
Steamers Track between St. John's & Glasgow
Steamers Track between St. John's & Galway
Submarine Telegraph Cable between America & Europe
St. Track between St. Johns
Track of U.S. Steamer ARCTIC 1856
Cork, Milford Haven, Liverpool & Southampton, Havre, London, Antwerp, Bremen
between St. John's Newfoundland & St. Johns New Brunswick, Portland, Boston, New York & Philadelphia
LANTIC CABLE (natural size)
Profile of the bottom of the Atlantic as sounded 1856 by the U. S. Steamer "Arctic."
ST. JOHN'S
VALENTIA BAY
Lith. 12 Frankfort St., N.Y.
Entered according to Act of Congress in 1857 by Korff Brothers, in the Clerk's Office of the District Court of the U.S. of the Southern District of New York.

a. Shannon





The Bell System Technical Journal

Vol. XXVII *July, 1948* *No. 3*

A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

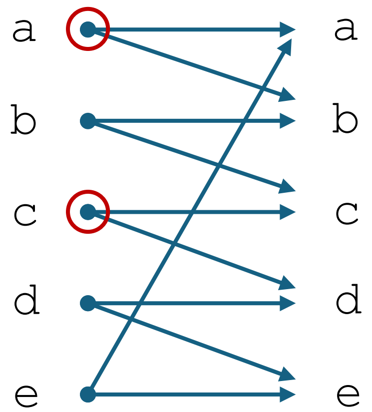
THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and

- large n (using the channel many times)
- redundancy and error-correction

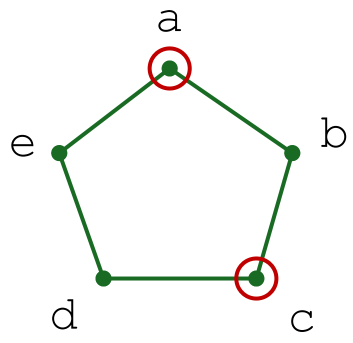
Graphs

zero-error communication, Shannon (1956)

noisy typewriter



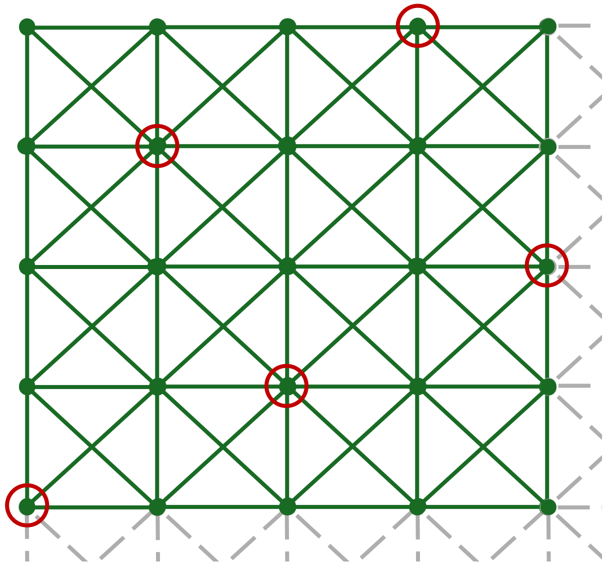
confusability graph



$\alpha(C_5) = 2$

$\{a, c\}$

using the channel twice



$\alpha(C_5^{\boxtimes 2}) = 5$

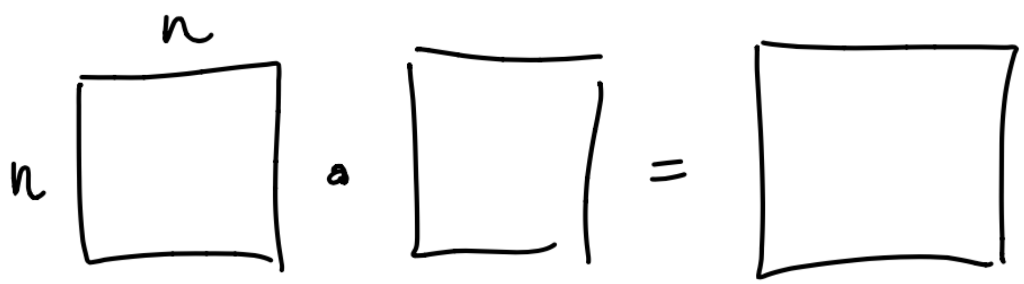
$\{(a, a), (c, b), (e, c), (b, d), (d, e)\}$

