Asymptotic spectra and applications II

II.1. Summary of part I
2. Elements in the asympt. spectrum of graphs, X(graphs)
3. Elements in the asympt. spectrum of tensors, X(tensors)
4. Combinatorial aspects of X(tensors)

1. Summary of part I

1.1 Motivations

- matrix multiplication

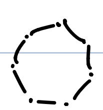
Kronecker product

$$\langle 2,2,2\rangle := \sum_{i,j,k=1}^{2} e_{ij}\otimes e_{jk}\otimes e_{ki},\quad \underset{\sim}{R}(\langle 2,2,2\rangle) := \inf_n R(\langle 2,2,2\rangle^{\otimes n})^{1/n}$$

$\underset{\sim}{R}(\langle 2,2,2\rangle) \overset{!!}{=} 2^{\omega}$; R = tensor rank

$2 \le \omega \le 2.37$

- Shannon capacity

$C_7 =$  $\quad \Theta(C_7) := \sup_n \alpha(C_7^{\boxtimes n})^{1/n}$

$\boxtimes$: strong graph product ; α: independence number

$3.25 \le \Theta(C_7) \le 3.31$

- Set (the game)

What is the maximum size of any subset $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ such that $\forall x,y,z \in A \ \nexists\, u,v \in (\mathbb{Z}/3\mathbb{Z})^n,\ v\neq 0$ $(x,y,z) = (u, u+v, u+2v)$

$$2.4 \le \sup_n |A|^{1/n} \le 2.755$$

1.2. Duality: Strassen's spectral theorem

- $(S,+,\cdot,0,1)$ semiring
- $\leq$ Strassen preorder
- $R(a) = \min\{n \in \mathbb{N} : a \leq n\}$ rank
- $Q(a) = \max\{n \in \mathbb{N} : n \leq a\}$ subrank
- asymptotic spectrum

$$X(S,\leq) = \{\phi \in \mathrm{Hom}(S,\mathbb{R}_{\geq 0}) : \forall a,b \in S \;\; a \leq b \Rightarrow \phi(a) \leq \phi(b)\}$$

Then* $\forall a,b \in S$

- $a^N \leq b^{N+o(N)}$ iff $\forall \phi \in X(S,\leq)$ $\phi(a) \leq \phi(b)$
- $\underset{\sim}{R}(a) := \inf_n R(a^n)^{1/n} \overset{!}{=} \max_{\phi \in X(S,\leq)} \phi(a)$. asymp. rank
- $\underset{\sim}{Q}(a) := \sup_n Q(a^n)^{1/n} \overset{!}{=} \min_{\phi \in X(S,\leq)} \phi(a)$ asymp. subrank

*under some mild assumptions

<u>Interaction theorem</u> $\underset{\sim}{Q}(ab) = \underset{\sim}{Q}(a)\underset{\sim}{Q}(b)$ iff $\underset{\sim}{Q}(a+b) = \underset{\sim}{Q}(a) + \underset{\sim}{Q}(b)$.
[Holzman]

<u>Subsemiring theorem</u> $S \subseteq T$ subsemiring, then $X(S,\leq|_S) \twoheadleftarrow X(T,\leq)$.

1.3 Instances

$(\{\text{tensors}\}, \oplus, \otimes, I_0, I_1)$

$S \leq T$ if $\exists A,B,C$ $S = (A,B,C)\cdot T$

$(\{\text{graphs}\}, \sqcup, \boxtimes, K_0, K_1)$

$G \leq H$ if there is a graph hom $\bar{G} \to \bar{H}$

$Q(G) = \alpha(G)$ $\quad R(G) = \bar{\chi}(G)$

$\underset{\sim}{Q}(G) = \Theta(G)$ $\quad \underset{\sim}{R}(G) = \underset{\sim}{\bar{\chi}}(G)$

2. Elements in the asymptotic spectrum of graphs

$$\{h_f^{\mathbb{F}}, \theta, \bar{\xi}_f, \bar{\chi}_f\} \subseteq X(\text{graphs})$$
$$= \{\phi : \{\text{graphs}\} \to \mathbb{R}_{\geq 0} \mid \boxtimes\text{-mult}, \sqcup\text{-add}, \overline{K_n}\text{-norm}, \leq\text{-mon}\}$$

