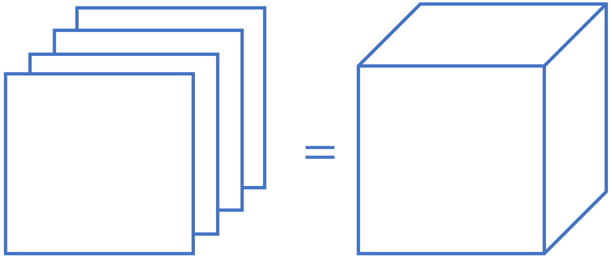
Geometric Rank of Tensors and Subrank of Matrix Multiplication

Swastik Kopparty (Rutgers)
Guy Moshkovitz (DIMACS, Rutgers and IAS)
Jeroen Zuiddam (IAS)

Tensors are 3-dimensional arrays

Matrix

Tensor



Tensors play a central role in Computer Science, Mathematics and Physics

- Algebraic complexity theory

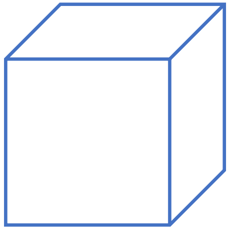
  Matrix Multiplication

- Quantum information theory

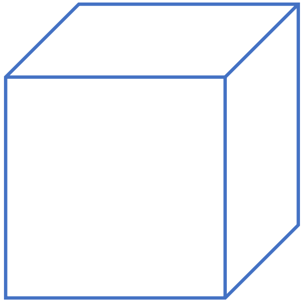
  Entanglement

- Extremal combinatorics

  Cap set problem, Sunflower problem



Motivated by these problems we introduce a new tensor parameter

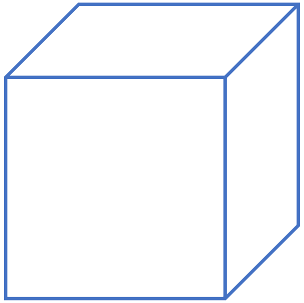


↦ nonnegative integer

Geometric Rank

Geometric Rank extends classical Matrix Rank

Matrix Rank



Geometric Rank

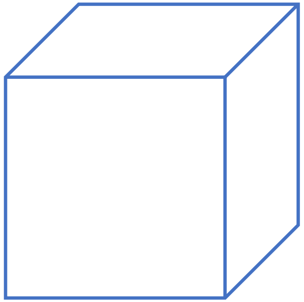
Tensor Rank

Slice Rank [Tao]

Subrank [Strassen]

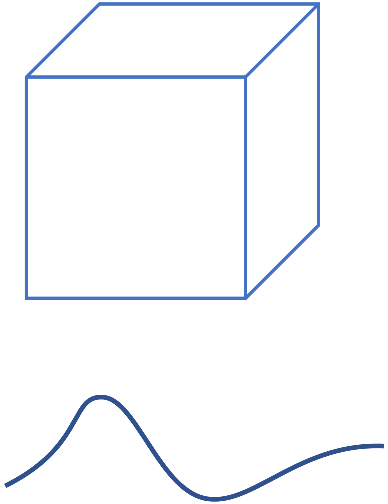
Analytic Rank [Gowers–Wolf, Lovett]

Geometric Rank is the geometric counterpart to Analytic Rank



$1, -1, -1, 1, 1, 1, -1, 1, \ldots$

Analytic Rank



Geometric Rank

Main results on Geometric Rank

- Basic properties and invariances
- Develop tools to reason about, and sometimes exactly compute it
- Intimate connections to the other important notions of rank for tensors
- Answer a question of Strassen (1987) on the Subrank of matrix multiplication

Applications: Geometric Rank provides new interesting route to

- Prove upper bounds on Subrank of tensors

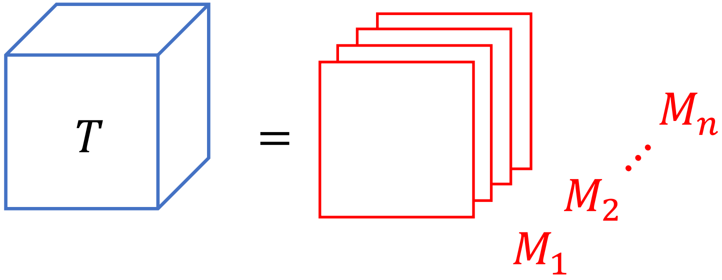
  important in complexity theory in the context of fast matrix multiplication and barriers

