Proportionality for Constrained Binary Decisions

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Constrained Binary Decisions

Typically, scenarios where voters make a decision of either **yes** or **no**.

- ⋆ Activities that a group of friends will partake in.
- ⋆ Candidates to be part of a committee.
- ⋆ Public projects to be implemented in an instance of participatory budgeting.

Plus some constraints.

- ⋆ Going to the museum leaves no time to go to the beach.
- ⋆ Cannot hire too many candidates with similar expertise.
- ⋆ Building a park bench leaves no space to build a fountain.

How do we ensure fair outcomes?

Talk Outline

- The Model
- Justified Representation
- Priceability

The Model

- ⋆ Issues $\mathcal{I} = \{1, \ldots, m\}$.
- ⋆ Voters $V = \{v_1, \ldots, v_n\}$.
- ⋆ Each voter $v_i \in V$ submits a ballot $\boldsymbol{b}_i = (b_i^1, \ldots, b_i^m) \in \{0, 1\}^m$.
- ⋆ An outcome is a vector $\boldsymbol{w} = (w_1, \ldots, w_m) \in \{0, 1\}^m$.
- ⋆ A constraint $\mathcal{C}$ is a set of feasible outcomes.
- ⋆ Voter satisfaction $u_i(\boldsymbol{w}) = |\{t \in \mathcal{I} \mid b_i^t = w_t\}|$.

Justified Representation without Constraints

Definition (T-cohesiveness)

A group of voters V' is T-cohesive for a set of issues T if:

- All voters agree on the decisions of all issues in T.
- $|V'| \geqslant |T| \cdot \frac{n}{m}$.

Definition (Extended Justified Representation, EJR)

An outcome $\boldsymbol{w}$ provides EJR if for every T-cohesive group of voters V', there exists a voter $v_i \in V'$ such that:

$$u_i(\boldsymbol{w}) \geqslant |T|.$$

Justified Representation with Constraints

Definition (Feasible deviation)

A group of voters V' has an $(S, \boldsymbol{w})$-deviation if S is non-empty, and:

- These voters agree on all decisions in S.
- Outcome $\boldsymbol{w}$ disagrees with these voters on all issues in S.
- It is feasible to 'flip' outcome $\boldsymbol{w}$'s decisions on all issues in S.

Example

- Constraint $\mathcal{C} = \{(0,0),(0,1)\}$.
- Three voters with $\boldsymbol{b}_1 = (1,0)$, $\boldsymbol{b}_2 = (1,1)$ and $\boldsymbol{b}_3 = (0,1)$.
- Suppose outcome is $\boldsymbol{w} = (0,0)$.
- Voters $\{v_1, v_2\}$ have no deviation.
- Voters $\{v_2, v_3\}$ have a deviation for $S = \{2\}$ to outcome $\boldsymbol{w}' = (0,1)$.

Justified Representation with Constraints

Definition (Constrained EJR, c-EJR)

An outcome $\boldsymbol{w}$ provides c-EJR if for every T-cohesive group of voters V that has an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$, there exists a voter $v_i \in V'$ such that:

$$u_i(\boldsymbol{w}) \geqslant |T|.$$

Example

- Constraint $\mathcal{C} = \{(0, 1), (0, 0)\}$.
- Two voters with $\boldsymbol{b}_1 = (1, 1)$ and $\boldsymbol{b}_2 = (1, 0)$.

Constraint $\mathcal{C}$ has the NFD property if no issue's decision is fixed by the constraint.

Does the situation improve with the NFD property?

c-EJR

With NFD, c-EJR can always be provided when $|\mathcal{I}| \in \{2, 3\}$.

With NFD, c-EJR can always be provided when $|\mathcal{C}| = 2$.

Unfortunately, we can't do better.

Example

- Constraint $\mathcal{C} = \{(0000), (0111), (1111), (1000)\}$.
- Four voters with $\boldsymbol{b}_1 = (0000)$, $\boldsymbol{b}_2 = (0111)$, $\boldsymbol{b}_3 = (1111)$ and $\boldsymbol{b}_4 = (1000)$.

What next? Let us look at a weaker version of EJR.

Weakening of EJR

Definition (EJR Up to One Issue, EJR-1)

An outcome $\boldsymbol{w}$ provides EJR-1 if for every T-cohesive group of voters V', there exists a voter $v_i \in V'$ such that:

$$u_i(\boldsymbol{w}) \geqslant |T| - 1.$$

Is EJR-1 always satisfiable?

Method of Equal Shares

Definition (Method of Equal Shares, MES)

- Each voter has a budget of m.
- Each decision $d \in \{0, 1\}$ on an issue t costs n.
- In every round, compute for every undecided issue t, the minimum value for $\alpha(t, d)$ such that the supporters of decision d on issue t can afford the price n, by each paying $\alpha(t, d)$ or the rest of their funds.
- If, for every pair (t, d), there exists no such value $\alpha(t, d)$, then stop.
- Otherwise, we select the pair (t, d) with a minimal value $\alpha(t, d)$, set decision d on issue t.

MES satisfies EJR-1.