2.1 Fractional functions

- $\bar{\chi}(G) = \min\{n \in \mathbb{N} : \exists f : V(G) \to [n],$
  $\forall u \neq v, \{u,v\} \notin E(G)\ f(u) \neq f(v)\}$

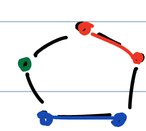
- $\bar{\xi}(G) = \min\{n \in \mathbb{N} : \exists f : V(G) \to \mathbb{R}^n \setminus \{0\}$
  $\forall u \neq v, \{u,v\} \notin E(G)\ \langle f(u), f(v)\rangle = 0\}$

- $h^{\mathbb{F}}(G) = \min\{R(M) : M \in \mathbb{F}^{V(G) \times V(G)}$
  $\forall v\ M_{vv} \neq 0$
  $\forall u \neq v, \{u,v\} \notin E(G)\ M_{uv} = 0\}$

<u>Lemma</u>: The above are submult, add, mon, norm.

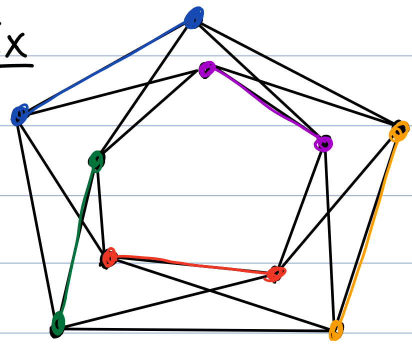
<u>Proof that $\alpha(G) \leq h^{\mathbb{F}}(G)$</u> Let $S \subseteq V(G)$ be an independent set. Let M be feasible. Then $|S| \leq R(M|_{S \times S}) \leq R(M)$.

<u>Ex</u> $\underset{\sqrt{5}}{\alpha(C_5^{\boxtimes 2})} \leq \Theta(C_5) \leq \underset{3}{\bar{\chi}(C_5)}$



Lemma. Let $\phi : \{\text{graphs}\} \to \mathbb{R}_{\geq 0}$ submult, add, mon, norm
Let
$$\phi_f(G) := \inf_d \phi(G[\overline{K_d}])/d$$
Then ϕ_f is submult, add, mon, norm.

Ex



$\alpha(C_5) \leq \bar{\chi}_f(C_5)$

$\frac{5}{2}$

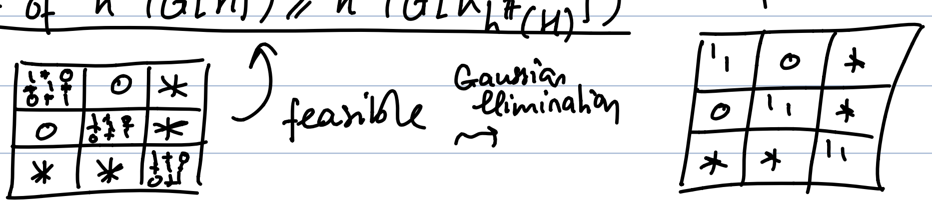
Lexicographic product $V(G[H]) = V(G) \times V(H)$

$E(G[H]) = \{((g,h),(g',h')) : \{g,g'\} \in E(G) \text{ or } (g=g' \text{ and } \{h,h'\} \in E(H))\}$

Lemma Let $\phi : \{\text{graphs}\} \to \mathbb{R}_{\geq 0}$ submult, add, mon, norm.
If $\forall G, H$ $\phi(G[H]) \geq \phi(G[\overline{K_{\phi(H)}}])$ then
$\phi_f \in X(\text{graphs})$.

