Asymptotic spectra and Applications

IAS

I. 1. Motivations
  1.1 Matrix multiplication
  1.2 Shannon capacity
  1.3 Set (the game)
 2. The asymptotic spectrum of matrices
 3. The asymptotic spectrum of tensors
 4. The asymptotic spectrum of graphs
 5. The asymptotic spectrum of an abstract semiring with respect to a Strassen preorder.

II. 1. Summary of part I
  2. Elements in X(graphs)
  3. Elements in X(tensors)
  4. Matchings, CW method.

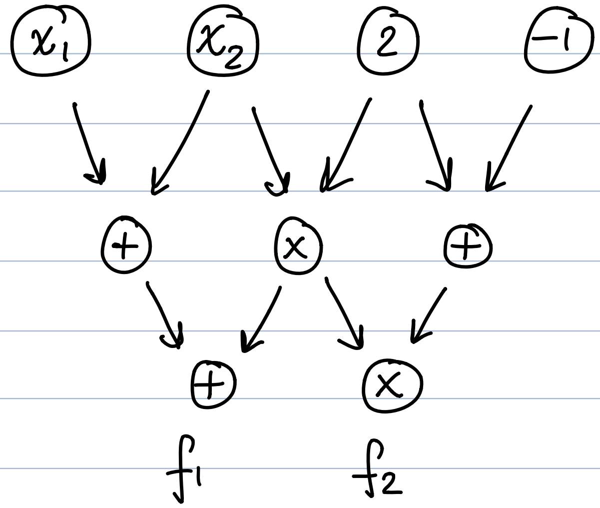
Literature

- Strassen. 1986, 1987, 1988, 2005
- Christandl, Vrana, Zuiddam. 2016 and 2017
- Zuiddam. The asymptotic spectrum of graphs and the Shannon capacity, 2018
- Zuiddam. phd thesis, 2018

1. Motivations.

1.1 Matrix multiplication

Algebraic circuit



input gates: variables and constants

computation gates: +, ×
(fan-in 2)

