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Splitting Tychonoff cubes into homeomorphic and homogeneous parts

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ABSTRACT

Let τ be an infinite cardinal. We prove that $\mathbb{I}$ and the Tychonoff cube $\mathbb{I}^\tau$ can be split into two homeomorphic and homogeneous parts. If τ is uncountable, such a partition cannot consist of spaces homeomorphic to topological groups.

1. Introduction

It is known that the real line $\mathbb{R}$ can be partitioned into two homeomorphic and homogeneous parts, [10]. Although it is not mentioned in [10], this was an answer to a question posed by the late Maarten Maurice. Since then, various similar results were obtained. Shelah [14] and, independently, van Engelen [6], showed that $\mathbb{R}$ can be partitioned into two homeomorphic rigid parts. Here a space is called *rigid* if the identity map is its only homeomorphism. See also [7] and [13] for other results in the same spirit.

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It was asked by the second author of the present paper whether the closed unit interval $\mathbb{I} = [0, 1]$ can be partitioned into two homogeneous and homeomorphic parts. In the present paper we give an affirmative answer to this question in Theorem 2.7, and this immediately leads to the following result:

Theorem 1.1. *Let τ be any infinite cardinal. Then the Tychonoff cube $\mathbb{I}^\tau$ can be partitioned into two homogeneous and homeomorphic parts.*

We do not know whether a similar result holds for the finite dimensional cubes $\mathbb{I}^n$, where $1 < n < \omega$. Theorem 1.1 suggests the question whether the homeomorphic parts can actually be chosen to be (homeomorphic to) a topological group. For uncountable τ, the answer is in the negative.

Theorem 1.2. *Let τ be any uncountable cardinal. Then for every subspace A of $\mathbb{I}^\tau$ which is (homeomorphic to) a topological group, we have that $\mathbb{I}^\tau \setminus A$ and A are not homeomorphic.*

2. The closed unit interval can be conveniently split

We begin by reviewing the construction from [10]. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$.

Lemma 2.1. *[10, 2.3] If $X \subseteq \mathbb{R}$ is such that $X = X + \mathbb{Q}$, then X is homogeneous.*

In [10, §3], a subset $A \subseteq \mathbb{R}$ was constructed having the following properties:

(1) A is dense in $\mathbb{R}$, and so is $B = \mathbb{R} \setminus A$,
(2) $\mathbb{Q} \subseteq A$ and $A + \mathbb{Q} = A$ (hence $B + \mathbb{Q} = B$),
(3) the map $\phi\colon \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x+\pi$ sends A onto B.

These sets A and B are fixed here and used everywhere in the remaining part of the paper. Let $\mathbb{D} = \pi+\mathbb{Q}$. Then $\mathbb{D}$ is dense in B, and $\phi(\mathbb{Q}) = \mathbb{D}$. If $s, t \in \mathbb{D}$ and $s < t$, then $[s, t]_A = [s, t] \cap A$ is called a *clopen arc* in A. Moreover, if $p, q \in \mathbb{Q}$ and $p < q$, then $[p, q]_B = [p, q] \cap B$ is called a *clopen arc* in B. Observe that clopen arcs in A respectively B are clopen subsets of A respectively B. If $C = [s, t]_A$ is a clopen arc in A, then $\lambda(C) = t-s$ denotes its length. Observe that $\lambda(C) \in \mathbb{Q}$. If $\mathscr{C}$ is a family of pairwise disjoint clopen arcs in A, then $\lambda(\bigcup \mathscr{C}) = \sum_{C\in\mathscr{C}} \lambda(C)$. Similarly for B.

We use some ideas in [11].

Lemma 2.2. *If C_0 and C_1 are clopen arcs in A such that $\lambda(C_0) = \lambda(C_1)$, then C_0 and C_1 are homeomorphic. Similarly for B. Moreover, if C is a clopen arc in A and D is a clopen arc in B such that $\lambda(C) = \lambda(D)$, then C and D are homeomorphic.*

Proof. Let $C_0 = [r_0, t_0]_A$ and $C_1 = [r_1, t_1]_A$. Define $f\colon C_0 \to C_1$ by $f(t) = (t-r_0) + r_1$. Since $r_1-r_0 \in \mathbb{Q}$ and $A + \mathbb{Q} = A$, it easily follows that f is a homeomorphism. Similarly for B.

