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Connected and/or topological group pd-examples ☆

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The pinning down number $\mathrm{pd}(X)$ of a topological space X is the smallest cardinal κ such that for every neighborhood assignment $\mathscr{U}$ on X there is a set of size κ that meets every member of $\mathscr{U}$. Clearly, $\mathrm{pd}(X) \le \mathrm{d}(X)$ and we call X a pd-example if $\mathrm{pd}(X) < \mathrm{d}(X)$. We denote by $\mathbf{S}$ the class of all singular cardinals that are not strong limit. It was proved in [6] that TFAE:

(1) $\mathbf{S} \ne \emptyset$;
(2) there is a 0-dimensional T_2 pd-example;
(3) there is a T_2 pd-example.

The aim of this paper is to produce pd-examples with further interesting topological properties like connectivity or being a topological group by presenting several constructions that transform given pd-examples into ones with these additional properties.

We show that $\mathbf{S} \ne \emptyset$ is also equivalent to the existence of a connected and locally connected T_3 pd-example, as well as to the existence of an abelian T_2 topological group pd-example.

However, $\mathbf{S} \ne \emptyset$ in itself is not sufficient to imply the existence of a connected $T_{3.5}$ pd-example. But if there is $\mu \in \mathbf{S}$ with $\mu \ge \mathfrak{c}$ then there is an abelian T_2 topological group (hence $T_{3.5}$) pd-example which is also arcwise connected and locally arcwise connected. Finally, the same assumption $\mathbf{S} \setminus \mathfrak{c} \ne \emptyset$ even implies that there is a locally convex topological vector space pd-example.

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1. Introduction

The *pinning down number* $\mathrm{pd}(\mathscr{A})$ of a family of sets $\mathscr{A}$ is defined to be the smallest cardinality of a set that intersects every non-empty member of $\mathscr{A}$. For a topological space X, the *pinning down number of* X, abbreviated $\mathrm{pd}(X)$, is defined as follows:

$$\mathrm{pd}(X) = \sup\left\{\mathrm{pd}(\mathscr{U}) : \mathscr{U} = \{U_x : x \in X\} \text{ is a neighborhood assignment on } X\right\}.$$

Clearly, $\mathrm{c}(X) \le \mathrm{pd}(X) \le \mathrm{d}(X)$. Here $\mathrm{c}(X)$ and $\mathrm{d}(X)$ denote the cellularity and density of X, respectively.

The cardinal function $\mathrm{pd}(X)$ has been introduced recently, under different names, by Aurichi and Bella in [2] and independently by Banakh and Ravsky in [3]. The latter showed among other things that if $|X| < \aleph_\omega$, then $\mathrm{pd}(X) = \mathrm{d}(X)$, [3, Theorem 5.2].

The *Weak Generalized Continuum Hypothesis* (abbreviated WGCH) is the statement that 2^κ is a finite successor of κ for any cardinal κ, i.e. in symbols: $(\forall\,\kappa)(2^\kappa < \kappa^{+\omega})$. Clearly, WGCH is equivalent to the statement that every singular cardinal is strong limit. Answering some problems raised in [3], the following was proved recently in Juhász, Soukup and Szentmiklóssy [6, Theorem 1.2]:

Theorem 1.1. *The following statements are equivalent:*

(1) WGCH*;*
(2) $\mathrm{d}(X) = \mathrm{pd}(X)$ *for every* T_2 *space* X*;*
(3) $\mathrm{d}(X) = \mathrm{pd}(X)$ *for every 0-dimensional* T_2 *space* X*.*

Let us call the topological space X a *pd-example* if $\mathrm{pd}(X) < \mathrm{d}(X)$ and denote by $\mathbf{S}$ the class of all singular cardinals that are not strong limit. Thus Theorem 1.1 says that (0-dimensional) T_2 pd-examples exist iff $\mathbf{S} \neq \emptyset$. The aim of this paper is to examine what is needed to obtain T_2, T_3, or $T_{3.5}$ pd-examples with the additional properties of connectivity and/or homogeneity. Here is our first main result:

Theorem 1.2. *TFAE with the negation of the statements in Theorem 1.1, in particular with* $\mathbf{S} \neq \emptyset$*:*

(I) *There is a connected and locally connected* T_3 *pd-example;*
(II) *there is an abelian* T_2 *topological group pd-example.*

It is conspicuous that this result does not provide us with a connected $T_{3.5}$ pd-example. In fact, we shall show in Corollary 3.6 that under suitable assumptions on cardinal arithmetic we have $\mathbf{S} \neq \emptyset$ but no connected $T_{3.5}$ pd-example exists. However, our second main theorem says that if there is $\mu \in \mathbf{S}$ with $\mu \ge \mathfrak{c}$ then there is a connected $T_{3.5}$ pd-example and, in fact, much more is true.

Theorem 1.3. *If there is* $\mu \in \mathbf{S}$ *with* $\mu \ge \mathfrak{c}$ *then there is an abelian* T_2 *topological group pd-example which is also arcwise connected and locally arcwise connected.*

Of course, any locally convex topological $\mathbb{R}$-vector space is an abelian T_2 topological group which is arcwise connected and locally arcwise connected. Thus our third main result is actually a strengthening of Theorem 1.3. Still, we decided to present the two results separately because their proofs are based on two very different constructions that are both interesting in their own rights.

Theorem 1.4. *If there is* $\mu \in \mathbf{S}$ *with* $\mu \ge \mathfrak{c}$ *then there is a pd-example which is a locally convex topological* $\mathbb{R}$*-vector space.*

On the other hand, we shall show in 2.10 that if *Shelah's strong hypothesis* (in short: SSH) holds then already the existence of a connected $T_{3.5}$ pd-example implies that there is $\mu \in \mathbf{S}$ with $\mu \geq \mathfrak{c}$. Consequently, under SSH the converses of both 1.3 and 1.4 are also valid.

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2. Preliminary observations and definitions

If X is any space then $\tau(X)$ denotes its topology and $\tau^+(X) = \tau(X) \setminus \{\emptyset\}$. We also put $\triangle(X) = \min\{|G| : G \in \tau^+(X)\}$ and call X *neat* if $\triangle(X) = |X|$.

First in this section we collect some facts that will be important later when we calculate the pinning down numbers of various spaces.

Lemma 2.1. *Let X be a space and let $\mathscr{U} \subseteq \tau^+(X)$ with $|\mathscr{U}| \leq \triangle(X)$. Then there is a neighborhood assignment on X that contains $\mathscr{U}$ in its range, hence* $\operatorname{pd}(\mathscr{U}) \leq \operatorname{pd}(X)$.

