Proportionality in Multiwinner Voting with Weighted Seats

Julian Chingoma

LILAC Seminar

j.z.chingoma@uva.nl

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Institute of Logic, Language and Computation (ILLC)
Complex Domains

- **Multiwinner Voting**: A job panel must produce a shortlist of $k$ candidates to continue to the next interview stage.

- **Participatory Budgeting**: Citizens must decide on the public projects, each coming with a cost, that are to be implemented by the local municipality, subject to a budget.

  We look *another* such complex domain.
Talk Outline

- Standard Multwinner Voting (MWV) Model
- Proportionality in MWV.
- MWV with Weighted Seats.
Candidates $C = \{a, b, c, \ldots\}$.

Agents $N = \{1, \ldots, n\}$.

Each agent submits an approval ballot $A_i \subseteq C$.

Outcome is a committee $W \subseteq C$ of size $k$. 
Proportionality in MWV

**Definition (ℓ-cohesiveness)**

For an integer $\ell \in \{1, \ldots, k\}$, a group of agents $N' \subseteq N$ is $\ell$-cohesive if $|N'| \geq n \cdot \frac{\ell}{k}$ and $|\bigcap_{i \in N'} A_i| \geq \ell$.

**Example**

- Candidates $C = \{a, b, c, d\}$ with $k = 3$.
- Agents $N = \{1, 2, 3\}$.
- Approval ballots are $A_1 = \{a, b\}$, $A_2 = \{a, b, c\}$ and $A_3 = \{c, d\}$.
- $\{1, 2\}$ is 2-cohesive.
- $\{2, 3\}$ and $\{3\}$ are 1-cohesive.
Proportionality in MWV

**Natural axiom:** if a group is \( \ell \)-cohesive then \( \ell \) of their common candidates should be elected to the committee.

**Definition (Strong Justified Representation (SJR))**

A committee \( W \) provides SJR if for every \( \ell \)-cohesive group \( N' \), it holds that 
\[
|W \cap \bigcap_{i \in N'} A_i| \geq \ell.
\]

However, this requirement is too strong, even when \( \ell = 1 \).

**Example**

- Candidates \( C = \{a, b, c, d\} \) with \( k = 3 \).
- Agents \( N = \{1, \ldots, 9\} \).
- Suppose 2 agents approve \( \{a\} \), another 2 agents approve \( \{d\} \), and 1 agent each approves of \( \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\} \).
- Each candidate \( c \in \{a, b, c, d\} \) must be elected to provide SJR.
Proportionality in MWV

A weaker axiom: if a group is $\ell$-cohesive then at least one group member should be represented by $\ell$ committee members.

**Definition (Extended Justified Representation (EJR))**

A committee $W$ provides EJR if for every $\ell$-cohesive group $N'$, there exists an agent $i \in N'$ such that $|W \cap A_i| \geq \ell$. 
Multiwinner Voting with Weighted Seats

Joint work with Ulle Endriss, Ronald de Haan, Adrian Haret and Jan Maly.
MWV with Weighted Seats

Example
Each seat represents a role and some roles are more valuable than others.
- The committee has 5 seats with the following roles:
  (chair, treasurer, secretary, member, member).

Example
Each seat has an associated budget that is available for the seat’s elected candidate to spend.
- The committee has 5 seats with the following budgets:
  ($3278, $1400, $560, $100, $4).
Model

🌟 Candidates $C = \{a, b, c, \ldots\}$.
🌟 Agents $N = \{1, \ldots, n\}$.
🌟 Each agent submits an approval ballot $A_i \subseteq C$.
🌟 A weight vector $\mathbf{w} = (w_1, \ldots, w_k)$ with a weight for each of the $k$ seats.
🌟 $W$ is the sum of all the weights.
🌟 Outcome is a committee $\mathbf{c} = (c_1, \ldots, c_k)$.
🌟 For any set of candidates $A \subseteq C$, the satisfaction from a committee $\mathbf{c}$ is 
$$\text{sat}(A, \mathbf{c}) = \sum_{j=1}^{k} \mathbb{1}_{c_j \in A} \cdot w_j.$$
Proportionality

For weight vector $\mathbf{w}$, the set of all possible satisfaction values is $\text{SAT}(\mathbf{w})$.

Example

If $\mathbf{w} = (5, 3, 1)$, then $\text{SAT}(\mathbf{w}) = \{1, 3, 4, 5, 6, 8, 9\}$.

Definition ($\ell$-WS-cohesiveness)

For an integer $\ell \in \text{SAT}(\mathbf{w})$, a group of agents $N'$ is $\ell$-WS-cohesive if $|N'| \geq n \cdot \frac{\ell}{W}$ and there exists a $C' \subseteq \bigcap_{i \in N'} A_i$ with $|C'| = t$ such that there exists a committee $c$ where $\text{sat}(C', c) \geq \ell$, and $|N'| \geq n \cdot \frac{t}{k}$.