Assume that $C = [r, t]_A$ and $D = [p_1, q_1]_B$. Let $r = \pi + p_0$ and $t = \pi + q_0$. Then ϕ^{-1} sends C homeomorphically onto the clopen arc $[p_0, q_0]_B$ of B. By the above, $[p_0, q_0]_B$ and $[p_1, q_1]_B$ are homeomorphic, hence we are done. $\square$

Lemma 2.3. *Let $\mathscr{C}$ be a family of pairwise disjoint clopen arcs in A such that $\varepsilon = \lambda(\bigcup \mathscr{C}) \in \mathbb{Q}$. Then $\bigcup \mathscr{C}$ is homeomorphic to the clopen arc $[\pi, \pi + \varepsilon]_A$. Similarly, let $\mathscr{D}$ be a family of pairwise disjoint clopen arcs in D such that $\delta = \lambda(\bigcup \mathscr{D}) \in \mathbb{Q}$, then $\bigcup \mathscr{D}$ is homeomorphic to the clopen arc $[0, \delta]_B$.*

Proof. We assume that $\mathscr{C}$ is infinite. The proof when $\mathscr{C}$ is finite is entirely similar. Assume that

$$\mathscr{C} = \{[\pi+r_0, \pi+t_0]_A, [\pi+r_1, \pi+t_1]_A, \dots, [\pi+r_n, \pi+t_n]_A, \dots\}.$$

By Lemma 2.2,

$$\begin{aligned} &[\pi+r_0, \pi+t_0]_A \approx [\pi, \pi+(t_0 - r_0)]_A, \\ &[\pi+r_1, \pi+t_1]_A \approx [\pi+(t_0 - r_0), \pi+(t_0 - r_0)+(t_1 - r_1)]_A, \\ &\qquad\vdots \\ &[\pi+r_n, \pi+t_n]_A \approx [\pi+\sum_{j\leq n-1}(t_j - r_j), \pi+\sum_{j\leq n}(t_j - r_j)]_A, \\ &\qquad\vdots \end{aligned}$$

Since all sets involved are clopen, the union of these homeomorphisms gives us that

$$\bigcup \mathscr{C} \approx [\pi, \pi + \sum_{j<\omega}(t_j - r_j)]_A = [\pi, \pi + \varepsilon]_A.$$

The proof for B is entirely similar. $\square$

Corollary 2.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be collections of pairwise disjoint clopen arcs in A respectively B such that $\lambda(\bigcup \mathscr{C}) = \lambda(\bigcup \mathscr{D}) \in \mathbb{Q}$. Then $\bigcup \mathscr{C}$ and $\bigcup \mathscr{D}$ are homeomorphic.*

Proof. Let $\gamma = \lambda(\bigcup \mathscr{C}) = \lambda(\bigcup \mathscr{D})$. By Lemma 2.3,

$$\bigcup \mathscr{C} \approx [\pi, \pi + \lambda]_A, \quad \bigcup \mathscr{D} \approx [0, \lambda]_B.$$

Hence we are done by Lemma 2.2. $\square$

In the proof of the next result, we use the well-known result from Calculus, that for every $t \in \mathbb{I}$ there is a subset A of $\mathbb{N}$ such that $\sum_{n\in A} 2^{-n} = t$. For more on this topic, see Ferdinands [8].

Lemma 2.5. *Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{0\} \cup [0, q]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0, q]_B$.*

Proof. Put $q_0 = q$. For every $n \geq 1$, put $q_n = 2^{-n}q$. Moreover, put $t_0 = q$ and for $n \geq 1$, $t_n = t_{n-1} - q_n$.

Let $x \in B \cap (2, 3)$. Pick $r \in \mathbb{Q}$ such that $r < x < r + q$. Let $F \subseteq \mathbb{N}$ be such that $\sum_{n\in F} q_n = x - r$. Observe that F has to be infinite since x is irrational. Put $G = \mathbb{N} \setminus F$. Then $\sum_{n\in G} q_n = r + q - x$. It also follows that G is infinite.

Put $r_0 = r$. There clearly is a sequence $(r_n)_{n\geq 1}$ of rational numbers in (r, x) such that $(r_n)_n \nearrow x$ while moreover for every $n \geq 1$ we have

$$r_n - r_{n-1} = q_{\mu(n)},$$

where $\mu(n)$ is the n-the element of F (ordered as a subset of $\mathbb{N}$). Put $s_0 = r+q$. There similarly is a sequence $(s_n)_{n\geq 1}$ of rational numbers in $(x, r + q)$ such that $(s_n)_n \searrow x$ while moreover for every $n \geq 1$ we have

$$s_{n-1} - s_n = q_{\nu(n)},$$

where $\nu(n)$ is the n-the element of G (ordered as a subset of $\mathbb{N}$).