Th [Butch-Cox, Shannon, ...]
$$\{h_f^{\#}, \bar{\vartheta}_f, \bar{\chi}_f\} \subseteq X(\text{graphs})$$

Proof of $h^{\#}(G[H]) \geq h^{\#}(G[\overline{K_{h^{\#}(H)}}])$ $\leftarrow$ feasible



Rem [Bukh-Cox] $h_f^{\#}$ is an infinite family

Recall: $\Theta = \underset{\sim}{Q} \notin X$

Theorem [See Schrijver] $\bar{\chi}_f = \underset{\sim}{\chi} = \underset{\sim}{R} \in X(\text{graphs})$

Rem $\bar{\chi}_f(G) = \min\{ \sum_C x(C) \mid x : \{\text{cliques}\} \to \mathbb{R}_{\geq 0}$
$\qquad \forall v \in V(G) \ \sum_{C \ni v} x(C) \geq 1 \}$

$\bar{\chi}_f(G)$ is large(!) linear program.

2.2 Lovász theta function

$\vartheta(G) := \min\{ \Lambda(M) : M \in \mathbb{R}^{V(G)\times V(G)}$ symmetric

$\uparrow$

largest eigenvalue $\qquad \forall v \ M_{vv} = 1$
$\qquad \forall u \neq v, \{u,v\} \notin E(G) \ M_{uv} = 1 \}$

Theorem [Lovász] $\vartheta \in X(\text{graphs})$

Rem $\vartheta(G)$ computable by small (!) semidefinite program.

Ex $A :=$ adjacency matrix of C_5

$M := J - \frac{1}{2}(5 - \sqrt{5})A$

$\Theta(C_5) \leq \vartheta(C_5) \leq \Lambda(M) = \sqrt{5}$

Appendix

Proof that $\alpha(G) \le \theta(G)$ Let $S \subseteq V(G)$ be an independent set. Let M be feasible in the definition of $\theta(G)$. Then $M|_{S\times S} = J_{|S|}$. Since $\Lambda(J_{|S|}) = |S|$, $\Lambda(M) \geq |S|$. $\square$

Proof that $\theta(G \boxtimes H) \le \theta(G)\theta(H)$ Let M, N be feasible in the def for $\theta(G), \theta(H)$ resp.

Let $\lambda = \Lambda(M)$, $\mu = \Lambda(N)$

Then $\lambda I - M$, $\mu I - N$, J and J are PSD.

So also the following matrix is PSD

$$(\lambda I - M) \otimes (\mu I - N) + (\lambda I - M) \otimes J + J \otimes (\mu I - N)$$

$$= \lambda\mu\, I \otimes I - \underbrace{\Big(\lambda I \otimes (N - J) + \mu (M - J) \otimes I - (M - J) \otimes (N - J) + J \otimes J \Big)}_{A}$$

A is feasible for $\theta(G \boxtimes H)$

$\lambda\mu\, I \otimes I - A$ is PSD, so $\Lambda(A) \le \lambda\mu$. $\square$

3. Elements in the asymptotic spectrum of tensors

$X(\text{tensors}) = \{\phi : \{\text{tensors}\} \to \mathbb{R}_{\geq 0} : \otimes\text{-mult}, \oplus\text{-add}, I_n\text{-norm}, \leq\text{-mon}\}$

$S \leq T$ if $\exists$ matrices A, B, C s.t $S = (A,B,C) \cdot T$

$S \cong T$ if $\exists$ invertible matrices A, B, C s.t $S = (A,B,C) \cdot T$

$S \trianglelefteq T$ if $\exists\ S_1, S_2, \dots \to S$ $\forall_i\ S_i \leq T$

- R^1, R^2, R^3 flattening ranks
- J^θ Strassen functions $\qquad \theta \in \mathbb{R}^3_{\geq 0}$, $\sum_i \theta_i = 1$
- F^θ quantum functions

3.1 Entropy

- $P = (P_1, \dots, P_d)$
- $\sum_{i=1}^{d} P_i = 1 \quad \forall_i P_i \geq 0$
- $\exists n\ \forall_i\ np_i \in \mathbb{N}$

$$\binom{n}{nP} := \frac{n!}{(nP_1)! \cdots (nP_d)!} = \text{number of strings in } [d]^n \text{ of type } P$$

$$2^{nH(P) - o(n)} \leq \binom{n}{nP} \leq 2^{nH(P)}$$

$$H(P) := -\sum P_i \log_2 P_i$$

3.2 Strassen functions

$T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$

$\mathrm{supp}(T) = \{(i,j,k) : T_{ijk} \neq 0\} \subseteq [n_1] \times [n_2] \times [n_3]$

$$\zeta^{\theta}(T) := \min_{S \cong T} \max \{ 2^{\sum_{i=1}^{3} \theta(i) H(P_i)} : P \in \mathcal{P}(\mathrm{supp}(S)) \}$$