Proof. Since every nonempty open subset of X has size at least $\triangle(X)$ and $|\mathscr{U}| \leq \triangle(X)$, by a trivial transfinite recursion, it is possible to pick for every $U \in \mathscr{U}$ a point $x_U \in U$ such that for distinct $U, V \in \mathscr{U}$ the points x_U and x_V are different. Put $A = \{x_U : U \in \mathscr{U}\}$. Then $x_U \mapsto U$ is a 'partial' neighborhood assignment on A which can be extended to the required global neighborhood assignment on X. $\square$

Corollary 2.2. *Let X be a neat space with* $\operatorname{pd}(X) \geq \omega$ *and* $\kappa \leq \operatorname{pd}(X)$ *be a cardinal such that* $|X^\kappa| = |X|$. *Then* $\operatorname{pd}(X^\kappa) = \operatorname{pd}(X)$.

Proof. Since X is a continuous image of X^κ, we clearly have $\operatorname{pd}(X) \leq \operatorname{pd}(X^\kappa)$. To prove the converse inequality, let $U\colon X^\kappa \to \tau(X^\kappa)$ be any neighborhood assignment. We may clearly assume that each $U(y)$ belongs to the standard base for the product topology on X^κ. This means that for every $y \in X^\kappa$ there are a finite set of indices $F(y) \subseteq \kappa$ and for all $\alpha \in F(y)$ open sets $U^y_\alpha \in \tau(X)^+$ such that

$$U(y) = \{z \in X^\kappa : (\forall\, \alpha \in F(y))(z_\alpha \in U^y_\alpha)\}.$$

Now put

$$\mathscr{U} = \{U^y_\alpha : y \in X^\kappa, \alpha < \kappa\}.$$

Then $|\mathscr{U}| \leq |X|^\kappa = |X| = \Delta(X)$, hence by Lemma 2.1 we have $\operatorname{pd}(\mathscr{U}) \leq \operatorname{pd}(X)$. Let A be a subset of X of size $\operatorname{pd}(X)$ that meets every member of $\mathscr{U}$. Fix $p \in X$ and put

$$\sigma = \{y \in X^\kappa : (\exists\, F \in [\kappa]^{<\omega})(\forall \alpha \in F)(y_\alpha \in A)\,\&\,(\forall\, \alpha \notin F)(y_\alpha = p)\}.$$

Then $|\sigma| = \left|[A]^{<\omega}\right| = |A| = \operatorname{pd}(X)$ and for every $y \in X$ we have $\sigma \cap U(y) \neq \emptyset$. Hence we are done. $\square$

The following obvious result will play an essential role in our constructions.

Corollary 2.3. *For every neat T_2 space X and for every $0 < n < \omega$ we have* $\operatorname{pd}(X^n) = \operatorname{pd}(X)$.

But can the assumption of neatness of X be dropped here? Clearly yes if $\operatorname{pd}(X) = \operatorname{d}(X)$, hence a counterexample is a T_2 pd-example. And in fact, the existence of a T_2 pd-example, that we know is equivalent to $\mathbf{S} \neq \emptyset$, does yield a counterexample.

Example 2.4. If $\mathbf{S} \neq \emptyset$ then there is a 0-dimensional T_2 space X such that $\mathrm{pd}(X^2) > \mathrm{pd}(X)$.

Proof. If $\mu \in \mathbf{S}$ then there is a cardinal λ satisfying $\mathrm{cf}(\mu) \leq \lambda < \mu$ and $2^\lambda > \mu$. The construction theorem [6, Theorem 3.3], in fact a simplified version of it, then yields a 0-dimensional T_2 space Y such that $\mathrm{pd}(Y) \leq \lambda < \mathrm{d}(Y) = \mu$ and $w(Y) \leq 2^\lambda$.

Let Z denote the Cantor cube of weight 2^λ. We claim that $\mathrm{pd}(Y \times Z) = \mu$. Then the topological sum $X = Y \oplus Z$ is the space we are looking for. Indeed, $\mathrm{pd}(Y) \leq \lambda$ and $\mathrm{pd}(Z) = \mathrm{d}(Z) \leq \lambda$ obviously imply $\mathrm{pd}(X) \leq \lambda$, while $\mathrm{pd}(X^2) \geq \mu = \mathrm{pd}(Y \times Z)$ holds because $Y \times Z$ is an open subspace of X^2.

Now, let $\mathscr{B}$ be a base of Y of size $\leq 2^\lambda$ and consider the collection

$$\mathscr{V} = \{B \times Z : B \in \mathscr{B}\} \subset \tau^+(Y \times Z).$$

Then $|\mathscr{V}| = |\mathscr{B}| \leq 2^\lambda$ and we have $\Delta(Y \times Z) = \Delta(Z) = 2^\lambda$, hence Lemma 2.1 implies $\mathrm{pd}(Y \times Z) \geq \mathrm{pd}(\mathscr{V})$. But if $A \subset Y \times Z$ and $|A| < \mu$ then the projection P of A to Y is not dense in Y, hence there is a $B \in \mathscr{B}$ with $B \cap P = \emptyset$. Consequently, $(B \times Z) \cap A = \emptyset$ as well, hence we have $\mathrm{pd}(\mathscr{V}) \geq \mu$. $\square$

Note that any neat T_2 space that is not a singleton is infinite, consequently if X is such a space then $\mathrm{d}(X^\omega) = \mathrm{d}(X)$. On the other hand, our next example shows that $\mathrm{pd}(X) < \mathrm{pd}(X^\omega)$ may hold for a neat T_2 space X. This also shows that the assumption $|X^\kappa| = |X|$ cannot be dropped from Lemma 5.4 either. Again, this space must be a T_2 pd-example.

Example 2.5. It is consistent that there is a neat 0-dimensional T_2 space X such that $\mathrm{pd}(X) < \mathrm{pd}(X^\omega)$.

Proof. By [6, Theorem 3.3] again, it is consistent that there is a neat 0-dimensional T_2 pd-example X such that $\aleph_\omega = |X| < w(X) \leq \mathfrak{c}$. But then X^ω is not a pd-example because $w(X^\omega) = w(X) \leq \mathfrak{c} = \Delta(X^\omega)$, hence $\mathrm{pd}(X^\omega) = \mathrm{d}(X^\omega) = \mathrm{d}(X) > \mathrm{pd}(X)$. $\square$

It is obvious that, for any space X, if Y is dense open in X then $\mathrm{d}(X) = \mathrm{d}(Y)$. It is natural to ask if this also holds with pd instead of d. Of course, if $X \supset Y$ form a counterexample to this then Y must be a pd-example. Even though it will not be used in our later constructions, we present below such an example that is by no means trivial.

Example 2.6. It is consistent that there is a $T_{3.5}$ space X with a dense open subspace Y such that $\mathrm{pd}(Y) < \mathrm{pd}(X)$.

Proof. By [6, Theorem 3.3], it is consistent that there is a 0-dimensional neat T_2 space Z such that $\mathrm{pd}(Z) = \omega < \mathrm{d}(Z)$, and by [6, Theorem 4.9] we have $|Z| = \triangle(Z) < 2^{\mathrm{pd}(Z)} = \mathfrak{c}$. Hence Z has at most $2^\mathfrak{c}$ many cozero-sets. Let $\mathscr{B}$ denote the collection of all nonempty cozero-sets in Z.