Let $\mu(n) \in A$. By Lemma 2.2 we may pick a homeomorphism

$$g_n\colon [t_{\mu(n)}, t_{\mu(n)-1}]_B \to [r_{n-1}, r_n]_B.$$

Similarly, if $\nu(n) \in B$, we may pick a homeomorphism

$$h_n\colon [t_{\nu(n)}, t_{\nu(n)-1}]_B \to [s_n, s_{n-1}]_B.$$

Since all sets involved are clopen, the function $f\colon \{0\} \cup [0, q]_B \to [r, r+q]_B$ defined by

$$f(x) = \begin{cases} g_n(x) & (t_{\mu(n)} < x < t_{\mu(n)-1}), \\ h_n(x) & (t_{\nu(n)} < x < t_{\nu(n)-1}), \\ x & (t = 0), \end{cases}$$

is a homeomorphism. Hence we are done by Lemma 2.2. $\square$

The following can be proved with the same method.

Lemma 2.6. *Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{1\} \cup [1-q, 1]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0, q]_B$.*

We now come to the main result in this section.

Theorem 2.7. *The closed unit interval $\mathbb{I} = [0, 1]$ can be partitioned into two homogeneous and homeomorphic sets.*

Proof. Put $E = (0, 1) \cap A$ and $F = [0, 1]_B = (0, 1) \cap B$, respectively. Observe that E and F are homogeneous being both open subsets of zero-dimensional homogeneous spaces (this is folklore). Also, both E and F are the union of a family of pairwise disjoint clopen arcs in A respectively B such that $\lambda(E) = \lambda(F)$. Hence $E \approx F$ by Corollary 2.4.

Let us now consider the space F, and let $0 < q < 1/2$ be rational. Then by Lemmas 2.5, 2.6 and 2.2 we have that $\{0\} \cup [0, q]_B \approx [0, q]_B$ and $\{1\} \cup [1, 1-q]_B \approx [q, 2q]_B$. Moreover, $[q, 1-q]_B$ is homeomorphic to $[2q, 1]_B$, again by Lemma 2.2. Hence we conclude that $\{0\} \cup F \cup \{1\}$ is homeomorphic to F.

The partition $\{E, F \cup \{0, 1\}\}$ of $\mathbb{I}$ is consequently the one we are after since we already observed that E and $F \approx F \cup \{0, 1\}$ are homeomorphic and homogeneous. $\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Keller's Theorem [9] (see also [12]), $\mathbb{I}^\omega$ is homogeneous. This implies that $\mathbb{I}^\tau$ is homogeneous for every infinite cardinal τ. Let $\{E, F\}$ be the partition of $\mathbb{I}$ into homeomorphic and homogeneous parts from Theorem 2.7. Then $\{E \times \mathbb{I}^\tau, F \times \mathbb{I}^\tau\}$ is the required partition of $\mathbb{I} \times \mathbb{I}^\tau \approx \mathbb{I}^\tau$. $\square$

3. Topological groups

We show here that Theorem 1.1 for uncountable cardinals cannot be improved to the case of a splitting into homeomorphic topological groups. For information on topological groups, see Arhangel'skii and Tkachenko [4].

The following result is well-known, its proof is included for completeness sake.

Lemma 3.1. *Let G be a topological group. If S is a G_δ-subset of G containing the neutral element e of G, then there is a closed subgroup N of G such that*

(1) $N \subseteq S$,
(2) N is a G_δ-subset of G.

Proof. Write S as $\bigcap_{n<\omega} U_n$, where each U_n is open in G and $e \in U_n$. Recursively, pick open symmetric neighborhoods V_n of e such that $V_{n+1}^2 \subseteq V_n \subseteq U_n$ (see [4]), and let $N = \bigcap_{n<\omega} V_n$. □

Theorem 3.2. *If G is a dense subset of $\mathbb{I}^\tau$, where τ is uncountable, such that $\mathbb{I}^\tau \setminus G$ is Lindelöf, then G is not a topological group.*

Proof. Striving for a contradiction, assume that G is a topological group.

We may assume by homogeneity that the element of $\mathbb{I}^\tau$ with constant coordinates 0 is the neutral element e of G. Since $\mathbb{I}^\tau \setminus G$ is Lindelöf, there is a compact G_δ-subset S_0 of $\mathbb{I}^\tau$ such that $e \in S_0 \subseteq G$.

There is a countable subset A_0 of τ such that

$$S_1 = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_0)(x_\alpha = 0)\} \subseteq S_0.$$

By Lemma 3.1, we may pick a closed subgroup N_1 of G which is a G_δ-subset of G such that $N_1 \subseteq S_1$. Clearly, N_1 is a G_δ-subset of S_1 and hence is a compact G_δ-subset of $\mathbb{I}^\tau$. There is a countable subset A_1 of τ such that $A_0 \subseteq A_1$ while moreover

$$S_2 = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_1)(x_\alpha = 0)\} \subseteq N_1.$$

Continuing in this way, it is easy to construct by recursion countable subsets A_n of τ and closed subgroups N_n of G such that for every n,

(1) $A_n \subseteq A_{n+1}$,
(2) $S_{n+1} = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_{n+1})(x_\alpha = 0)\} \subseteq N_n \subseteq S_n$.