<u>Theorem</u> [Str] ζ^{θ} submult, add, mon, norm

<u>Ex</u> $W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$

$$\underset{\sim}{Q}(W) \leq \zeta^{(1/3, 1/3, 1/3)}(W) = 2^{\sum \frac{1}{3} H(\frac{1}{3}, \frac{2}{3})} = 2^{H(\frac{1}{3}, \frac{2}{3})} = 1.88\ldots$$

<u>Ex</u> $T = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3$
$+ e_1 \otimes e_2 \otimes e_3$ + permutations over $\mathbb{F}_3$

$$\underset{\sim}{Q}(T) \leq \zeta^{(1/3, 1/3, 1/3)}(T) = 2.755\ldots \quad \text{[Tao]}$$

<u>Proof of $\underset{\sim}{Q}(T) \leq \zeta^{\theta}$ via slicerank</u>

$$SR(T) = \min \{ R^1(X) + R^2(Y) + R^3(Z) : T = X + Y + Z \}$$

elementwise sum

<u>Rem</u> SR is not submult and not supermult!

Lemma

i) $Q(T) \leq SR(T)$

ii) $Q(T)^n \leq SR(T^{\otimes n})$

iii) $\widetilde{SR}(T^{\otimes n}) \leq J^\theta(T)^{n+o(n)}$

Proof of iii) $J^\theta(T) = \min_{S \simeq T} \max_{p \in \mathcal{P}(\mathrm{supp}(S))} 2^{\sum \theta_i H(P_i)}$.

Let $S \simeq T$ min and p max.

$$S = \sum_{(a,b,c) \in \mathrm{supp} S} S_{abc}\, e_a \otimes e_b \otimes e_c$$

$$S^{\otimes n} = \sum_{\substack{a \in [d_1]^n \\ b \in [d_2]^n \\ c \in [d_3]^n}} S_{a_1 b_1 c_1} \cdots S_{a_n b_n c_n} (e_{a_1} \otimes \cdots \otimes e_{a_n}) \otimes (e_{b_1} \otimes \cdots \otimes e_{b_n}) \otimes (e_{c_1} \otimes \cdots \otimes e_{c_n})$$

- Each summand corresponds to n samples from $\mathrm{supp}(S)$ $(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots$
- Think of (a,b,c) as (empirical) distribution on $\mathrm{supp}(S)$
- Think of $a = (a_1, a_2, \ldots)$, b, c as marginal distr.
- $H(a), H(b)$ or $H(c) \leq \sum \theta_i H(P_i)$!

$$S^{\otimes n} = \boxed{\sum_{\substack{a,b,c \\ H(a) \leq \sum \theta_i H(P_i)}}} + \sum_{\substack{a,b,c\,: \\ H(b) \leq \sum \theta_i H(P_i)}} \cdots + \sum_{\substack{a,bc: \\ H(c) \leq \sum \theta_i H(P_i)}} \cdots$$

$$R'(\cdot) \leq \#\{a \in [d_1]^n : H(a) \leq \sum \theta_i H(P_i)\}$$

$$= \sum_{\substack{P \in \mathcal{P}([d_1]): \\ Pn \in \mathbb{N}^{d_1};\ H(P) \leq \sum \theta_i H(P_i)}} \binom{n}{pn} = \mathrm{poly}(n) \cdot 2^{n \sum \theta_i H(P_i)} = \mathrm{poly}(n)\, J^\theta(T)^n$$

$\square$.

$T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ is <u>oblique</u> if $\exists S \simeq T$ s.t. $\mathrm{supp}(S) \subseteq [n_1] \times [n_2] \times [n_3]$ is an antichain in the product order.

<u>Ex</u> W

<u>Theorem</u> [Str] $\zeta^\theta \in X$(oblique tensors over $\mathbb{F}$).