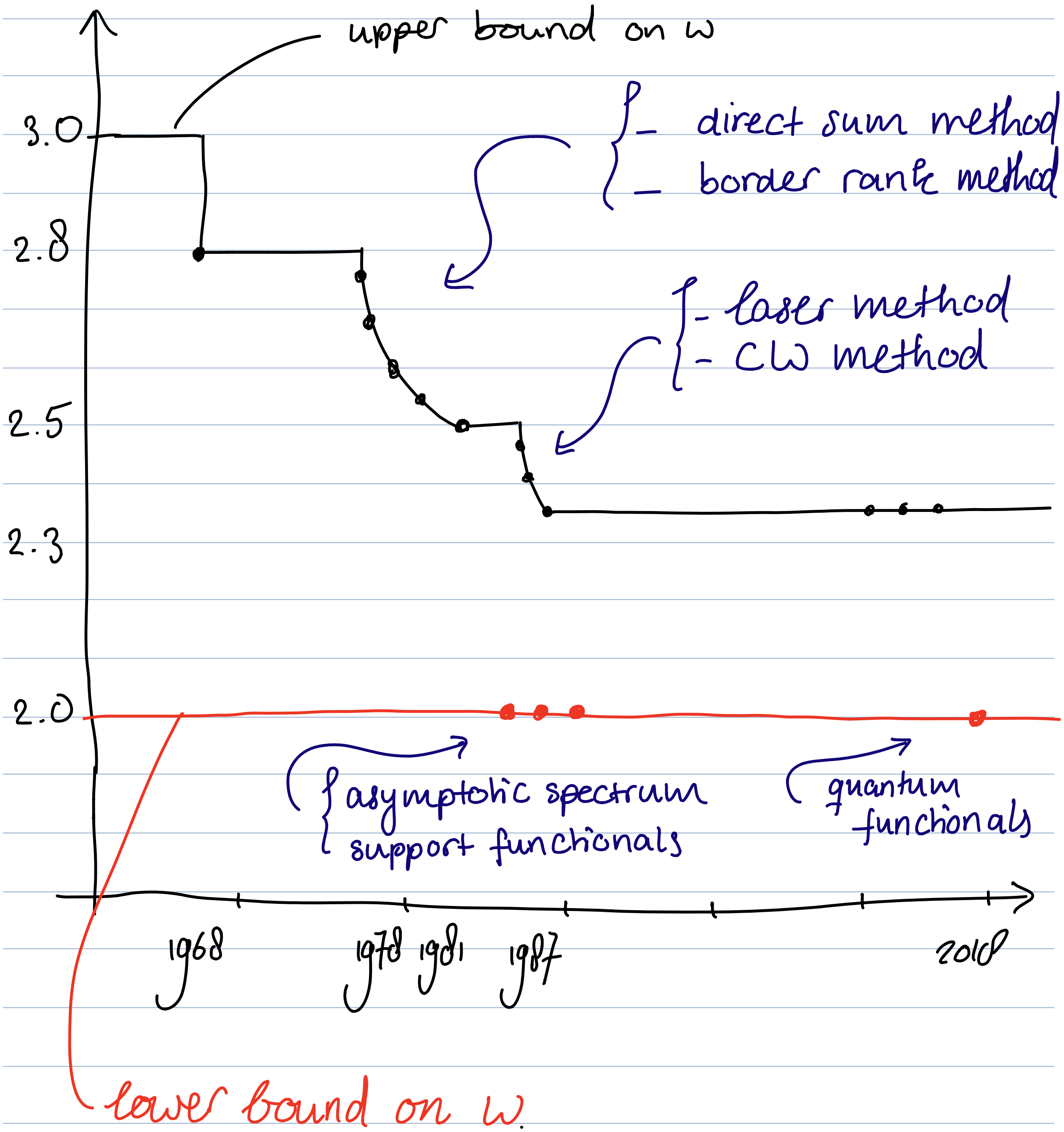
output gates

Matrix multiplication

$$n\underbrace{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & \ddots \end{bmatrix}}_{n} \cdot \underbrace{\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & \ddots \end{bmatrix}}_{n} = \underline{\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & \ddots \end{bmatrix}}$$

- How large a circuit do we need? $n^2 \le \cdot \le 2n^3$
- Simpler: what is the optimal exponent ; $c \cdot n^{\omega}$
- $2 \le \omega \le 2.37\ldots$ algorithms, constructions

non-existence of algorithms, obstructions


upper bound on ω
direct sum method
border rank method
laser method
CW method
3.0
2.8
2.5
2.3
2.0
asymptotic spectrum
support functionals
quantum functionals
1968
1978
1981
1987
2010
lower bound on ω.

Tensors $T = (T_{ijk}) \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ outer product

- T is *simple* if $\exists u, v, w$ $T = u \otimes v \otimes w := (u_i v_j w_k)_{ijk}$

- *rank* $R(T)$ is minimal $r \in \mathbb{N}$ s.t
  T is a sum of r simple tensors

- $\langle n, n, n \rangle = \sum_{ijk=1}^{n} e_{ij} \otimes e_{jk} \otimes e_{ki} \in \mathbb{F}^{n^2 \times n^2 \times n^2}$

- $R(\langle n,n,n \rangle) \approx$ min. circuit size of $n \times n$ ma.mu.

- self-reducible: $\langle n,n,n \rangle \boxtimes \langle m,m,m \rangle = \langle nm, nm, nm \rangle$
  Kronecker product

- $\omega := \inf_n \log_n R(\langle n,n,n \rangle)$
- $2^{\omega} = \inf_m R(\langle 2,2,2 \rangle^{\boxtimes m})^{1/m} =: \underset{\sim}{R}(\langle 2,2,2 \rangle)$

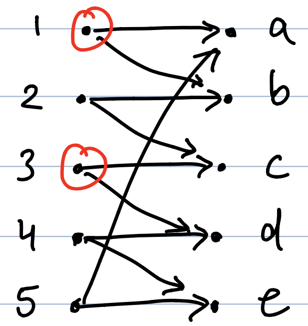
- $R(\langle 2,2,2 \rangle) \leq 7$ $(\omega \leq \log_2 7 \approx 2.8)$

  $\langle 2,2,2 \rangle = -(e_{-0\;0+\;11} + \text{cycl. perms})$
  $+(e_{-1\;1+\;00} + \text{cycl. perms})$
  $+(e_{00} + e_{11})^{\otimes 3}$