Put $A = \bigcup_{n<\omega} A_n$. Then since τ is uncountable,

$$\bigcap_{n<\omega} N_n = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A)(x_\alpha = 0)\} \approx \mathbb{I}^\tau.$$

Hence $\mathbb{I}^\tau$ is a topological group and consequently does not have the fixed-point property (no nontrivial translation has a fixed-point), which contradicts the Brouwer Fixed-Point Theorem. □

We are now in the position to present a proof of Theorem 1.2. We use a factorization result in [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [4, Corollary 1.7.8 (see also Theorem 1.7.7)]. This result is also stated and applied in the book [2, Lemma 0.2.3]. It implies that every continuous realvalued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Therefore, if A is a dense pseudocompact subset of some Tychonoff cube $\mathbb{I}^\tau$, then $\mathbb{I}^\tau$ is the Čech-Stone-compactification βA of A. Indeed, for every continuous function $f\colon A \to \mathbb{R}$ there is by Corollary 1.7.8 in [4], a countable subset L of τ and a continuous function $g\colon \pi_L(A) \to \mathbb{R}$, where $\pi_L\colon \mathbb{I}^\tau \to \mathbb{I}^L$ is the projection, such that $g(\pi_L(a)) = f(a)$ for all $a \in A$. However, since A is pseudocompact, $\pi_L(A) = \mathbb{I}^L$, which evidently implies that f can be extended over $\mathbb{I}^\tau$.

As usual, a space is called *nowhere locally compact* if no point in it has a compact neighborhood.

Proof of Theorem 1.2. Assume the contrary. First observe that A is nowhere locally compact. Indeed, if A would contain a point with a compact neighborhood (that is, A is somewhere locally compact), it would be locally compact at all points by homogeneity and so its complement would be compact implying that A would be compact; this is clearly impossible. This also gives us that A is dense. For if A would not be dense, $\mathbb{I}^\tau \setminus A$ would be somewhere locally compact, and so A would be somewhere locally compact.

The Dichotomy Theorem from [3] implies that $B = \mathbb{I}^\tau \setminus A$ is pseudocompact or Lindelöf. But it cannot be Lindelöf by Theorem 3.2. Hence B is pseudocompact and so A is pseudocompact. Since A is dense in $\mathbb{I}^\tau$, it follows by the above that $\mathbb{I}^\tau = \beta A$.

We complete the proof now in two ways. The first proof is as follows. Since A is a pseudocompact topological group, βA is a topological group by the Comfort-Ross theorem [5]. But $\mathbb{I}^\tau$ is not a topological group (this is the same argument as at the end of the proof of Theorem 3.2).

The second proof is more direct and avoids the use of the complicated Comfort-Ross Theorem.

We have already shown in the first part of the proof of this theorem that $\mathbb{I}^\tau = \beta A$. It follows similarly that $\mathbb{I}^\tau = \beta B$.

Fix a homeomorphism f of A onto B. This homeomorphism f can be extended to a continuous mapping $h\colon \mathbb{I}^\tau \to \mathbb{I}^\tau$.

Claim: This mapping h is a homeomorphism of $\mathbb{I}^\tau$ onto itself.

Indeed, f^{-1} is a homeomorphism of B onto A. Since $\mathbb{I}^\tau = \beta B$, this homeomorphism f^{-1} can be extended to a continuous mapping $g\colon \mathbb{I}^\tau \to \mathbb{I}^\tau$. The composition $g \circ h$ is a continuous mapping of $\mathbb{I}^\tau$ onto itself such that $g(h(a)) = a$, for every $a \in A$. Since A is dense in the cube and $g \circ h$ is continuous, it follows that $g(h(x)) = x$ for each $x \in \mathbb{I}^\tau$. Evidently, this implies that h and g are homeomorphisms and $g = h^{-1}$. The Claim is proved.

So h has no fixed-points, since A and B are disjoint and $h(A) = B$. Hence, the proof can be completed as at the end of the proof of Theorem 3.2. □

In the zero-dimensional case, the case of Cantor cubes instead of Tychonoff cubes, Theorem 1.2 does not hold. Indeed, let κ be an infinite cardinal, and let p be a free ultrafilter on κ. The set

$$A = \{x \in \{0,1\}^\tau : \{\alpha : x_\alpha = 1\} \in p\}$$

is a subgroup of $\{0,1\}^\tau$ of index 2. Hence A as well as its complement are homeomorphic to topological groups.

We do not know whether every compact topological group can be split into two homeomorphic and homogeneous parts.

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