Problem: Determine $\Theta(G) \coloneqq \lim_{n\to\infty} \alpha(G^{\boxtimes n})^{1/n}$

Lower bounds: mostly ad hoc constructions
Upper bounds: Lovász theta, Haemers bound

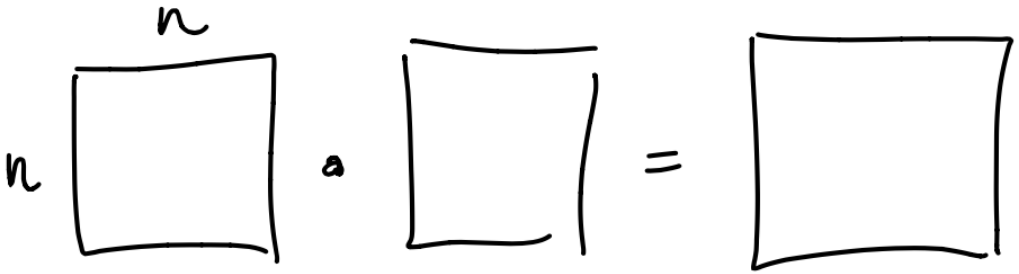
b. Matrix multiplication

n^2 arithmetic operations



n^3 arithmetic operations ?

Strassen

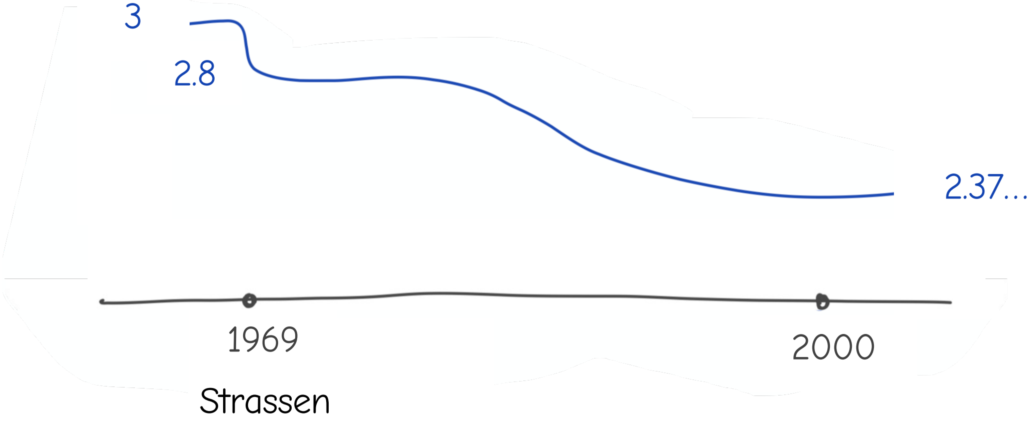


Standard algorithm: n^3 operations

Strassen's algorithm: $cn^{2.8}$ operations, better for large n (1969)

cn^{ω}

Optimal exponent is not known



Problem: Determine ω

Upper bounds: various intricate constructions

Lower bounds: "flattening rank"

Tensors

Survey: Bläser, Fast matrix multiplication

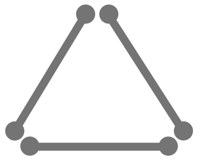
matrix multiplication map

$M_n : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$

tensor

$M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$

$$M_n = \sum_{i\in[n]} \sum_{j\in[n]} \sum_{k\in[n]} e_{i,j} \otimes e_{j,k} \otimes e_{k,i}$$



Problem: Determine $\underset{\sim}{R}(T)$

Asymptotic rank conjecture: equal to "flattening rank"

matrix multiplication algorithms

tensor rank decompositions

tensor rank $R(M_n)$

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

asymptotic rank $\underset{\sim}{R}(T) := \lim_{n\to\infty} R\left(T^{\otimes n}\right)^{1/n}$

$2^\omega = \underset{\sim}{R}(M_2)$

c. Quantum entanglement

tripartite pure states: $T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$

SLOCC transformation: $T \geq S$ iff $\exists A_1, A_2, A_3, (A_1 \otimes A_2 \otimes A_3)T = S$

stochastic local operations
and classical communication

$R(T) \leq r$ iff $T \leq I_r := \sum_{i=1}^{r} |i\rangle \otimes |i\rangle \otimes |i\rangle$

asymptotic SLOCC transformation: $T \gtrsim S$ iff $T^{\otimes(n+o(n))} \geq S^{\otimes n}$