We claim that we may assume without loss of generality that Z is nowhere separable. For let $\mathscr{A}$ denote a maximal pairwise disjoint collection of nonempty open separable subsets of Z. Then since $c(Z) \leq \mathrm{pd}(Z) = \omega$, $\mathscr{A}$ is countable. Hence any nonempty open subset V of Z that misses $\bigcup \mathscr{A}$ is nowhere separable and clearly has the property that $\mathrm{pd}(V) = \omega$.

Consider the space $Y = \omega \times Z$, its Čech-Stone compactification βY, and put

$$X = Y \cup \{p \in \beta Y : p \notin \bigcup_{n<\omega} \overline{\{n\} \times Z}\}.$$

(Here closures are taken in βY.) Then each $\{n\} \times Z$ is clopen in X, hence Y is dense open in X. Clearly, $\mathrm{pd}(Y) = \omega$ and we claim that $\mathrm{pd}(X) > \omega$.

Striving for a contradiction, assume that $\mathrm{pd}(X) = \omega$. For every $S \in [\omega]^\omega$ and $f \in \mathscr{B}^S$, let $V(f)$ be the largest open subset of βY such that $V(f) \cap Y = \bigcup_{n \in S} \{n\} \times f(n)$. We claim that $V(f) \cap X$ has size at least $2^{\mathfrak{c}}$. Indeed, for every $n \in S$ let Z_n be a nonempty zero-set in Z that is contained in $f(n)$. Then $\bigcup_{n \in S} \{n\} \times Z_n$ is a zero-set in Y that is disjoint from the zero-set $Y \setminus V(f)$. Hence these two zero-sets have disjoint closures in βY. For every $n \in S$, pick $p_n \in Z_n$. Then $V(f)$ contains the closure of the discrete and closed subset $P = \{(n, p_n) : n \in S\}$ of Y. Observe that $\overline{P}$ is homeomorphic to $\beta\omega$ and has therefore size $2^{\mathfrak{c}}$. Finally, $\overline{P} \setminus P$ is contained in X and so indeed $V(f) \cap X$ has size at least $2^{\mathfrak{c}}$.

By a simple transfinite induction we can consequently pick points $z(f) \in V(f) \cap X$ such that for all $S, T \in [\omega]^\omega$, $f \in \mathscr{B}^S$ and $g \in \mathscr{B}^T$, if $V(f) \cap X \neq V(g) \cap X$, then $z(f) \neq z(g)$. Since $\mathrm{pd}(X) = \omega$, there is a countable subset A of X which meets every $V(f) \cap X$.

Next observe that $X \setminus Y$ is the Čech-Stone remainder of the noncompact, locally compact and σ-compact space $\bigcup_{n \in \omega} \overline{\{n\} \times Z}$. Hence, as is well-known, the density of $X \setminus Y$ is uncountable (it follows from the existence of an uncountable almost disjoint family of infinite subsets of ω that even the cellularity of $X \setminus Y$ is uncountable, [11, p. 121]). A moments reflection shows that there is an infinite subset S of ω and an element $f \in \mathscr{B}^S$ such that $(V(f) \cap X) \cap (A \cap (X \setminus Y)) = \emptyset$. Since Z is nowhere separable, for every $n \in S$ we may pick an element $B_n \in \mathscr{B}$ such that $B_n \subseteq f(n) \setminus A$. Let $g \in \mathscr{B}^S$ denote the function $n \mapsto B_n$. Then $(V(g) \cap X) \cap A = \emptyset$, which is a contradiction. □

We end this section with a few simple but useful results.

Lemma 2.7. *Let $\mathscr{A}$ be any cover of the space X. Then* $\mathrm{pd}(X) \leq \sum\{\mathrm{pd}(A) : A \in \mathscr{A}\}$.

Proof. Let $U\colon X \to \tau(X)$ be a neighborhood assignment on X. Then, for every $A \in \mathscr{A}$, the function $V_A\colon A \to \tau(A)$ defined by $V_A(a) = U(a) \cap A$ is a neighborhood assignment on A, hence $\mathrm{pd}(\{U(a) : a \in A\}) \leq \mathrm{pd}(A)$. The rest is obvious. □

It is immediate from Lemma 2.7 that $\mathrm{pd}(X \times Y) \leq |Y| \cdot \mathrm{pd}(X)$ holds for any product $X \times Y$, hence $\mathrm{pd}(X \times Y) = \mathrm{pd}(X)$ if $|Y| \leq \mathrm{pd}(X)$. The following result yields a similar implication in which $\mathrm{d}(Y) \leq \mathrm{pd}(X)$ is sufficient instead of $|Y| \leq \mathrm{pd}(X)$.

Lemma 2.8. *Let X and Y be spaces such that* $\mathrm{d}(Y) \leq \mathrm{pd}(X)$ *and* $|Y| \leq \Delta(X) = |X|$. *Then* $\mathrm{pd}(X \times Y) = \mathrm{pd}(X)$.

Proof. Let $(x, y) \mapsto U_{(x,y)} \times V_{(x,y)}$ be a neighborhood assignment on $X \times Y$ such that $U_{(x,y)}$ (resp. $V_{(x,y)}$) is an open neighborhood of x in X (resp. y in Y). Consider the collection $\mathscr{U} = \{U_{(x,y)} : x \in X, y \in Y\}$. Then, by our assumptions, $|\mathscr{U}| \leq |X| = \triangle(X)$, hence by Lemma 2.1 there is a set $B \in [X]^{\leq \mathrm{pd}(X)}$ that meets every member of $\mathscr{U}$. Let $D \subseteq Y$ be dense such that $|D| \leq \mathrm{pd}(X)$. Then $(B \times D) \cap (U_{(x,y)} \times V_{(x,y)}) \neq \emptyset$ for all $(x, y) \in X \times Y$ and $|B \times D| \leq \mathrm{pd}(X)$, hence we conclude $\mathrm{pd}(X \times Y) \leq \mathrm{pd}(X)$. But $\mathrm{pd}(X \times Y) \geq \mathrm{pd}(X)$ holds because X is the continuous image of $X \times Y$. □

The following proposition is just Lemma 2.2 from [6]. It is added here because it will be used quite frequently.

Proposition 2.9 *([6, Lemma 2.2]). Every pd-example X has a neat open subspace Y that is also a pd-example.*

We may combine this proposition with some other results of [6] to obtain the result that we promised at the end of section 1.