- (As a result) prove upper bounds on Independence Number of hypergraphs

  central in combinatorics in the context of the cap set problem and Erdős–Szemerédi sunflower problem

I. Geometric Rank

Geometric Rank



$$\rightsquigarrow$$

$$(x_1, \dots, x_n)\ M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$(x_1, \dots, x_n)\ M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$\vdots$$

$$(x_1, \dots, x_n)\ M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

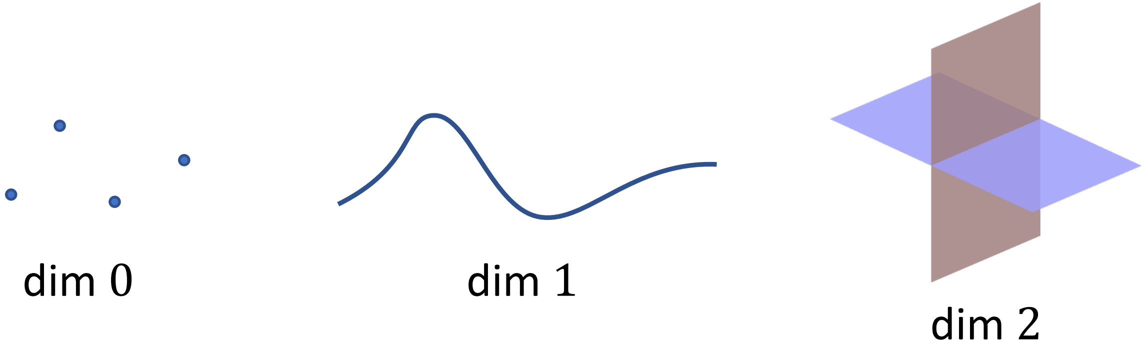
$$\rightsquigarrow$$

set of all solutions

$V(T)$

$\mathrm{GR}(T) = 2n -$ dimension of set of solutions $V(T)$

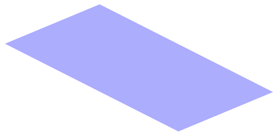
Dimension measures continuous degrees of freedom



“length of maximal chain of irreducible subvarieties”

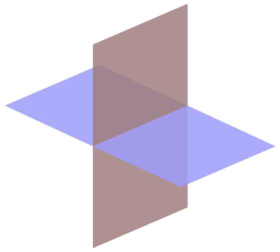
Computational intuition for dimension

- Dimension of linear space equals the notion of dimension from linear algebra



dim 2

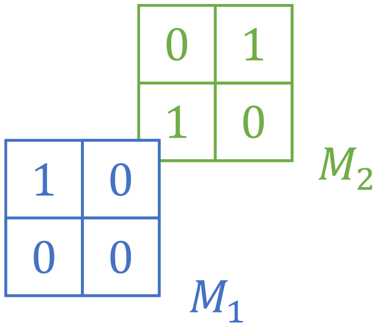
- Dimension of a finite union equals the maximum of the dimensions



dim 2

- Dimension does not increase under taking subsets

Example



$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}\ M_2$$

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}\ M_1$$

$$\rightsquigarrow$$

$$(x_1, x_2)\ M_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \ x_2 y_1 + x_1 y_2 = 0$$

$$(x_1, x_2)\ M_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \ x_1 y_1 = 0$$

Union of linear spaces of dimension 2:

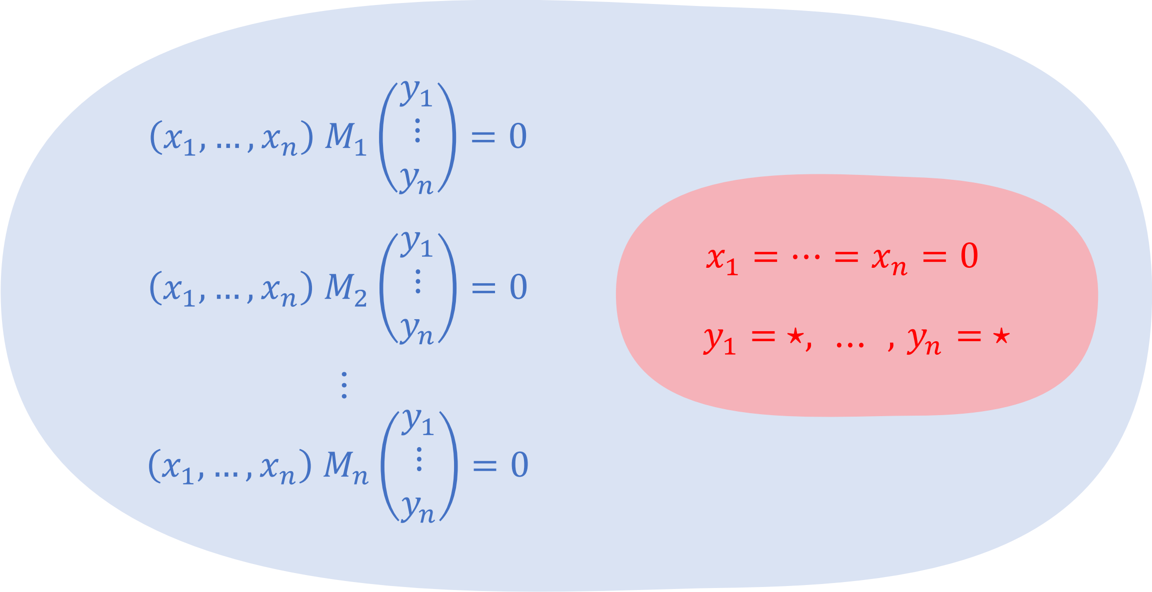
$\{x_1 = 0, y_1 = 0\}$

$\{y_1 = 0, y_2 = 0\}$

$\{x_1 = 0, x_2 = 0\}$

$\mathrm{GR}(T) = 4 - 2 = 2$

Observation: Geometric Rank takes values between 0 and n



$$(x_1, \dots, x_n)\, M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$(x_1, \dots, x_n)\, M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

$$\vdots$$

$$(x_1, \dots, x_n)\, M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$$

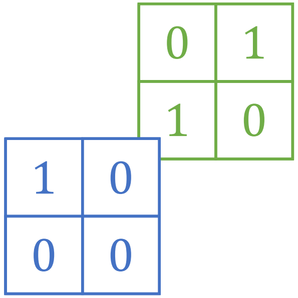
$$x_1 = \cdots = x_n = 0$$

$$y_1 = \star, \;\; \dots \;\; , y_n = \star$$

$$n \;\le\; \dim V(T) \;\le\; 2n$$

$$0 \;\le\; 2n - \dim V(T) \;\le\; n$$

Computing Geometric Rank is easy in practice for small tensors



$\rightsquigarrow$

system of equations:

$$x_2 y_1 + x_1 y_2 = 0$$
$$x_1 y_1 = 0$$

$\rightsquigarrow$

dimension:

2

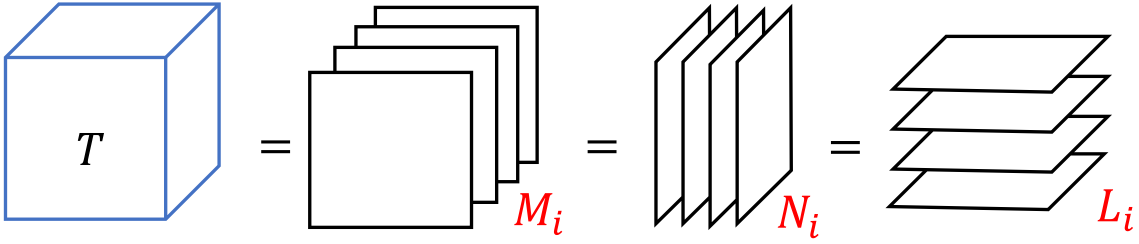
Macaulay2

```
R = CC[x1,x2,y1,y2];
dim ideal(x1*y1, x2*y1 + x1*y2)
```

Sage

```
A.<x1,x2,y1,y2> = AffineSpace(4, CC);
Ideal([x1*y1, x2*y1 + x1*y2]).dimension()
```

We do not know whether computing dimension of bilinear system is NP-hard.