MES works. How about for constraints?

Constrained version of EJR-1

Definition (c-EJR-1)

An outcome $\boldsymbol{w}$ provides *c-EJR-1* if for every T-cohesive group of voters V' that has an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$, there exists a voter $v_i \in V'$ such that:

$$u_i(\boldsymbol{w}) \geqslant |T| - 1.$$

Unfortunately, this is also not always satisfiable.

MES for constraints

Definition (λ-MES)

- Each voter has a budget of m.
- In every round, each decision $d \in \{0, 1\}$ on an issue t costs $\lambda(t, d)$.
- In every round, compute for every undecided issue t, the minimum value for $\alpha(t, d)$ such that the supporters of decision d on issue t could afford the price $\lambda(t, d)$, by each paying $\alpha(t, d)$ or the rest of their funds.
- If there exists no such value $\alpha(t, d)$ for every pair (t, d), then stop.
- Otherwise, we select the pair (t, d) with a minimal value $\alpha(t, d)$, set decision d on issue t, if it is feasible.

Now, what type of constraints to look at?

Restricted class of constraints

Definition (Budget-like constraints)

A constraint $\mathcal{C}$ is *budget-like* if there exists a cost function c on issue-decision pairs such that the following conditions hold for every $\boldsymbol{w} = (w_1, \ldots, w_m) \in \mathcal{C}$:

- $c(t, d) + c(t, 1 - d) = 2n$ for every issue t and decision $d \in \{0, 1\}$.
- $\sum_{w_t \in \boldsymbol{w}} c(t, w_t) \leqslant mn$.
- $\sum_{w_t \in \boldsymbol{w}} c(t, w_t) > mn - 2q$ where $q = \max\{|n - c(t, d)| \mid (t, d) \in \mathcal{I} \times \{0, 1\}\}$.

How does MES do on this class of constraints?

λ-MES and Budget-like constraints

Definition

For budget-like constraints for cost function c, λ_{b}-MES uses prices defined by the cost function c.

Given a constraint $\mathcal{C}$ that is budget-like for some cost function c, then for every outcome $\boldsymbol{w}$ returned by λ_{b}-MES, it holds for every T-cohesive group of voters V' that has an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$, that there exists a voter $v_i \in V'$ such that:

$$u_i(\boldsymbol{w}) \geqslant \frac{n}{n+q} \cdot |T| - 1$$

where $q = \max\{|n - c(t, d)| \mid (t, d) \in \mathcal{I} \times \{0, 1\}\}$.

Not easy to provide justified representation. What else can we do?

Priceability

Definition (Priceability)

Suppose that each voter has a personal budget of m and each issue-decision pair (t, d) has a price $\pi(t, d)$.
A price system $(\{p_i\}_{v_i \in V}, \{\pi(t, d)\}_{(t,d) \in \mathcal{I} \times \{0,1\}})$ supports an outcome $\boldsymbol{w} = (w_1, \ldots, w_m)$ if all the following hold:

- Voters only pay for they agree with.
- No voter exceeds their budget of m.
- For each (t, w_t), payments by its supporters must equal its price $\pi(t, w_t)$.
- For each $(t, 1 - w_t)$, there are no payments for it.
- There exists no group of voters V' with an $(S, \boldsymbol{w})$-deviation such that V' collectively hold more in funds than the sum of $\max\{\pi(t, w_t), \pi(t, 1 - w_t)\}$ over all $t \in S$.

An outcome is priceable if there exists a price system that supports it.

Priceability

Example

- Constraint $\mathcal{C} = \{(0000), (0111), (1111), (1000)\}$.
- Four voters with $\boldsymbol{b}_1 = (0000)$, $\boldsymbol{b}_2 = (0111)$, $\boldsymbol{b}_3 = (1111)$ and $\boldsymbol{b}_4 = (1000)$.
- Suppose the outcome is $\boldsymbol{w} = (0000)$.
- Priceable with prices being $\pi(1, d) = 4$ for $d \in \{0, 1\}$, and $\pi(t, d) = 1^1/_3$ for $t \in \{2, 3, 4\}$ and $d \in \{0, 1\}$.

For a constraint $\mathcal{C}$ that is budget-like for some cost function c, then every outcome $\boldsymbol{w}$ returned by λ_b-MES is priceable.

Variant of MES

Definition (c-MeCorA)

- Each voter has a budget of m.
- At the start, an arbitrary outcome $\boldsymbol{w}$ is selected and each issue costs 0.
- In every round, a group of voters with an $(S, \boldsymbol{w})$-deviation may 'flip' outcome $\boldsymbol{w}$'s decisions on the issues in S (must lead to a feasible outcome). But to do so, they must spend their funds to raise the price of every issue in S (by at least ϵ).
- If no such group exists, the rule stops.
- Otherwise, 'flip' the decisions for the group of voters where each voter pays the least (as in MES).

c-MeCorA always returns priceable outcomes.

Future Work

- Study other EJR weakenings like PJR.
- Adapt more rules such as PAV or Sequential Phragmén.
- Stable Pricebility.

Thanks!