Corollary 2.10. *Under SSH the existence of a connected $T_{3.5}$ pd-example implies that there is $\mu \in \mathbf{S}$ with $\mu \geq \mathfrak{c}$.*

Proof. Assume that SSH holds and X is a connected $T_{3.5}$ pd-example. It is well-known that then $\Delta(X) \geq \mathfrak{c}$. By Proposition 2.9 there is a neat open subspace $Y \subseteq X$ that is also a pd-example. Then clearly we have

$$|Y| = \Delta(Y) \geq \Delta(X) \geq \mathfrak{c}.$$

By [6, Theorem 3.5] and the remark made after it SSH implies that every neat pd-example has singular cardinality. In particular, then $|Y|$ must be singular. Since Y is a T_2 pd-example, then [6, Theorem 3.1] implies that $|Y|$ cannot be strong limit, hence we have both $|Y| \in \mathbf{S}$ and $|Y| \geq \mathfrak{c}$. $\square$

A couple of remarks are of order here. First, note that in Corollary 3.6 below we formulate some simple cardinal arithmetic conditions that imply $\mathbf{S} \neq \emptyset$ but $\mu < \mathfrak{c}$ for all $\mu \in \mathbf{S}$. Secondly, we note that large cardinals, hence going beyond ZFC, are needed to refute SSH.

3. Proof of Theorem 1.2 part (I): connected pd-examples

In [5], de Groot associated to every T_1 space X a certain extension λX which he called the *superextension* of X. We will briefly describe his construction.

A system of sets $\mathscr{L}$ is called a *linked system* if any two of its members meet. A *maximal linked system* (or *mls*) on X is a system of closed subsets of X which is maximal with respect to being linked. We denote the collection of all mls's on X by λX. For any $A \subseteq X$ we write

$$A^+ = \{\mathscr{M} \in \lambda X : (\exists\, M \in \mathscr{M})(M \subseteq A)\}.$$

We then take the collection

$$\{A^+ : A \text{ is closed in } X\}$$

as a *closed* subbase for the topology of λX. With this topology, λX is a (super)compact T_1-space that contains X as a subspace, hence λX is an extension of X. In fact, the function $i\colon X \to \lambda X$ defined by

$$i(x) = \{x\}^+ = \{A \subseteq X : A \text{ is closed and } x \in A\}$$

is an embedding of X into λX. We identify each point $x \in X$ with the mls $i(x)$. The closure of X in λX is the familiar *Wallman compactification* of X.

(V0) ([10, II.1.6)]) The collection $\{U^+ : U \in \tau(X)\}$ is a subbase for the open subsets of λX.

A *defining set* for $\mathscr{M} \in \lambda X$ is a subset S of X with the following property: for every $M \in \mathscr{M}$ there exists $M' \in \mathscr{M}$ such that $M' \subseteq M \cap S$. An mls $\mathscr{M} \in \lambda X$ is called *finitely generated* (or an *fmls*) if it has a finite defining set. The subspace of λX consisting of all fmls's is denoted by $\lambda_f(X)$.

Clearly, $i(x)$ is an fmls whenever $x \in X$, having $\{x\}$ as defining set. Thus we have $X \subseteq \lambda_f(X)$.

We shall need the following results that were proved for $\lambda_f(X)$ by Verbeek [10].

(V1) ([10, IV.3.4(iii)]) If X is T_2 then X is closed in $\lambda_f(X)$.
(V2) ([10, IV.3.4(v)+(vi)]) If X is T_2 then so is $\lambda_f(X)$. Similarly for $T_{3.5}$.
(V3) ([10, IV.3.4(viii)]) If X is connected then $\lambda_f(X)$ is both connected and locally connected.
(V4) ([10, III.2.5(b)]) $\lambda_f(X)$ can be represented as the countable union of subspaces each of which is a continuous image of some finite power of X.
(V5) ([10, III.4.3(iv)]) $\mathrm{d}(X) = \mathrm{d}(\lambda_f(X))$.

We shall need the fact that if X is T_3 then so is $\lambda_f(X)$. This is not stated explicitly in Verbeek [10], hence we provide a (simple) proof.

Lemma 3.1. *If X is T_3 then so is $\lambda_f(X)$.*

Proof. Let $\mathscr{M} \in \lambda_f(X)$, and let $G \subseteq \lambda_f(X)$ be open such that $\mathscr{M} \in G$. In addition, let S be a finite defining set for $\mathscr{M}$. Since G is open, by (V0) there is a finite collection $\mathscr{U}$ of open subsets of X such that $\mathscr{M} \in \bigcap_{U \in \mathscr{U}} U^+ \subseteq G$. For every $U \in \mathscr{U}$, there exists $F(U) \subseteq U \cap S$ such that $\mathscr{M} \in \bigcap_{U \in \mathscr{U}} F(U)^+$. For every $U \in \mathscr{U}$ let $V(U)$ be an open subset of X such that $F(U) \subseteq V(U) \subseteq \overline{V(U)} \subseteq U$. (Here we use that X is T_3.) Then $\bigcap_{U \in \mathscr{U}} \overline{V(U)}^+$ is a closed neighborhood of $\mathscr{M}$ in $\lambda_f(X)$ that is contained G. □

Lemma 3.2. *For any T_2 space X we have* $\mathrm{pd}(X) \leq \mathrm{pd}(\lambda_f(X))$. *If, in addition, X is neat then* $\mathrm{pd}(X) = \mathrm{pd}(\lambda_f(X))$.

Proof. We first prove that $\mathrm{pd}(X) \leq \mathrm{pd}(\lambda_f(X))$. This is trivial if X is finite because then λX is discrete, so we assume that X is infinite. Note that then $\lambda_f(X)$ and hence $\mathrm{pd}(\lambda_f(X))$ are also infinite. Let $U\colon X \to \tau(X)$ be any neighborhood assignment on X. Define $V\colon X \to \tau(\lambda_f(X))$ by $V(x) = U(x)^+$. Extend V to a neighborhood assignment W on $\lambda_f(X)$ by putting $W(x) = V(x)$ for $x \in X$ and $W(\mathscr{M}) = \lambda_f(X) \setminus X$ for $\mathscr{M} \in \lambda_f(X) \setminus X$. Then there is a subset A of $\lambda_f(X)$ that meets every element of the collection $\{V(x) : x \in X\}$ and has size $\mathrm{pd}(\lambda_f(X))$. For every $\mathscr{M} \in A$ let $F(\mathscr{M})$ be a finite defining set for $\mathscr{M}$. Put

$$B = \bigcup \{F(\mathscr{M}) : \mathscr{M} \in A\}.$$

Then $|B| \leq \omega \cdot |A| = |A|$, and we claim that B meets $U(x)$ for each $x \in X$. Indeed, take any $\mathscr{M} \in A \cap U(x)^+$. This means that there is a subset G of $F(\mathscr{M})$ such that $G \in \mathscr{M}$ and $G \subseteq U(x)$. Hence $\emptyset \neq G \subseteq U(x) \cap B$, and this completes the proof.