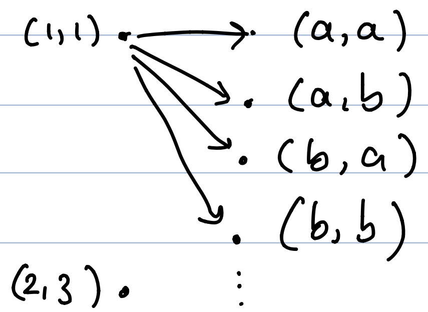
- $R(\langle 3,3,3 \rangle) \leq 23$
- $R(\langle 4,4,4 \rangle) \leq 49$

1.2 Shannon capacity

Communication channel $\mathcal{C}$



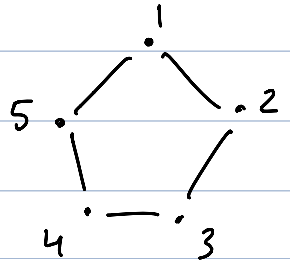
one-shot protocol: 2 symbols: 1, 3



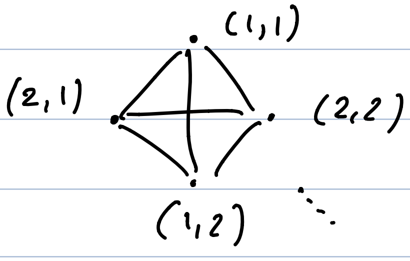
two-shot protocol: 5 symbols: (1,1), (2,3), (3,5), (4,2), (5,4)

- Shannon capacity: $\sup_n \left(n\text{-shot capacity of } \mathcal{C}\right)^{1/n}$

Confusability graph $\quad \mathcal{C} \mapsto G = (V,E)$



$= C_5$



$= C_5^{\boxtimes 2}$

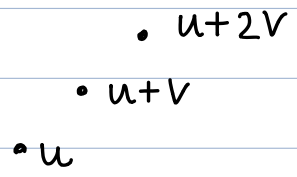
- <u>independent set</u> in G is set of pairwise nonadjacent vertices.
- <u>independence number</u> $\alpha(G)$ = size of largest ind. set
- $\Theta(G) := \sup_n \alpha(G^{\boxtimes n})^{1/n}$.

<u>Ex</u> $\Theta(C_5) = \sqrt{5}$ [Lovász]

[Polak-Schrijver] $3.25 \underset{\substack{\uparrow \\ \text{construction}}}{\le} \Theta(C_7) \underset{\substack{\uparrow \\ \text{obstruction}}}{\le} 3.31$ [Lovász]

1.3 Set (the game)

- $x, y, z \in (\mathbb{Z}/3\mathbb{Z})^{\overset{\swarrow n}{4}}$ form a <u>line</u> (a "Set") if $(x, y, z) = (u, u+v, u+2v)$ for some $u, v \in (\mathbb{Z}/3\mathbb{Z})^{\overset{\swarrow n}{4}}$, $v \neq 0$.



- How large can a line-free subset $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ be?

- $\underset{\substack{\text{[Edel]}}}{2.4^{n-o(n)}} \underset{\substack{\uparrow \\ \text{construction}}}{\le} |A| \underset{\substack{\uparrow \\ \text{obstruction.}}}{\le} 2.755^{n+o(n)}$ [Ellenberg, Gijswijt]

2. The asymptotic spectrum of matrices

- matrices $M=(M_{ij})_{ij}$ $N=(N_{ij})_{ij}$
- $M \leq N$ iff $\exists\, A, B$ $M = ANB^T =: (A,B)\cdot N$

- matrix rank $R(M) = \min\{n\in\mathbb{N} : M \leq I_n\}$

$$= \max\{n\in\mathbb{N} : I_n \leq M\}$$

$$\underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{n} n$$

- $$M \otimes N = \begin{bmatrix} M_{11}\cdot N & M_{12}\cdot N \\ M_{21}\cdot N & \ddots \end{bmatrix}$$

- $$M \oplus N = \begin{bmatrix} \boxed{M} & \\ & \boxed{N} \end{bmatrix}$$

<u>Th</u> Let $X(\text{matrices})$ = set of maps $\phi : \{\text{matrices}\} \to \mathbb{R}$ with

1. $\phi(M\otimes N) = \phi(M)\phi(N)$
2. $\phi(M\oplus N) = \phi(M) + \phi(N)$
3. $\phi(I_n) = n$
4. $M \leq N \Rightarrow \phi(M) \leq \phi(N)$

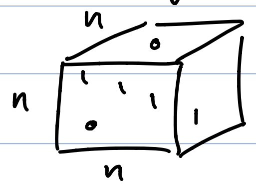
Then $X(\text{matrices}) = \{R\}$.