Problem: Determine if $T \gtrsim S$

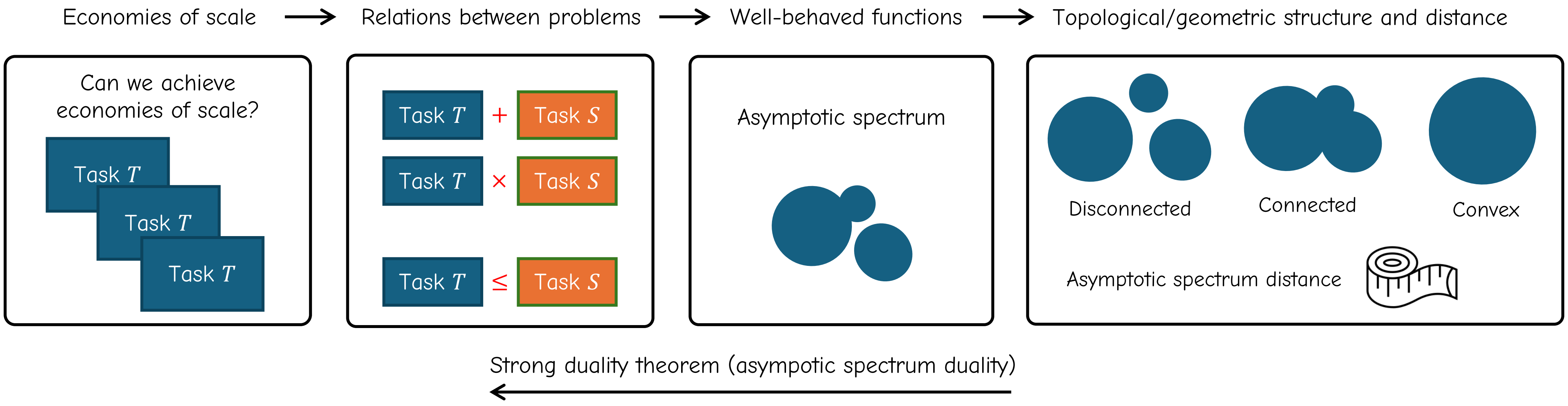
$\omega = 2$ iff $I_4 \gtrsim M_2$

2. Asymptotic spectrum duality

Strassen (1986–1991)
Wigderson–Zuiddam (2025)

Shannon capacity	Asymptotic tensor rank	Asymptotic SLOCC
$\Theta(G)$	$\underset{\sim}{R}(T)$	$\gtrsim$

overarching theory



Asymptotic spectrum of tensors

Definition

$X =$ set of all functions $\phi : \{\text{tensors}\} \to \mathbb{R}_{\geq 0}$ that are

$\geq$-monotone, $\otimes$-multiplicative, $\oplus$-additive, and normalised

Theorem

$T \gtrsim S \quad \Leftrightarrow \quad \phi(T) \geq \phi(S)$ for every $\phi \in X$

$\underset{\sim}{R}(T) \quad = \quad \max_{\phi \in X} \phi(T)$

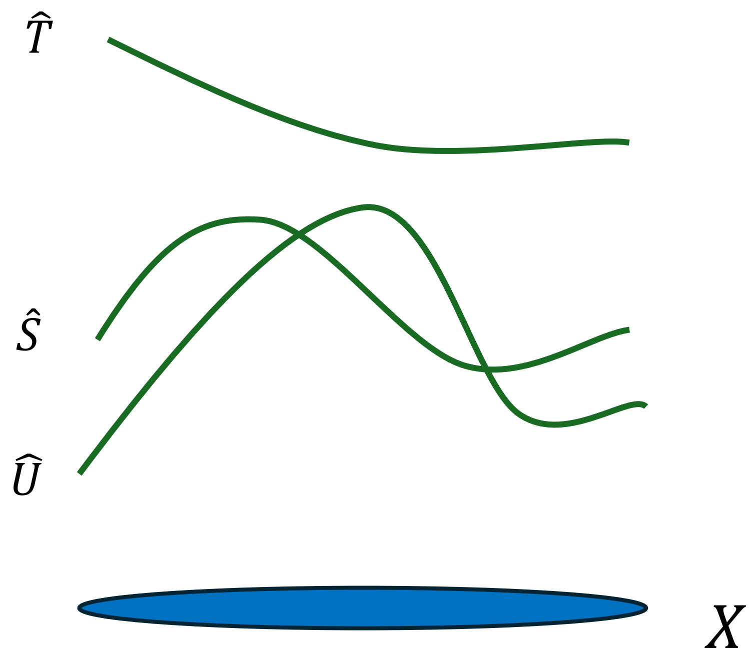
knowing X solves our problems!