Theorem 1

Slicing the tensor in a different direction gives the same notion of Geometric Rank

“Fundamental Theorem of Multilinear Algebra”

II. Main technical result: Monotonicity

Gaussian elimination

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & {}^{2}/_{3} \\ 0 & {}^{-2}/_{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

“Gaussian order” on Matrices

$$\boxed{M} \geq \boxed{N} \quad \text{if} \quad \underbrace{\boxed{A}}_{\text{some matrix}} \cdot \boxed{M} \cdot \overbrace{\boxed{B}}^{\text{some matrix}} = \boxed{N}$$

by taking some linear combinations of the rows and columns of M we obtain N

Example

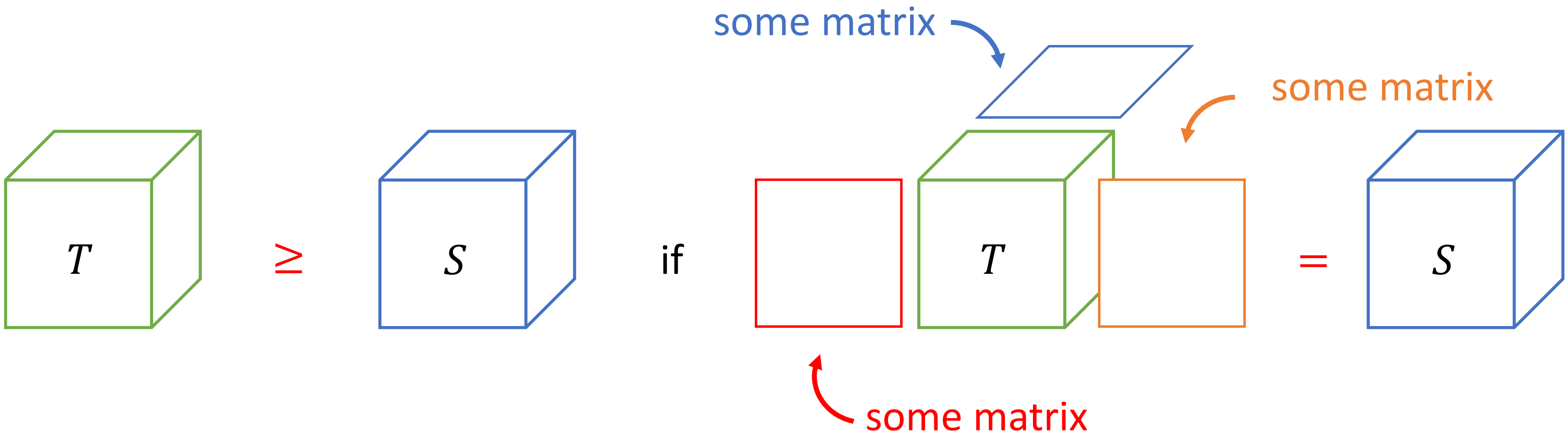
$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & {}^{2}/_{3} \\ 0 & {}^{-2}/_{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \geq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Rank completely determines the Gaussian order

$M \geq N$ if and only if $\mathrm{R}(M) \geq \mathrm{R}(N)$

Gaussian order on Tensors generalizes row and column operations



T ≥ S if

some matrix

some matrix

T = S

some matrix

by taking some linear combinations of the slices of T we obtain S

Gaussian order in Mathematics, Physics and Computer Science

- Complexity of Matrix Multiplication

identity tensor $\geq$ matrix multiplication tensor

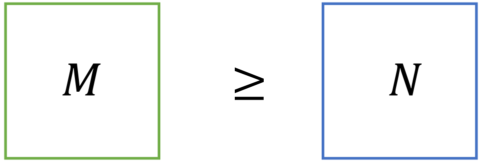
- Classifying Quantum Entanglement

tensor $\geq$ tensor

- 3-Uniform Hypergraph Independence Number

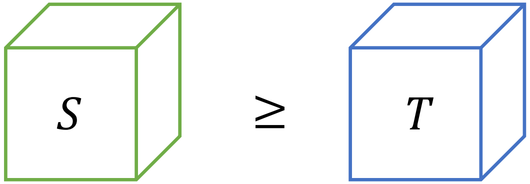
tensor fitting hypergraph $\geq$ identity tensor