<u>Rem</u>: $\underbrace{\{\text{oblique}\} \subseteq \{\text{tensors}\}}_{\text{subsemiring}}$ so $X(\text{obique}) \twoheadleftarrow X(\text{tensors})$.

$\phi|_{\{\text{oblique}\}} \leftarrow\!\shortmid \phi$

3.3 Quantum functions

3.3.1 Moment polytope of $T \in \mathbb{C}^{d_1 \times d_2 \times d_3}$

Flattening spectra description

$S \in \mathbb{C}^{d_1 \times d_2 \times d_3}$

flatten: S^1

singular values: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{d_1} \geq 0 \in \mathbb{R}$

probability vector: $\mathrm{spec}(S^1) = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) / \sum \sigma_i^2$

moment polytope:

$$P(T) := \{ (\mathrm{spec}(S^1), \mathrm{spec}(S^2), \mathrm{spec}(S^3)) : S \in \mathbb{C}^{d_1 \times d_2 \times d_3}, S \trianglelefteq T \}$$

Schur-Weyl type description

$T \in \mathbb{C}^{d_1 \times d_2 \times d_3} \simeq \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$

$T^{\otimes n} \in \mathbb{C}^{d_1^n \times d_2^n \times d_3^n} \simeq (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n}$

S_n-invariant subspace

- $\mathbb{C}^d = \bigoplus_{a \in [d]} e_a \mathbb{C}$
- $(\mathbb{C}^d)^{\otimes n} = \bigoplus_{a \in [d]^n} e_{a_1} \otimes \cdots \otimes e_{a_n} \mathbb{C}$
- $(\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} = \bigoplus_{\substack{a \in [d_1]^n \\ b \in [d_2]^n \\ c \in [d_3]^n}} (e_{a_1} \otimes \cdots \otimes e_{a_n}) \otimes (e_{b_1} \otimes \cdots \otimes e_{b_n}) \otimes (e_{c_1} \otimes \cdots \otimes e_{c_n}) \mathbb{C}$
- $(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} [(\mathbb{C}^d)^{\otimes n}]_\lambda \leftarrow$ S_n-, GL_d-invariant subspace

Ex $(\mathbb{C}^d)^{\otimes 2} = [(\mathbb{C}^d)^{\otimes 2}]_{(2,0)} \oplus [(\mathbb{C}^d)^{\otimes 2}]_{(1,1)}$

$e_1\otimes e_1,\ e_2\otimes e_2,$ $\quad e_1\otimes e_2 - e_2\otimes e_1$

$e_1\otimes e_2 + e_2\otimes e_1$

$$(\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} = \bigoplus_{\substack{\lambda \vdash_{d_1} n \\ \mu \vdash_{d_2} n \\ \nu \vdash_{d_3} n}} [(\mathbb{C}^{d_1})^{\otimes n}]_\lambda \otimes [(\mathbb{C}^{d_2})^{\otimes n}]_\mu \otimes [(\mathbb{C}^{d_3})^{\otimes n}]_\nu$$

$$T^{\otimes n} = \bigoplus [T^{\otimes n}]_{\lambda\mu\nu}$$

$$P(T) \overset{!}{=} \text{closure}\left\{ \left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n}\right) : \exists n\ [T^{\otimes n}]_{\lambda\mu\nu} \neq 0 \right\}.$$

[Mumford]

3.3.2 Quantum function F^θ

Let $\theta \in \mathbb{R}^3_{\geq 0}$, $\theta_1 + \theta_2 + \theta_3 = 1$

Let $T \in \mathbb{C}^{n_1 \times n_2 \times n_3}$

$$F^\theta(T) = \sup\left\{ 2^{\sum \theta_i H(x_i)} : (x_1, x_2, x_3) \in P(T) \right\}$$

Th [CVZ] $F^\theta \in X$(tensors over $\mathbb{C}$).