3. The asymptotic spectrum of tensors

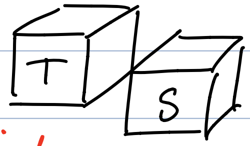
- tensors $T=(T_{ijk})_{ijk}$ $\quad S=(S_{ijk})_{ijk}$
- $T\le S$ iff $\exists A,B,C$ $\quad T=(A,B,C)\cdot S$

$$:=\Big(\sum_{abc} A_{ia}B_{jb}C_{kc}S_{abc}\Big)_{ijk}$$

- rank $R(T)=\min\{n\in\mathbb{N} : T\le I_n\}$
- subrank $Q(T)=\max\{n\in\mathbb{N} : I_n\le T\}$



- $(T\otimes S)_{(i,a)\,(j,b)\,(k,c)} = T_{ijk}\cdot S_{abc}$
- $T\oplus S$ =



- $\underset{\sim}{R}(T)=\underset{n\to\infty}{\overset{\inf}{\lim}}\; R(T^{\otimes n})^{1/n}$ $\qquad \underset{\sim}{Q}(T)=\underset{n\to\infty}{\overset{\sup}{\lim}}\; Q(T^{\otimes n})^{1/n}$
- $Q(T)\le \underset{\sim}{Q}(T)\le \underset{\sim}{R}(T)\le R(T)$

in fact, $\underset{\sim}{R}(T)\le \underline{R}(T)$

Ex $T=e_1\otimes e_1\otimes e_2+e_1\otimes e_2\otimes e_1+e_2\otimes e_1\otimes e_1$

$$1 < 2^{h(1/3)} < 2 < 3$$

Rem [CW] In general, $\forall n$ $\;\underset{\sim}{R}(T) < R(T^{\otimes n})^{1/n}$.

$X := X(\text{tensors}) :=$ set of maps $\phi : \{\text{tensors}\} \to \mathbb{R}_{\geq 0}$ with

1. $\phi(T \otimes S) = \phi(T)\phi(S)$
2. $\phi(T \oplus S) = \phi(T) + \phi(S)$
3. $\phi(I_n) = n$
4. $T \leq S \Rightarrow \phi(T) \leq \phi(S)$

__Lemma__ For $\phi \in X$ $\quad \underset{\sim}{Q}(T) \leq \phi(T) \leq \underset{\sim}{R}(T)$.

__Proof__ $\cdot$ $\underset{\sim}{R}(T) = \lim_{n \to \infty} R(T^{\otimes n})^{1/n}$

- $R(T^{\otimes n}) \leq \underset{\sim}{R}(T)^{n+o(n)}$
- $T^{\otimes n} \leq I_{\underset{\sim}{R}(T)^{n+o(n)}}$
- $\phi(T)^n \leq \underset{\sim}{R}(T)^{n+o(n)}$
- $\phi(T) \leq \underset{\sim}{R}(T)$ $\quad \square$

__Th__ (Strassen) $\underset{\sim}{Q}(T) = \min_{\phi \in X} \phi(T) \quad \underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$

__Remark__ $\underset{\sim}{Q}, \underset{\sim}{R} \notin X$ $\qquad$ __Ex__ $\langle 1,1,2 \rangle \otimes \langle 1,2,1 \rangle \otimes \langle 2,1,1 \rangle$
$= \langle 2,2,2 \rangle$

__Ex__ Three flattenings of T:

$T^1 := (T_{ijk})_{i(jk)}$
$T^2 := (T_{ijk})_{j(ik)}$
$T^3 := (T_{ijk})_{k(ij)}$

$$R^i : T \mapsto R(T^i) \qquad \in X$$

$\uparrow$ matrix rank.

Open problem Is there a tensor T with $\max_i R(T^i) < \underset{\sim}{R}(T)$?

Next week infinite families in

- $X(\text{tensors}/_{\mathbb{C}})$
- $X(\text{oblique tensors}/_{\mathbb{F}})$

and relation to Set.

general theory

$\swarrow \qquad \searrow$

Break.