Now assume that X is also neat. First observe that then $\mathrm{pd}(X^n) = \mathrm{pd}(X)$ for every $n < \omega$ by Corollary 2.3. Hence we get $\mathrm{pd}(\lambda_f(X)) \leq \mathrm{pd}(X)$ applying (V4) and Lemma 5.1. □

We now describe a very general version of the well-known 'cone' construction that can be used to obtain connectifications of spaces in a natural way. The input of the construction consists of an arbitrary topological space X and an infinite connected T_1 space P with a distinguished point $p \in P$. The output is the space $Z = Z(X, P, p)$ whose underlying set is

$$\big(X \times (P \setminus \{p\})\big) \cup \{p\},$$

where, of course, $p \notin X \times (P \setminus \{p\})$. The topology of Z is defined as follows: Basic neighborhoods of points of $X \times (P \setminus \{p\})$ in Z are just the standard product neighborhoods. A basic neighborhood of p in Z has the form

$$\big(X \times (U \setminus \{p\})\big) \cup \{p\},$$

where U is any neighborhood of p in P. It is easy to check that this is indeed a topology. It is also straightforward to show for every point $q \in P \setminus \{p\}$ that $X \times \{q\}$ is a closed homeomorphic copy of X in Z and that for every $x \in X$ the subspace $(\{x\} \times P \setminus \{p\}) \cup \{p\}$ is a homeomorphic copy of P in Z, hence Z is connected. We leave it to the reader to check these facts as well as the following proposition.

Proposition 3.3. *If both X and P are T_2 (or T_3, or $T_{3.5}$) then so is $Z(X, P, p)$.*

The following lemma provides us with a procedure that transforms any T_2 (resp. T_3) pd-example into a connected and locally connected T_2 (resp. T_3) pd-example. This, of course, will complete the proof of part (I) of Theorem 1.2.

Lemma 3.4.

(1) *Let X be an infinite neat T_2 space and P be a countably infinite connected T_2-space. Then for any $p \in P$, the T_2 space $Y = \lambda_f(Z(X, P, p))$ is connected and locally connected, moreover* $\mathrm{d}(X) = \mathrm{d}(Y)$ *and* $\mathrm{pd}(X) = \mathrm{pd}(Y)$.

(2) *If X is an infinite neat T_3 space and P is a separable connected T_3-space of cardinality ω_1 then for any $p \in P$ the T_3 space $Y = \lambda_f(Z(X, P, p))$ is connected and locally connected, moreover* $\mathrm{d}(X) = \mathrm{d}(Y)$ *and* $\mathrm{pd}(X) = \mathrm{pd}(Y)$.

Proof. For (1), fix a countably infinite connected T_2-space P and a point $p \in P$. (The existence of such a space P was first proved by Urysohn [9].) Then $Z = Z(X, P, p)$ is connected, T_2 and, since P is countable, we clearly have $\mathrm{d}(X) = \mathrm{d}(X \times P) = \mathrm{d}(Z)$.

Next we show that $\mathrm{pd}(Z) = \mathrm{pd}(X)$. It is obvious that $\mathrm{pd}(Z) = \mathrm{pd}(X \times (P \setminus \{p\}))$. But by Lemma 2.8 and because X a continuous image of $X \times (P \setminus \{p\})$, we have $\mathrm{pd}(X) = \mathrm{pd}(X \times (P \setminus \{p\}))$. Since X is neat and P is countable it is obvious that Z is neat as well.

The superextension $Y = \lambda_f(Z)$ is a connected and locally connected T_2 space by (V1) and (V3). Moreover, $\mathrm{d}(\lambda_f(Z)) = \mathrm{d}(Z) = \mathrm{d}(X)$ by (V5). Since Z is neat, we get by the above and by Lemma 3.2 that $\mathrm{pd}(X) = \mathrm{pd}(Z) = \mathrm{pd}(\lambda_f(Z))$, hence we are done.

For (2) we have to do a little more work. It is well-known that there is a connected T_3 space of size ω_1. This is the smallest such cardinality possible since every T_3 countable space is 0-dimensional. An example is Hewitt's Condensed Corkscrew and there are others. For details, see Steen and Seebach [8, p. 111].

However, for our construction we need a *separable* connected T_3 space P of size ω_1. Luckily for us, it was recently proved by Ciesielski and Wojciechowsk [4, Theorem 8] that there is such a space. Observe that then P is neat. For let us assume that it contains a countable nonempty open subset U. Then it contains by regularity a nonempty open subset V such that $\overline{V} \subseteq U$. Hence V is a countable regular space, hence normal, and so it is 0-dimensional. But then V contains a nonempty subset C which is clopen in $\overline{V}$. But this C would be clopen in P. This argument actually yields that for every non-singleton connected T_3 space X we have $\Delta(X) \geq \omega_1$.

Now, as in (1), consider $Z = Z(X, P, p)$ for some $p \in P$. Since both X and P are neat, it is not difficult to check that Z is neat as well. Since P is separable, we obviously have $\mathrm{d}(X) = \mathrm{d}(X \times P) = \mathrm{d}(Z)$.

We next show that $\mathrm{pd}(Z) = \mathrm{pd}(X)$. If X is countable then $X \times P$ is separable, hence so is Z. But then $\mathrm{pd}(Z) = \mathrm{pd}(X) = \omega$. If $|X| \geq \omega_1$ then, since $|P| = \omega_1$ and P is separable, we may apply Lemma 2.8 to obtain $\mathrm{pd}(X \times (P \backslash \{p\})) = \mathrm{pd}(X)$. But we also have $\mathrm{pd}(Z) = \mathrm{pd}(X \times (P \backslash \{p\}))$, hence we are done.

Now, the space $Y = \lambda_f(Z)$ is as we want by following the same argumentation as in part (1). We, of course, have to use that $\lambda_f(Z)$ is T_3 by Lemma 3.1. □

This completes the proof of part (I) of Theorem 1.2. The following results explain why T_3 cannot be replaced in it with $T_{3.5}$.

Theorem 3.5. *Assume that* WGCH *holds from the cardinal κ on, i.e. $2^\lambda < \lambda^{+\omega}$ for all $\lambda \geq \kappa$. Then for every T_2 pd-example X we have $\Delta(X) < \kappa^{+\omega}$. If, in addition,* GCH *holds in the interval $[\kappa, \kappa^{+\omega})$, i.e. $2^\lambda = \lambda^+$ for all λ with $\kappa \leq \lambda < \kappa^{+\omega}$, and X is T_3 then we even have $\Delta(X) \leq \kappa$.*

Proof. Assume, on the contrary, that $\Delta(X) \geq \kappa^{+\omega}$. We may also assume, without any loss of generality, that X is neat because, by Proposition 2.9, we can replace X with its open subspace Y that is a neat pd-example and, of course, satisfies $\Delta(Y) \geq \Delta(X)$.

But then $|X| = \Delta(X) = \mu^{+n}$, where $\mu \geq \kappa^{+\omega}$ is a limit and hence a strong limit cardinal. This, however, contradicts [6, Theorem 3.1] which says that in this case $\mathrm{pd}(X) = \mathrm{d}(X)$.