Proof idea of $F^\theta(S\otimes T) = F^\theta(S)F^\theta(T)$

$\geq$ $\quad P(S\otimes T) \supseteq P(S) \otimes P(T)$

$x \in P(S)$ $\quad x = (\mathrm{spec}(\mathcal{U}^i))_i$ $\quad \mathcal{U} \trianglelefteq S$

$y \in P(T)$ $\quad y = (\mathrm{spec}(V^i))_i$ $\quad V \trianglelefteq T$

$x\otimes y = (x_1\otimes y_1, x_2\otimes y_2, x_3\otimes y_3)$ $\quad \mathcal{U}\otimes V \trianglelefteq S\otimes T$

$= (\mathrm{spec}((\mathcal{U}\otimes V)^i))_i \in P(S\otimes T)$

$F^\theta(S\otimes T) \geq 2^{\sum \theta_i H(x_i\otimes y_i)} = 2^{\sum\theta_i(H(x_i)+H(y_i))} = F^\theta(S)\, F^\theta(T).$

$\leq$ $\quad P(S\otimes T) \subseteq P(S) \odot P(T)$

$:= \mathrm{closure}\{ (\frac{\nu_1}{n}, \frac{\nu_2}{n}, \frac{\nu_3}{n}) : \exists (\frac{\lambda_1}{n}, \frac{\lambda_2}{n}, \frac{\lambda_3}{n}) \in P(S),\ (\frac{\mu_1}{n}, \frac{\mu_2}{n}, \frac{\mu_3}{n}) \in P(T)$

Kronecker coefficient $\longrightarrow$ $k(\lambda_i, \mu_i, \nu_i) \neq 0 \quad \forall i \}$

$H(\nu_i) \leq H(\lambda_i) + H(\mu_i) \quad \forall i$

$F^\theta(S\otimes T) = 2^{\sum\theta_i H(z_i)} \leq 2^{\sum\theta_i (H(x_i)+H(y_i))} \leq F^\theta(S)\ F^\theta(T)$ □

Th [CVZ]. $F^\theta(T) \leq \mathcal{J}^\theta(T)$

T is free if $\exists S \simeq T$ $\forall a\neq b \in \mathrm{supp}(S)$ a, b differs in at least 2 coords

Th [CVZ] If T is free, then $F^\theta(T) = \mathcal{J}^\theta(T)$

Th [CVZ] $\underset{\sim}{SR}(T) = \min_\theta F^\theta(T)$

4. Combinatorial aspects of X(tensors)

Combinatorial viewpoint on

- $Q(T) = \max\{n \in \mathbb{N} : I_n \leq T\}$
- $\underset{\sim}{Q}(T) = \lim_{n\to\infty} Q(T^{\otimes n})^{1/n}$

$T \in \mathbb{F}^{n_1 \times n_2 \times n_3} \rightsquigarrow \mathrm{supp}(T) \subseteq [n_1] \times [n_2] \times [n_3]$

- <u>3-graph</u> $\Phi \subseteq V_1 \times V_2 \times V_3$ (3-partite 3-uniform hypergraph)
- Φ is <u>matching</u> if $\forall a \neq b \in \Phi\ \forall i\ a_i \neq b_i$

<u>Ex</u>. $\{(1,1,1),(2,2,2),(3,3,3)\}$ $\{(1,2,3),(2,3,1),(3,1,2)\}$

- subset $\Psi \subseteq \Phi$ is <u>induced</u> if $\Psi = \Phi \cap (\Psi_1 \times \Psi_2 \times \Psi_3)$ where $\Psi_i = \{a_i : a \in \Psi\}$.
- <u>subrank</u> or <u>induced matching number</u> $Q(\Phi)$ is the size of the largest induced $\Psi \subseteq \Phi$ that is a matching.

<u>Ex</u> $Q(\{(1,1,1),(2,2,2),(3,3,3),(1,2,3)\}) = 2$

- Kronecker product
$$\Phi \otimes \Psi = \{((a_1,b_1),(a_2,b_2),(a_3,b_3)) : a \in \Phi, b \in \Psi\}$$
- $\underset{\sim}{Q}(\Phi) = \lim_{n\to\infty} Q(\Phi^{\otimes n})^{1/n}$

Lemma

- $Q(\operatorname{supp}(T)) \leq Q(T)$
- $\operatorname{supp}(S \otimes T) = \operatorname{supp}(S) \otimes \operatorname{supp}(T)$
- $\underset{\sim}{Q}(\operatorname{supp}(T)) \leq \underset{\sim}{Q}(T)$