Matrix Rank completely determines the Gaussian order on matrices



$$M \geq N \iff \mathrm{R}(M) \geq \mathrm{R}(N)$$

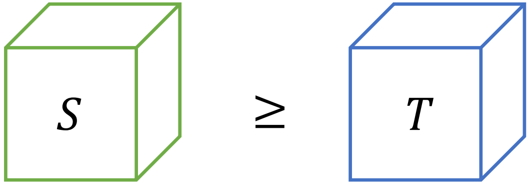
For tensors that level of complete understanding is out of reach



$$S \geq T \iff ?$$

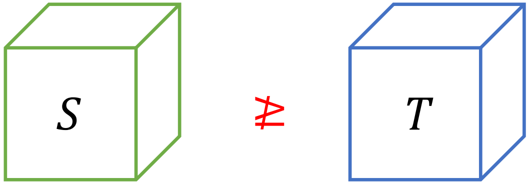
(NP-hard problem)

An important question is to find monotones for the Gaussian order on tensors:



$$S \geq T \quad \Longrightarrow \quad F(S) \geq F(T)$$

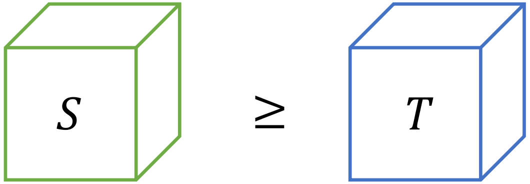
Monotones give obstructions:



$$S \ngeq T \quad \Longleftarrow \quad F(S) < F(T)$$

Theorem 2

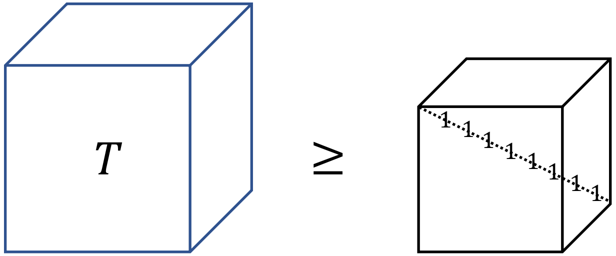
Geometric Rank is monotone



$$\Rightarrow \qquad \mathrm{GR}(S) \geq \mathrm{GR}(T)$$

III. Applications: Subrank and Independence number

Subrank $\mathrm{Q}(T)$ of T is the size of the largest identity tensor smaller than T



- Strassen (1987): central in theory of fast matrix multiplication

- Naturally leads to Haemers bound for hypergraphs:

Subrank $\mathrm{Q}(T)$ $\geq$ Independence number of *hypergraph for which T fits*

Geometric Rank upper bounds Subrank

Geometric Rank $\geq$ Subrank

$\mathrm{GR} \geq \mathrm{Q}$

Proof:

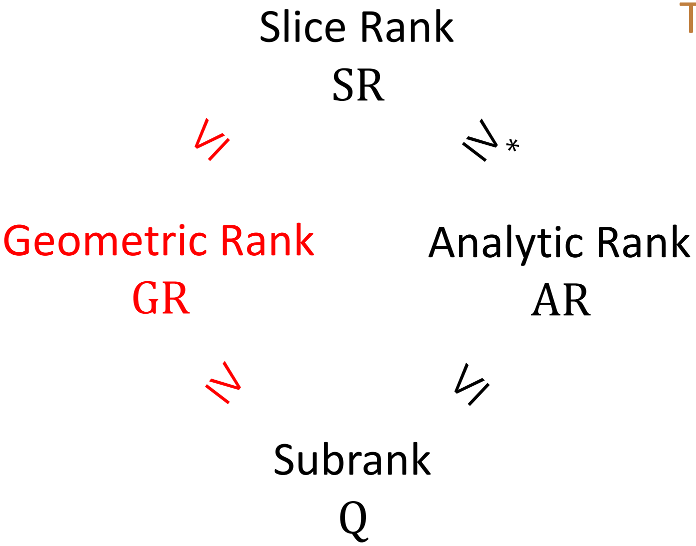
- Monotonicity
- Geometric Rank of diagonal tensor equals its size