$X(\text{matrices}) = \{R\}$ $\qquad$ $X(\text{graphs})$.

$X(\text{tensors}) \supseteq \{R^1, R^2, R^3\}$

$\underset{\sim}{Q}(T) = \min_{\phi \in X} \phi(T)$ $\qquad$ $\Theta(G) = \min_{\phi \in X} \phi(G)$

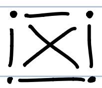
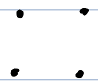
$\underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$ $\qquad$ $\underset{\sim}{\bar{\chi}}(G) = \max_{\phi \in X} \phi(G)$.

4. The asymptotic spectrum of graphs

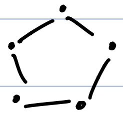
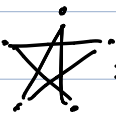
graph $G = (V, E)$ no self-loops.

Ex. $\bullet$ C_n $\quad V(C_n) = \{1, 2, \dots, n\}$

$E(C_n) = \{\{1,2\}, \{2,3\}, \dots, \{n,1\}\}$

- K_n $\quad V(K_n) = \{1, 2, \dots, n\}$ $\quad E(K_n) =$ all pairs.
- $K_4 =$  $\overline{K_4} =$ 

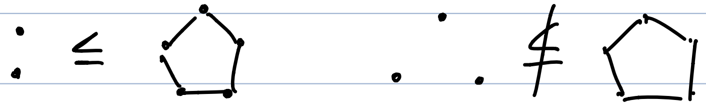
isomorphism

- $C_5 =$  $\overline{C_5} =$  $\cong C_5$

$G \leq H$ if there is a map $f : V(G) \to V(H)$ s.t. $\forall u, v \in V(G)$

$(u \neq v \wedge \{u, v\} \notin E(G)) \Rightarrow (f(u) \neq f(v) \wedge \{f(u), f(v)\} \notin E(H))$

Ex.



$Q(G) := \max\{n \in \mathbb{N} : \overline{K_n} \le G\}$

$R(G) := \min\{n \in \mathbb{N} : G \le \overline{K_n}\}$

Lemma $Q(G) = \alpha(G)$

Proof · $\overline{K_n} \le G$

- $f : V(\overline{K_n}) \to V(G)$ mapping non-edges to nonedges.
- f is injective
- G has independent set of size n
- $\alpha(G) \geq n$ □

strong graph product $V(G \boxtimes H) = V(G) \times V(H)$

$E(G \boxtimes H) = \{((g,h),(g',h')) : (g = g' \text{ or } \{g,g'\} \in E(G))$
$\text{and } (h = h' \text{ or } \{h,h'\} \in E(H))\}$

disjoint union $V(G \sqcup H) = V(G) \sqcup V(H)$

$E(G \sqcup H) = E(G) \sqcup E(H)$.

$\underset{\sim}{Q}(G) = \lim_{n\to\infty} Q(G^{\boxtimes n})^{1/n}$

$= \Theta(G)$

$X(\text{graphs})$ = set of maps $\phi : \{\text{graphs}\} \to \mathbb{R}_{\geq 0}$
with

1. $\phi(G \boxtimes H) = \phi(G)\phi(H)$
2. $\phi(G \sqcup H) = \phi(G) + \phi(H)$
3. $\phi(\overline{K_n}) = n$
4. $G \leq H \Rightarrow \phi(G) \leq \phi(H)$.

<u>Th</u> [Z] $\textcircled{H}(G) = \min\limits_{\phi \in X} \phi(G)$.

<u>Rem</u> $\textcircled{H} \notin X$

[Haemers] $\exists G, H \quad \textcircled{H}(G \boxtimes H) > \textcircled{H}(G)\textcircled{H}(H)$

[Alon] $\exists G, H \quad \textcircled{H}(G \sqcup H) > \textcircled{H}(G) + \textcircled{H}(H)$

We will come back to these examples later.

<u>Next week</u> Elements in $X(\text{graphs})$

5. The asymptotic spectrum of an abstract semiring with respect to a Strassen preorder.

(commutative) semiring $(S, +, \cdot, 0, 1)$

1. $+$ assoc, comm
2. $0 + a = a$
3. $\cdot$ assoc, comm
4. $1 \cdot a = a$
5. $\cdot$ distributes over $+$ $\quad a\cdot(b+c) = (a\cdot b) + (a\cdot c)$
6. $0 \cdot a = 0$

notation $\overbrace{1 + \cdots + 1}^{n} =: n \in S$

preorder on X $\lesssim$ relation

1. $\lesssim$ reflexive
2. $\lesssim$ transitive

Strassen preorder $\lesssim$ on semiring S is preorder st.