To see the second part, assume that X is any T_3 pd-example. Again we may also assume, without any loss of generality, that X is neat, hence, by the first part, we have

$$\mathrm{pd}(X) < \mathrm{d}(X) \leq |X| = \Delta(X) < \kappa^{+\omega}.$$

By [6, Theorem 4.8] $\mathrm{d}(X) < 2^{\mathrm{pd}(X)}$ holds for every T_3 space X. Thus $\mathrm{pd}(X) \geq \kappa$ and our cardinal arithmetic assumptions would imply $\mathrm{d}(X) < 2^{\mathrm{pd}(X)} = \mathrm{pd}(X)^+$, a contradiction. Hence we actually have $\mathrm{pd}(X) < \kappa$. But then we also have $2^{\mathrm{pd}}(X) \leq 2^\kappa = \kappa^+$, while by [6, Theorem 4.9] any T_3 pd-example X also satisfies $\Delta(X) < 2^{\mathrm{pd}}(X)$. Hence we indeed have $\Delta(X) \leq \kappa$. $\square$

Since for every non-singleton connected $T_{3.5}$ space X we have $\Delta(X) \geq \mathfrak{c}$, we immediately obtain the following.

Corollary 3.6. *If $\mathfrak{c} = \kappa^+ > \aleph_\omega$ then $\mathbf{S} \neq \emptyset$, in fact $\aleph_\omega \in \mathbf{S}$. But if, in addition,* WGCH *holds from κ on and* GCH *holds in the interval $[\kappa, \kappa^{+\omega})$, then there is no connected $T_{3.5}$ pd-example.*

Let us remark that the assumptions of Corollary 3.6 are satisfied in a generic extension obtained by adding κ^+ Cohen reals to a model of GCH for any $\kappa \geq \aleph_\omega$.

If X is any $T_{3.5}$ space with $|X| = \Delta(X) \geq \big|[0,1]\big| = \mathfrak{c}$, then clearly the ordinary cone $Z = Z(X,[0,1],1)$ is a connected $T_{3.5}$ space that is neat and, by Lemma 2.8 satisfies both $\mathrm{pd}(Z) = \mathrm{pd}(X)$ and $\mathrm{d}(Z) = \mathrm{d}(X)$. Consequently, then $Y = \lambda_f(Z)$ is a connected and locally connected $T_{3.5}$ pd-example if X is a pd-example. Now, the existence of such an X follows from $\mathbf{S} \setminus \mathfrak{c} \neq \emptyset$ (see the first paragraph in the proof of Example 2.4). However, we shall see later that this same assumption gives us much stronger $T_{3.5}$ pd-examples.

4. Proof of Theorem 1.2 part (II): topological group pd-examples

The idea of the proof of part (II) of Theorem 1.2 is very simple: We show that if X is a neat $T_{3.5}$ pd-example then $A(X)$, the free abelian topological group on X is a pd-example as well. Now let us see the details.

Free topological groups: If X is a $T_{3.5}$ space, then $F(X)$ and $A(X)$ denote the free topological group and the free abelian topological group on X. That is, $F(X)$ is a topological group containing (a homeomorphic copy of) X such that

(1) X generates $F(X)$ algebraically,
(2) every continuous function $f\colon X \to H$, where H is any topological group, can be extended to a continuous homomorphism $\bar{f}\colon F(X) \to H$.

Similarly for $A(X)$. The existence of these groups was proved by Markov [7]. See Arhangel'skii and Tkachenko [1, Chapter 7] for details and references. It is known that

(FG1) ([1, Theorem 7.1.13]) X is closed in $F(X)$ as well as $A(X)$,
(FG2) ([1, Theorem 7.1.5]) $F(X)$ and $A(X)$ are $T_{3.5}$ (being T_2 topological groups),

(FG3) ([1, Theorem 7.1.13]) $F(X)$ as well as $A(X)$ can be represented as a countable union of subspaces each of which is a continuous image of some finite power of X.

The first part of the following crucial result is probably well-known.

Proposition 4.1. *Let X be infinite and $T_{3.5}$. Then $\mathrm{d}(X) = \mathrm{d}(F(X)) = \mathrm{d}(A(X))$ and if X is neat, then $\mathrm{pd}(X) = \mathrm{pd}(F(X)) = \mathrm{pd}(A(X))$.*

Proof. We will only check this for $F(X)$, the proof for $A(X)$ is similar. That $\mathrm{d}(F(X)) \leq \mathrm{d}(X)$ is a direct consequence of (FG3). Now let D be dense in $F(X)$. Every element of $D \setminus \{e\}$, where e is the neutral element of $F(X)$, can be written uniquely in the form

$$d = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \tag{†}$$

where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for every $i = 1, 2, \ldots, n-1$. Let E be the set of all points $x \in X$ that appear in the expressions (†). We claim that E is dense in X. Indeed, assume that this is not true, and fix a continuous function $f\colon X \to \mathbb{R}$ such that $f(\overline{E}) \subseteq \{0\}$ and $f(x) = 1$, for some $x \in X \setminus \overline{E}$. We can extend f to a continuous homomorphism $\bar{f}\colon F(X) \to \mathbb{R}$. Then $\bar{f}(D) \subseteq \{0\}$, but $\bar{f}$ is not constant. This is clearly a contradiction. Hence $\mathrm{d}(X) \leq \mathrm{d}(F(X))$.

Now assume that X is neat. Then $\mathrm{pd}(X^n) = \mathrm{pd}(X)$ for every $n < \omega$ (Corollary 2.3). Hence by (FG3) and Lemma 2.7 we get $\mathrm{pd}(F(X)) \leq \mathrm{pd}(X)$. For the converse inequality, let $U\colon X \to \tau^+(X)$ be any neighborhood assignment on X. For every $x \in X$, let $f_x\colon X \to [0,1]$ be continuous such that $f_x(x) = 1$ and $f_x(X \setminus U(X)) \subseteq \{0\}$. Consider the continuous homomorphism $\bar{f}_x\colon F(X) \to \mathbb{R}$ that extends f. Put $V(x) = \bar{f}_x^{-1}((0,2))$. Then $V\colon X \to \tau^+(F(X))$ is a neighborhood assignment, which can be extended to a full neighborhood assignment by simply putting $V(y) = F(X)$ for every $y \in F(X) \setminus X$. Now let B be a subset of $F(X)$ of size at most $\mathrm{pd}(F(X))$ meeting every set of the form $V(x)$, for $x \in X$. We can write every $b \in B \setminus \{e\}$ uniquely in the form

$$b = y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}, \tag{‡}$$

where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $y_i \in X$, and $y_i \neq y_{i+1}$ for every $i = 1, 2, \ldots, n-1$. Let F be the set of all y's that appear in the expressions (‡). Then, clearly, $|F| \leq \mathrm{pd}(F(X))$. We claim that $F \cap U(x) \neq \emptyset$ for every $x \in X$. Indeed, striving for a contradiction, assume that $F \cap U(x) = \emptyset$ for some $x \in X$. Then $f_x(F) \subseteq \{0\}$, and so $\bar{f}_x(B \setminus \{e\}) \subseteq \{0\}$. But this is a contradiction, since $B \setminus \{e\}$ meets $V(x) = \bar{f}_x^{-1}((0,2))$. So we conclude that $\mathrm{pd}(X) \leq \mathrm{pd}(F(X))$, as desired. □

To prove part (II) of Theorem 1.2, we simply have to recall that, by Theorem 1.2 and Proposition 2.9, if $\mathbf{S} \neq \emptyset$ then there is a neat 0-dimensional T_2, hence $T_{3.5}$ pd-example. Then we may apply Proposition 4.1 to obtain from X the T_2 abelian topological group pd-example $A(X)$.