In fact, Geometric Rank upper bounds Border Subrank

$$\begin{array}{ccccc} \text{Geometric Rank} & & \text{Border Subrank} & & \text{Subrank} \\ \mathrm{GR} & \geq & \underline{\mathrm{Q}} & \geq & \mathrm{Q} \\ & & \uparrow & & \end{array}$$

approximative version of Subrank

How Geometric Rank connects to other Ranks



Slice Rank
SR

Geometric Rank
GR

Analytic Rank
AR

Subrank
Q

Tao: Cap set problem

Gowers–Wolf

Lovett

Briët: Application in combinatorics

Geometric Rank "extends" Analytic Rank to characteristic 0

Theorem

For any tensor T with integer coefficients:

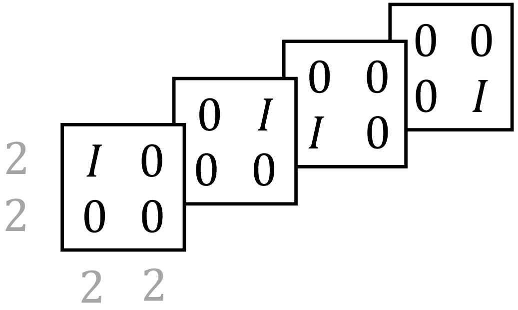
$$\mathrm{GR}(T) = \liminf_{p\to\infty} \mathrm{AR}(T \bmod p)$$

Proof ingredients:

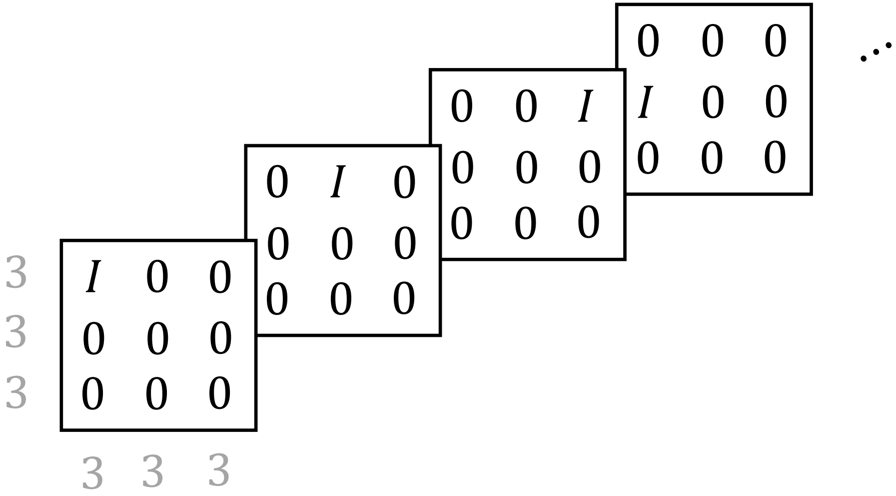
- Lang–Weil Theorem (good bounds on # $\mathbb{F}_p$-points for nice varieties in terms of dim)
- Bertini–Noether Theorem (relating $\mathbb{F}_p$-dimension to $\mathbb{C}$-dimension)
- Generalized Schwartz–Zippel lemma (coarse bound on # $\mathbb{F}_p$-points for all varieties) [Bukh–Tsimerman, Dvir–Kollár–Lovett]

Application of Geometric Rank: Matrix multiplication tensors

$\langle 2,2,2\rangle =$



$\langle 3,3,3\rangle =$



We compute the border subrank of matrix multiplication

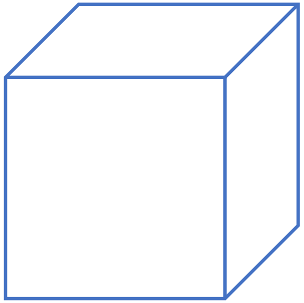
Theorem

$$\underline{\mathrm{Q}}(\langle n, n, n\rangle) \;=\; \left\lceil \tfrac{3}{4} n^2 \right\rceil$$

Proof:

- Lower bound: Strassen (1987)
- Upper bound: Geometric Rank

Matrix Rank



Geometric Rank