1. $\forall n, m \in \mathbb{N}$ $\quad n \leq m$ in $\mathbb{N}$ iff $n \lesssim m$
2. $\forall a, b, c \in S$ if $a \lesssim b$ then $a+c \lesssim b+c$ and $ac \lesssim bc$
3. $\forall a, b \in S, b \neq 0$ $\exists r \in \mathbb{N}$ $a \lesssim rb$.

asymptotic preorder $\underset{\sim}{\lesssim}$ of $\lesssim$

$$a_2 \underset{\sim}{\lesssim} a_1 \text{ if } \exists (x_N)_N \subseteq \mathbb{N} \left(\inf_N x_N^{1/N} = 1 \text{ and } \forall N \; a_2^N \lesssim x_N a_1^N \right)$$

Examples • $(\text{tensors}, \otimes, \oplus, I_0, I_1)$

• $(\text{graphs}, \boxtimes, \sqcup, K_0, K_1)$

Asymptotic spectrum

$X(S, \leq) :=$ $\leq$-monotone semiring homomorphisms $S \to \mathbb{R}_{\geq 0}$

Spectral theorem (Strassen) Let S semiring and $\leq$ on S a Strassen preorder. Then $\forall a, b \in S$

$$a \underset{\sim}{\leq} b \quad \text{iff} \quad \forall \phi \in X(S, \leq) \ \ \phi(a) \leq \phi(b).$$

rank $R(a) = \min\{n \in \mathbb{N} : a \leq n\}$

subrank $Q(a) = \max\{n \in \mathbb{N} : n \leq a\}$

asymptotic rank $\underset{\sim}{R}(a) = \lim_{n \to \infty} R(a^n)^{1/n}$

asymptotic subrank $\underset{\sim}{Q}(a) = \lim_{n \to \infty} Q(a^n)^{1/n}$

Cor. Let $a \in S$. Then

- $\min_{\phi \in X} \phi(a) = \underset{\sim}{Q}(a)$
- $\max_{\phi \in X} \phi(a) = \underset{\sim}{R}(a)$

Under mild conditions:

$\exists k \in \mathbb{N} : a^k \geq 2$

$\exists \phi \in X : \phi(a) \geq 1.$

Additive vs. multiplicative

Cor. $\underset{\sim}{Q}(ab) = \underset{\sim}{Q}(a)\underset{\sim}{Q}(b)$ iff $\underset{\sim}{Q}(a+b) = \underset{\sim}{Q}(a)+\underset{\sim}{Q}(b)$.
Same for $\underset{\sim}{R}$.

Proof For $a \in S$ let $\phi_a := \operatorname{argmin}_{\phi \in X} \phi(a)$.
So $\underset{\sim}{Q}(a) = \phi_a(a) \leq \phi_b(a)$ $\forall b \in S$.

Easy:

- $\underset{\sim}{Q}(a+b) = \phi_{a+b}(a+b) = \phi_{a+b}(a) + \phi_{a+b}(b)$
  $\geq \phi_a(a) + \phi_b(b) = \underset{\sim}{Q}(a) + \underset{\sim}{Q}(b)$
- $\underset{\sim}{Q}(ab) = \phi_{ab}(ab) = \phi_{ab}(a)\,\phi_{ab}(b)$
  $\geq \phi_a(a)\,\phi_b(b) = \underset{\sim}{Q}(a)\,\underset{\sim}{Q}(b)$.