Proof Let $\Psi \subseteq \operatorname{supp}(T)$ induced matching, so $\Psi = \operatorname{supp}(T) \cap (\Psi_1 \times \Psi_2 \times \Psi_3)$. Then $T|_{\Psi_1 \times \Psi_2 \times \Psi_3} \simeq I_{|\Psi|}$

$\square$

$\boxed{\text{Upper bounds on } \underset{\sim}{Q}(\Phi)}$

$$\underset{\sim}{Q}(\Phi) \leq \min_{\mathbb{F}} \min_{T \in \mathbb{F}^{n_1 \times n_2 \times n_3} : \operatorname{supp}(T) = \Phi} \underset{\sim}{Q}(T)$$

Ex Set (the game).

- $\forall x,y,z \in (\mathbb{Z}/3\mathbb{Z})^n \ (\exists u,v \ (x,y,z) = (u, u+v, u+2v) \iff x+y+z=0)$
- Let $\Phi = \{(x,y,z) \in (\mathbb{Z}/3\mathbb{Z})^n : x+y+z=0\}$
- Let $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ line-free
- $\forall x,y,z \in A \quad \exists u,v \ (x,y,z) = (u, u+v, u+2v) \Rightarrow x=y=z$
- $\forall x,y,z \in A \quad x+y+z=0 \Rightarrow x=y=z$
- Then $\Phi^{\otimes n}|_{A \times A \times A} = \{(a,a,a) : a \in A\}$, a matching!
- $\lim_{n\to\infty} |A|^{1/n} \leq \underset{\sim}{Q}(\Phi) \leq \underset{\sim}{Q}(T) = 2.755$

[Ellenberg, Gijswijt, Tao, Sawin]

$\boxed{\text{Lower bounds on } \underset{\sim}{Q}(\Phi)}$

$\Phi \subseteq V_1 \times V_2 \times V_3$ is <u>tight</u> if $\exists c \ \exists u_i : V_i \hookrightarrow \mathbb{Z}$ s.t.

$$\forall a \in \Phi \quad \sum_{i=1}^{3} u_i(a_i) = c$$

<u>Coppersmith-Winograd method</u> Let $\mathrm{supp}(T) = \Phi$ tight
Then

$$\underset{\sim}{Q}(\Phi) \overset{!}{=} \underset{\sim}{Q}(T) \overset{!}{=} \min_{\theta} \zeta^{\theta}(T)$$

<u>Ex.</u> $W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$
$\underset{\sim}{Q}(\mathrm{supp}(W)) = \underset{\sim}{Q}(W) = 2.755$

<u>Rem</u> [CVZ] We have an extension to k-graphs.

How about when Φ is not tight?
<u>Combinatorial degeneration</u> Let $\Psi \subseteq \Phi \subseteq V_1 \times V_2 \times V_3$.
$\Psi \trianglelefteq \Phi$ if $\exists c \ \exists u_i : V_i \to \mathbb{Z}$

$$\forall a \in \Psi \quad \sum u_i(a_i) = c$$
$$\forall a \in \Phi \setminus \Psi \quad \sum u_i(a_i) > c$$

<u>Theorem</u> [CVZ] If $\Psi \trianglelefteq \Phi$ then $\underset{\sim}{Q}(\Psi) \leq \underset{\sim}{Q}(\Phi)$.

Ex **Tri-colored sum-free sets**

$$\Phi = \{ (x,y,z) \in (\mathbb{Z}/3\mathbb{Z}) \ : \ x+y+z = \overset{2}{\cancel{0}} \}$$

$$\Psi = \{ (x,y,z) \in \{0,1,2\} \ : \ x+y+z = 2 \}$$ is clearly tight!

$$\Psi \trianglelefteq \Phi \qquad u_i : \mathbb{Z}/3\mathbb{Z} \to \{0,1,2\} : [a] \mapsto a.$$

$$2.755 \overset{CW}{=} \underset{\sim}{Q}(\Psi) \leq \underset{\sim}{Q}(\Phi) \leq 2.755$$

[Kleinberg, Sawin, Speyer, Norin, Pebody]