5. Proof of Theorem 1.3: connected topological group pd-examples

We have seen in the previous section that if X is a neat $T_{3.5}$ pd-example then the free (abelian) topological group over X is a pd-example as well. In fact, what we showed was that this construction preserves the values of both $\mathrm{pd}(X)$ and $\mathrm{d}(X)$. To obtain a proof of Theorem 1.3, we compose this construction with a procedure due to Hartman and Mycielski that, in turn embeds any (abelian) T_2 topological group G in a larger (abelian) T_2 topological group $G^\bullet$ that is pathwise connected and locally pathwise connected. Under certain conditions this procedure also preserves the values of both $\mathrm{pd}(G)$ and $\mathrm{d}(G)$ and, for an appropriate neat $T_{3.5}$ pd-example X, then $A(X)^\bullet$ is a pathwise connected and locally pathwise connected abelian T_2 topological group pd-example.

We now describe the Hartman-Mycielski construction, following its presentation in Arhangel'skii and Tkachenko [1, 3.8.1], and then prove a few new results about it that will be needed. All topological groups we consider are assumed to be T_2.

Let G be a topological group with neutral element e and with group operation written multiplicatively. $G^\bullet$ is defined to be the set of all step functions $f : J = [0, 1) \to G$ such that, for some sequence $0 = a_0 < a_1 < \cdots < a_n = 1$, the function f is constant on $[a_k, a_{k+1})$ for every $k = 0, \dots, n-1$. Define a binary operation $*$ on $G^\bullet$ by $(f * g)(x) = f(x) \cdot g(x)$ for all $f, g \in G^\bullet$ and $x \in G$. Then every $f \in G^\bullet$ has a unique inverse in $G^\bullet$, defined by $f^{-1}(x) = (f(x))^{-1}$. Then $(G^\bullet, *)$ is a group with identity $e^\bullet$, where $e^\bullet(r) = e$ for each $r \in J$. It is also easy to see that G can be identified with a subgroup of $G^\bullet$ via $x \mapsto x^\bullet$, where $x^\bullet(r) = x$ for every $r \in J$.

Let V be a neighborhood of e in G, and for every $\varepsilon > 0$, put

$$O(V, \varepsilon) = \{f \in G^\bullet : \mu(\{r \in J : f(r) \notin V\}) < \varepsilon\},$$

here μ denotes Lebesgue measure. The $O(V, \varepsilon)$ are the neighborhoods of the neutral element $e^\bullet$ of $G^\bullet$ that generate its group topology. The following facts are known:

(HM1) ([1, 3.8.2]) $G^\bullet$ is a topological group and is pathwise connected and locally pathwise connected.
(HM2) ([1, 3.8.3]) The function $i_G : G \to G^\bullet$ defined by $i_G(x) = x^\bullet$ is a topological isomorphism of G onto a closed subgroup of $G^\bullet$.
(HM3) ([1, 3.8.8(e)]) $\mathrm{d}(G^\bullet) \le \mathrm{d}(G)$.

We next prove that in (HM3) one actually has equality.

Lemma 5.1. $\mathrm{d}(G) \le \mathrm{d}(G^\bullet)$.

Proof. Let D be dense in $G^\bullet$, and put $E = \bigcup_{f \in D} f(J)$. Since the range of every element of D is finite, $|E| = |D|$. We claim that E is dense in G. If not, let V be an open neighborhood of e and let $x \in G$ be such that $xV \cap E = \emptyset$. The nonempty open subset $x^\bullet * O(V, \frac{1}{2})$ of $G^\bullet$ meets D, say in the point f. Let $g \in O(V, \frac{1}{2})$ be such that $f = x^\bullet * g$. Clearly, $\mathrm{range}(g) \cap V \ne \emptyset$, say $g(t) \in V$. Then $f(t) \in E$, but also $f(t) = x^\bullet(t)g(t) = xg(t) \in xV \subseteq G \setminus E$. This is a contradiction. □

Corollary 5.2. $\mathrm{d}(G) = \mathrm{d}(G^\bullet)$.

To obtain conditions under which we also have $\mathrm{pd}(G) = \mathrm{pd}(G^\bullet)$ we first make a little detour.

Lemma 5.3. *If the infinite $T_{3.5}$ space X is neat then so are both $A(X)$ and $F(X)$.*

Proof. We prove this for $F(X)$, the proof for $A(X)$ being entirely similar. First observe that by (FG3) from section 4, we have $|F(X)| = |X|$. Let U be any open subset of $F(X)$ containing the neutral element of $F(X)$. Fix $x \in X$, and observe that xU contains x, hence $xU \cap X$ is a nonempty open subset of X. This gives us that $|U| \le |F(X)| = |X|$ on one hand and $|X| = |xU \cap X| \le |U|$ on the other, hence we are done. □

The following is our crucial result concerning the Hartman-Mycielski construction.

Lemma 5.4. *If G is neat and $|G| \ge \mathfrak{c}$ then $G^\bullet$ is neat and* $\mathrm{pd}(G^\bullet) = \mathrm{pd}(G)$.

Proof. That $G^\bullet$ is neat follows by the same argument as in the proof of Lemma 5.3. The assumption $|G| \geq \mathfrak{c}$ ensures that we have $|G| = |G^\bullet|$.

Let $U\colon G \to \tau(G)$ be a neighborhood assignment on G. For every $x \in G$ we have then a neighborhood V_x of the neutral element e of G such that $xV_x \subseteq U(x)$. Observe that $V\colon G \to \tau(G^\bullet)$ defined by $V(x) = x^\bullet * O(V_x, \frac{1}{2})$ is a partial neighborhood assignment on $G^\bullet$ (note that we identify x with $x^\bullet$). Let A be a subset of $G^\bullet$ of size $\mathrm{pd}(G^\bullet)$ such that $A \cap V(x) \neq \emptyset$ for every $x \in G$ and put $B = \bigcup_{f\in A} f(J)$. Then $|B| \leq |A|$ because A is infinite and $f(J)$ is finite for all $f \in A$. Now take an arbitrary point $x \in G$ and let $f \in A$ be such that $f \in V(x) = x^\bullet * O(V_x, \frac{1}{2})$. There is $g \in O(V_x, \frac{1}{2})$ such that $f = x^\bullet * g$. Clearly, then $\mathrm{range}(g) \cap V_x \neq \emptyset$, say $g(t) \in V_x$. But then $f(t) = x(t)g(t) \in xV_x \subseteq U(x)$, hence B meets $U(x)$. This proves that $\mathrm{pd}(G) \leq \mathrm{pd}(G^\bullet)$. Note that this part only used that G is infinite. In the proof of the reverse inequality, however, the assumption $|G| \geq \mathfrak{c}$ that ensures $|G| = |G^\bullet|$ will play an essential role.