Interesting:

- Suppose $\underset{\sim}{Q}(a+b) \leq \underset{\sim}{Q}(a) + \underset{\sim}{Q}(b)$
- Then $\phi_{a+b}(a) + \phi_{a+b}(b) = \phi_{a+b}(a+b) \leq \phi_a(a) + \phi_b(b)$
- Recall $\phi_{a+b}(a) \geq \phi_a(a)$ and $\phi_{a+b}(b) \geq \phi_b(b)$
- Thus $\phi_{a+b}(a) = \phi_a(a)$ and $\phi_{a+b}(b) = \phi_b(b)$
- Therefore

$$\underset{\sim}{Q}(ab) = \phi_{ab}(ab) \leq \phi_{a+b}(ab) = \phi_{a+b}(a)\,\phi_{a+b}(b)$$
$$= \phi_a(a)\,\phi_b(b) = \underset{\sim}{Q}(a)\,\underset{\sim}{Q}(b).$$

- Similarly, $\underset{\sim}{Q}(ab) \leq \underset{\sim}{Q}(a)\underset{\sim}{Q}(b)$ implies $\underset{\sim}{Q}(a+b) \leq \underset{\sim}{Q}(a)+\underset{\sim}{Q}(b)$ $\square$

EX. The results of Haemers and Alon are equivalent.

EX $\underset{\sim}{R}(\bigoplus_i \langle n_i, n_i, n_i \rangle) = \sum_i \underset{\sim}{R}(\langle n_i, n_i, n_i \rangle) = \sum_i n_i^{\omega}$

Subsemirings. Let S semiring and $\lesssim$ Strassen preorder. Let $T \subseteq S$ be a subsemiring. If $\phi \in X(S, \lesssim)$, then $\phi|_T \in X(T, \lesssim|_T)$.

Th. $X(S, \lesssim) \to X(T, \lesssim|_T) : \phi \mapsto \phi|_T$ is surjective.

Ex next week: $\{\text{oblique tensors}\} \hookrightarrow \{\text{tensors}\}$

Topology. Let $X = X(S, \leq)$

For $a \in S$ let $\hat{a} : X \longrightarrow \mathbb{R}_{\geq 0} : \phi \mapsto \phi(a)$

Endow $\mathbb{R}_{\geq 0}$ with Eucl. topol.

Endow X with coarsest topology making every $\hat{a}$ continuous.

Th X is compact and Hausdorff

Ex
$$\begin{aligned} X(\mathbb{N}[\langle 2,2,2\rangle], \leq) &\overset{\widehat{\langle 2,2,2\rangle}}{\hookrightarrow} \mathbb{R}_{\geq 0} \\ \phi &\mapsto \phi(\langle 2,2,2\rangle) \end{aligned}$$

The image is compact i.e closed and bounded. Strassen proves it is in fact a single closed interval $[4, 2^\omega]$.

Appendix A Proof of the spectral theorem

Let $\mathcal{P}$ be the set of Strassen preorders on S, ordered by inclusion: $\lesssim_1 \subseteq \lesssim_2$ if $\forall a,b \in S \; a \lesssim_1 b \Rightarrow a \lesssim_2 b$.

Lemma If $\lesssim \in \mathcal{P}$ then $\underset{\sim}{\lesssim} \in \mathcal{P}$

Lemma Let $\lesssim \in \mathcal{P}$. Let $a_1, a_2, b \in S$.

i. if $a_2 + b \lesssim a_1 + b$ then $a_2 \underset{\sim}{\lesssim} a_1$
ii. if $a_2 b \lesssim a_1 b$ and $b \neq 0$ then $a_2 \underset{\sim}{\lesssim} a_1$
iii. if $a_2 \underset{\approx}{\lesssim} a_1$ then $a_2 \underset{\sim}{\lesssim} a_1$
iv. if $\exists s \in S \; \forall n \in \mathbb{N} \; n a_2 \lesssim n a_1 + s$ then $a_2 \lesssim a_1$

Lemma Let $\lesssim \in \mathcal{P}$ with $\lesssim = \underset{\sim}{\lesssim}$ and $a_2 \not\lesssim a_1$. Then there is an element $\lesssim \subseteq \lesssim_{a_1 a_2} \in \mathcal{P}$ with $a_1 \lesssim_{a_1 a_2} a_2$.

Proof sketch For $x_1, x_2 \in S$ let

$$x_1 \lesssim_{a_1 a_2} x_2 \quad \text{if} \quad \exists s \in S \quad x_1 + s a_2 \lesssim x_2 + s a_1$$

Then

- $\lesssim_{a_1 a_2}$ is a preorder satisfying (ii), (iii) of Strassen preorder definition.
- if $\lesssim_{a_1 a_2}$ does not satisfy (i) then $a_2 \lesssim a_1$. □

Lemma Let $\preccurlyeq$ maximal in $\mathcal{P}$. Then $\preccurlyeq = \underset{\sim}{\preccurlyeq}$.