analogously defined asymptotic spectrum of graphs characterizes Shannon capacity!

Topological point of view

$T \mapsto \hat{T} := [\phi(T)]_{\phi \in X} \in \mathbb{R}^X$

- $T \gtrsim S$ iff $\hat{T} \geq \hat{S}$ pointwise
- $\underset{\sim}{R}(T)$ is the pointwise max of $\hat{T}$



compact topological space

Holy grail: what is X?

Flattening ranks

$T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \quad \rightsquigarrow \quad T_1 \in \mathbb{C}^d \otimes (\mathbb{C}^d \otimes \mathbb{C}^d) \quad \rightsquigarrow \quad R_1(T) := R(T_1)$

Lemma

$R_i \in X$ for every $i = 1,2,3$

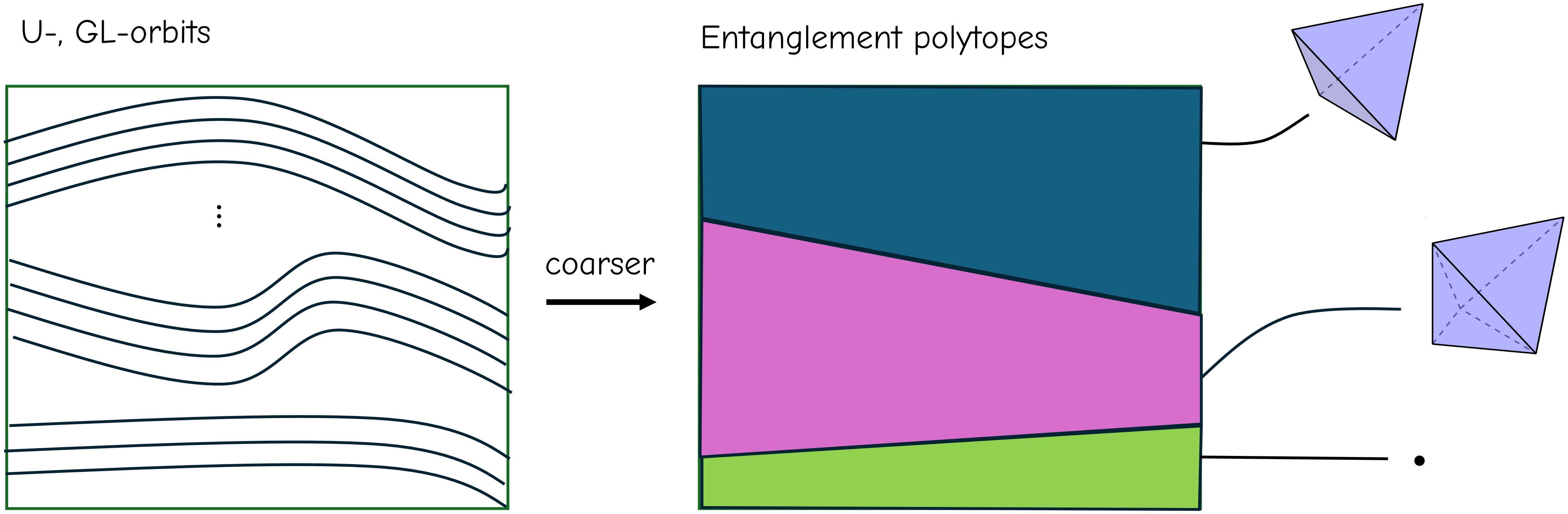
Question

Is this all of X? (If so, then $\omega = 2$ and many more consequences.)

No! We can make more using quantum information

3. Entanglement polytopes

How can we classify entanglement in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$?