Definition ($\ell$-WSJR)

A committee $c$ provides $\ell$-WSJR if for every $\ell$-WS-cohesive group $N'$, there exists an agent $i \in N'$ such that $\text{sat}(A_i, c) \geq \ell$. 

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Unfortunately, $\ell$-WSJR is not always satisfiable.

**Example**

- Candidates $C = \{a, b, c\}$.
- Agents $N = \{1, 2, 3\}$.
- Weight vector $w = (3, 2, 1)$.
- Approval ballots are $A_1 = \{a\}$, $A_2 = \{b\}$ and $A_3 = \{c\}$.

Also, even if such a committee exists, it is computationally hard to compute it. **What now?** Weaken the axiom.
Weakening $\ell$-WSJR: Part 1

**Intuition:** some cohesive group member is just one ‘swap’ away from the deserved satisfaction?

$I_c(A)$ is the vector of positions within the committee $c$ of candidates in $A$.

**Definition ($\ell$-WSJR-1)**

A committee $c$ provides $\ell$-WSJR-1 if for every $\ell$-WS-cohesive group $N'$, there exists an agent $i \in N'$ and some $j \in I_c(C \setminus A_i)$ such that either (i), we have $w_j + \text{sat}(A_i, c) \geq \ell$ if there exists some candidate $c \in A_i$ with $c \notin c$, or (ii), for some $h \in I_c(A_i)$, it holds that $w_j + \text{sat}(A_i, c) - w_h \geq \ell$.

Can $\ell$-WSJR-1 always be satisfied?
Inspired by the Method of Equal Shares (MES) rule in standard MWV. The rule works in $k$ rounds where agents pay to assign candidates to weights from $\mathbf{w} = (w_1, \ldots, w_k)$:

- In round $r \in \{1, \ldots, k\}$, agents consider assignments to weight $w_r$.
- $b_i(r)$ is agent $i$’s budget to start round $r$, and in round 1, we set $b_i(1) = \frac{W}{n}$.
- In round $r$, we say a pair $(c, w_r)$ is $q$-affordable for some $q \in \mathbb{R}_{\geq 0}$, with $c$ currently unelected, if:
  $$\sum_{i \in N: c \in A_i} \min(q, b_i(r)) \geq w_r.$$  
- If no pair is $q$-affordable then go to the next round, otherwise, for a $q$-affordable pair $(c, w_r)$ for a minimum $q$, assign $c$ to $w_r$ and continue to the next round.
**Good news** in the following *restricted setting*.

**Party-list elections**: An election where for every pair of agents $i, j \in N$, it holds that either $A_i = A_j$, or $A_i \cap A_j = \emptyset$, and for every agent $i$, we have $|A_i| \geq k$.

**Theorem**

*$w$-MES satisfies $\ell$-WSJR-1 on party-list elections.*
Weakening $\ell$-WSJR: Part 2

Use $\text{LowSAT}(w) = (\ell_1, \ell_2, \ldots, \ell_k)$ where $\ell_t = \sum_{j=1}^{t} w_j$.

Example

If $w = (5, 3, 3, 1)$, then $\text{LowSAT}(w) = (1, 4, 7, 12)$.

Definition (Lower $\ell$-WS-cohesiveness)

For an integer $\ell \in \text{LowSAT}(w)$, a group of agents $N'$ is lower $\ell$-WS-cohesive if $|N'| \geq n \cdot \frac{\ell}{W}$ and there exists a $C' \subseteq \bigcap_{i \in N'} A_i$ with $|C'| = t$ such that there exists a committee $c$ where $\text{sat}(C', c) \geq \ell$, and $|N'| \geq n \cdot \frac{t}{k}$.

Definition (Lower $\ell$-WSJR)

A committee $c$ provides lower $\ell$-WSJR if for every lower $\ell$-WS-cohesive group $N'$, there exists an agent $i \in N'$ such that $\text{sat}(A_i, c) \geq \ell$.
Bad news! $w$-MES does not satisfy lower $\ell$-WSJR.

Is lower $\ell$-WSJR is always satisfiable? Yes, use MES as in standard MWV.

- Treat all seats as having weight 1.
- Run MES where each agent $i$ has initial budget $b_i(1) = \frac{k}{n}$ instead of $\frac{W}{n}$.
- When a seat is bought for a candidate $c$, assign $c$ to some weight.
- MES ensures that cohesive groups get the seats that they deserve.
Future Work

⭐ Test more rules.
⭐ Define other fairness notions.
⭐ More axioms for the setting.