Proof $\preccurlyeq \subseteq \underset{\sim}{\preccurlyeq} \in \mathcal{P}$, so $\preccurlyeq = \underset{\sim}{\preccurlyeq}$.

Lemma Let $\preccurlyeq$ maximal in $\mathcal{P}$. Then $\preccurlyeq$ is total.

Proof Suppose not and $a_1 \not\preccurlyeq a_2$ and $a_2 \not\preccurlyeq a_1$. There is an element $\preccurlyeq \subsetneq \preccurlyeq_{a_1 a_2} \in \mathcal{P}$ with $a_1 \preccurlyeq_{a_1 a_2} a_2$, contrad. max. $\square$

Lemma Let $\preccurlyeq \in \mathcal{P}$ maximal. Then there is a semiring homomorphism $\phi : S \to \mathbb{R}_{\geq 0}$ with $a \preccurlyeq b \iff \phi(a) \leq \phi(b)$.

Proof sketch For $a \in S$ define

$$\phi(a) := \inf \{ r/s : r, s \in \mathbb{N},\ sa \preccurlyeq r \}$$

$$\psi(a) := \sup \{ u/v : u, v \in \mathbb{N},\ u \preccurlyeq va \}$$

We use $\preccurlyeq$ is total to show

- $\phi(a) = \psi(a)$
- ϕ is subadd and submult
- ψ is superadd and supermult
- ϕ is $\preccurlyeq$-monotone
- $\phi(1) = 1$, $\phi(0) = 0$

We use $\preccurlyeq = \underset{\sim}{\preccurlyeq}$ to show $a \not\preccurlyeq b \Rightarrow \phi(a) \not\leq \phi(b)$. Let $a \not\preccurlyeq b$. Then $\exists n$ $na \not\preccurlyeq nb+1$. By totality, $na \succcurlyeq nb+1$. Apply ϕ to get $\phi(a) \geqslant \phi(b) + \frac{1}{n}$. Then $\phi(a) \not\leq \phi(b)$. $\square$

__Proof of spectral theorem__ Let $a, b \in S$. Suppose that $a \lesssim b$. Then for all $\phi \in X$ we have $\phi(a) \leq \phi(b)$. Suppose $a \not\lesssim b$. Let $n \in \mathbb{N}_{\geq 1}$ with $na \not\lesssim nb+1$. There is an element $\lesssim \subseteq \lesssim_{nb+1,na} \in \mathcal{P}$ with $nb+1 \lesssim_{nb+1,na} na$ and we may assume it is maximal (Zorn's lemma). Then $a \not\lesssim_{nb+1,na} b$ so for the corresponding $\phi \in X(S, \leq)$ holds $\phi(a) > \phi(b)$. $\square$

Appendix B To prove $\lim_{n\to\infty} R(a^n)^{1/n} = \inf_n R(a^n)^{1/n}$ and $\lim_{n\to\infty} Q(a^n)^{1/n} = \sup_n Q(a^n)^{1/n}$ one uses:

Fekete's lemma Let $x_1, x_2, x_3, \ldots \in \mathbb{R}_{\geq 0}$ satisfy $x_{n+m} \leq x_n + x_m$. Then $\lim_{n\to\infty} x_n/n = \inf_n x_n/n$.

Proof. Let $y = \inf_n x_n/n$. Let $\varepsilon > 0$. Let $m \in \mathbb{N}_{\geq 0}$ with $x_m/m < y+\varepsilon$. Any $n \in \mathbb{N}$ can be written in the form $n = qm+r$ where r is an integer $0 \leq r \leq m-1$. Set $x_0 = 0$. Then

$$x_n = x_{qm+r} \leq \overbrace{x_m + x_m + \cdots + x_m}^{q} + x_r = q x_m + x_r$$

Therefore,

$$\frac{x_n}{n} = \frac{x_{qm+r}}{qm+r} \leq \frac{q x_m + x_r}{qm+r} = \frac{x_m}{m}\frac{qm}{qm+r} + \frac{x_r}{n}$$

Thus

$$y \leq \frac{x_n}{n} < (y+\varepsilon)\frac{qm}{n} + \frac{x_r}{n}$$

The claim follows because $x_r/n \to 0$ and $qm/n \to 1$ when $n \to \infty$. □