3. Entanglement polytopes

$T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$

Quantum information

$T_1 \in \mathbb{C}^d \otimes (\mathbb{C}^d \otimes \mathbb{C}^d)$

$r_i(T) = \mathrm{spec} \frac{T_i T_i^*}{\|T_i T_i^*\|}$

$$\Delta(T) = \{(r_1(S), r_2(S), r_3(S)) : S \in \overline{\mathrm{GL} \cdot T}\}$$

↑ marginals reachable by approximative SLOCC

Representation theory

Schur-Weyl duality

$(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes n} = \oplus_\lambda V_\lambda$

$$\Delta(T) = \mathrm{closure}\, \{\lambda/n : P_\lambda\, T^{\otimes n} \neq 0\}$$

Theorem

- These descriptions coincide
- $\Delta(T)$ is a boundedconvex polytope with rational coefficients
- If $S \in \overline{\mathrm{GL} \cdot T}$ then $\Delta(S) \subseteq \Delta(T)$

Quantum functionals

Christandl, Vrana, Zuiddam (QIP, JAMS, STOC 2018)

"Interpolate" between the flattening ranks $\{R_1, R_2, R_3\} \subseteq X$

$$F_\theta(T) \coloneqq \exp \ \max \{\theta_1 H(p_1) + \theta_2 H(p_2) + \theta_3 H(p_3) : p \in \Delta(T)\}$$

Lemma

$$R_1(T) = F_{(1,0,0)}(T)$$

Theorem

$F_\theta \in X$ for every probability vector θ

There are numerical algorithms (tensor scaling) to approximate $F_\theta(T)$

Applications: barriers for matrix multiplication algorithms.

Recent work

van den Berg et al. (STOCC 2025)

Algorithm for computing entanglement polytopes

- Based on a characterization of Franz

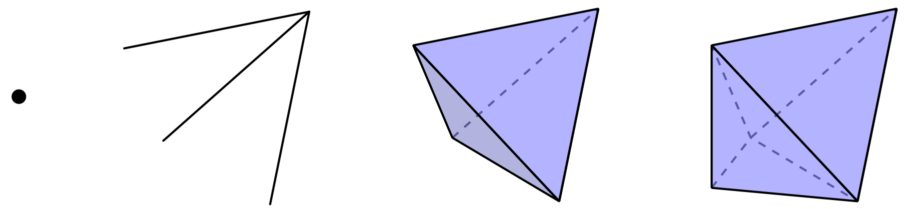
$$\Delta(T) = \cap_u \operatorname{conv} \operatorname{supp} (u \cdot g \cdot T)$$

- Not efficient, but practical for 3x3x3 and 4x4x4 tensors
- Several applications

Polytopes of all 3x3x3 tensors

With the algorithm we determined the entanglement polytopes of all 3x3x3 tensors

Previously: 2x2x2



New: 3x3x3

25 polytopes in dimension 2+2+2

https://github.com/qi-rub/explicit-tensor-moment-polytopes

Entanglement polytope separation

- Observed with algorithm
- Proved a separation between moment polytopes of matrix multiplication tensors and diagonal tensors

$$M_n = \sum_{i\in[n]} \sum_{j\in[n]} \sum_{k\in[n]} e_{i,j} \otimes e_{j,k} \otimes e_{k,i} \qquad I_r = \sum_{i\in[r]} e_i \otimes e_i \otimes e_i$$

Theorem
$\Delta(I_{n^2})$ is not contained in $\Delta(M_n)$ (more generally true for I_c with $n^2 - n + 1 < c$)

- That is, I_{n^2} can reach marginal spectra that cannot be reached from M_n
- Limitation on expressibility of tensor networks (bond dimension)

Explicit non-free tensors

Freeness plays an important role on the theory of tensors

- Any set $A \subseteq [d]^3$ is called free if every two elements differ in at least two coordinates, e.g. $\{(1,1,2),(1,2,1),(2,1,1)\}$ is free
- A tensor $T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ is called free if its support is free for in some basis
- Freeness is easy to certify, but only existence of non-free tensors was known

Theorem

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

is not free (and this extends to nxnxn)

Proof via moment polytopes

Open problems

1. What is the asymptotic spectrum of tensors?
2. What is its structure (convexity)?
3. What information can we get from entanglement polytopes?
4. Determine the entanglement polytope of matrix multiplication tensors, diagonal tensors