Let $U\colon G^\bullet \to \tau(G^\bullet)$ be a neighborhood assignment on $G^\bullet$. For every $f \in G^\bullet$ we may then take a neighborhood V_f of e in G and an $\varepsilon_f > 0$ such that $f * O(V_f, \varepsilon_f) \subseteq U(f)$. Consider the collection

$$\mathscr{V} = \{xV_f : f \in G^\bullet, x \in G\}$$

of open subsets of X which, using $|G| \geq \mathfrak{c}$, has size at most $|G|$. Since G is neat, $\mathscr{V}$ can be pinned down by an infinite set $D \subset G$ of size at most $\mathrm{pd}(G)$. Now let S be the set of all g in $G^\bullet$ for which there exist for some n rational numbers $0 = b_0 < b_1 < \cdots < b_{n-1} < b_n = 1$ and elements $d_0, \ldots, d_{n-1} \in D$ such that g takes the constant value d_k on $[b_k, b_{k+1})$ for every $k = 0, \ldots, n-1$. Clearly, we have then $|S| = |D|$.

For any fixed $f \in G^\bullet$ there exist numbers $0 = a_0 < a_1 < \ldots < a_n = 1$ such that the function f is constant x_k on $[a_k, a_{k+1})$ for each $0 \leq k < n$. We may then choose rational numbers $b_1, \ldots, b_{n-1}$ such that $a_k \leq b_k < a_{k+1}$ for each $1 \leq k < n$ and $\sum_{k=1}^{n-1}(b_k - a_k) < \varepsilon_f$. Put $b_0 = 0$ and $b_n = 1$. For every $0 \leq k < n$, we can choose a point $y_k \in D \cap f(a_k)V_f$, and then define an element $g \in S$ by letting $g(r) = y_k$ for each $r \in [b_k, b_{k+1})$, $0 \leq k < n$. We claim that then $g \in f * O(V_f, \varepsilon_f)$. To see this, it suffices to prove that the function $h\colon J \to G$ defined by $h(t) = f(t)^{-1}g(t)$ belongs to $O(V_f, \varepsilon_f)$, but this is clear from the easily checked fact that $\{r \in J : h(r) \notin V_f\} \subset \bigcup_{k=1}^{n-1}[a_k, b_k)$. Thus we have shown that S meets $f * O(V_f, \varepsilon_f) \subset U(f)$ for all $f \in G^\bullet$, proving that $\mathrm{pd}(G^\bullet) \leq \mathrm{pd}(G)$. $\square$

Corollary 5.5. *Let X be a neat $T_{3.5}$ space such that $|X| \geq \mathfrak{c}$. Then X admits a closed embedding into a $T_{3.5}$ topological group G such that*

(1) $\mathrm{d}(X) = \mathrm{d}(G)$,
(2) $\mathrm{pd}(X) = \mathrm{pd}(G)$,
(3) *G is neat,*
(4) *G is pathwise connected and locally pathwise connected.*

In particular, if X is a neat $T_{3.5}$ pd-example of size $\geq \mathfrak{c}$ then $A(X)^\bullet$ is a pathwise connected and locally pathwise connected abelian topological group pd-example.

Now, [6, Theorem 3.3] implies that if there is $\mu \in \mathbf{S}$ with $\mu \geq \mathfrak{c}$ then there is a neat 0-dimensional T_2, hence $T_{3.5}$ pd-example of size $\geq \mathfrak{c}$, completing the proof of Theorem 1.3.

6. Proof of Theorem 1.4

For every infinite $T_{3.5}$ space X one can define the free locally convex $\mathbb{R}$-vector space $L(X)$ on X. This is a space with similar properties as the free groups that we considered in section 4. The space $L(X)$ contains X as a closed subspace and at the same time X forms an $\mathbb{R}$-vector space basis for $L(X)$. Moreover, the following defining property holds: every continuous mapping f from X to a locally convex $\mathbb{R}$-vector space E can be extended to a continuous linear operator $\bar{f}\colon L(X) \to E$. The existence and uniqueness of $L(X)$ was proved by Markov in [7].

We can treat $L(X)$ in almost the same way as we treated the free topological groups $F(X)$ and $A(X)$. There is one important difference however: the statement (FG3) should be replaced by the following.

(FLC3) $L(X)$ can be represented as the countable union of subspaces each of which is a continuous image of some finite power of $X \times \mathbb{R}$.

So, to use Lemma 2.8 to conclude $\mathrm{pd}(X) = \mathrm{pd}(X \times \mathbb{R})$ we need $|X| = \Delta(X) \geq \mathfrak{c}$.

Proposition 6.1. *Let X be an infinite $T_{3.5}$ space. Then $\mathrm{d}(X) = \mathrm{d}(L(X))$ and if, in addition, X is neat and $|X| \geq \mathfrak{c}$ then $\mathrm{pd}(X) = \mathrm{pd}(L(X))$.*

Proof. That $\mathrm{d}(L(X)) \leq \mathrm{d}(X)$ is a direct consequence of (FLC3). The proof that $\mathrm{d}(X) \leq \mathrm{d}(L(X))$ is completely analogous to the proof that $\mathrm{d}(X) \leq \mathrm{d}(F(X))$ in Proposition 4.1. Now assume that X is neat and that $|X| \geq \mathfrak{c}$. Then $\mathrm{pd}(X \times \mathbb{R}) = \mathrm{pd}(X)$ by Lemma 2.8. Since $X \times \mathbb{R}$ is clearly neat, we have by Corollary 2.3 that $\mathrm{pd}((X \times \mathbb{R})^n) = \mathrm{pd}(X \times \mathbb{R}) = \mathrm{pd}(X)$ for every $0 < n < \omega$. Hence again by (FLC3) we get $\mathrm{pd}(L(X)) \leq \mathrm{pd}(X)$. That $\mathrm{pd}(X) \leq \mathrm{pd}(L(X))$ follows exactly as in the proof of Proposition 4.1 for $F(X)$. □

Hence if there is a $T_{3.5}$ pd-example X such that $\mathrm{pd}(X) \geq \mathfrak{c}$ then $L(X)$ is a locally convex $\mathbb{R}$-vector space that is also a pd-example. Such a space X exists by the construction given in [6, Theorem 3.3], provided that there is a cardinal $\mu \in \mathbf{S}$ with $\mu \geq \mathfrak{c}$. The proof of Theorem 1.4 is thus completed.

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