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THE POLISH TOPOLOGY OF ERDŐS SPACE

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Abstract. We show that Erdős space E is Polishable and prove that E with its Polish topology is homeomorphic to complete Erdős space.

1. Introduction

A topological group is *Polishable* if it admits a stronger Polish topology that is compatible with its group structure. If a topological group is Polishable then it is obviously Borel. But this is not sufficient, see Becker and Kechris [1, p. 12]. If a topological group is Polishable, then its Polish topology is unique; see Kechris [8, Theorem 9.10]. Hence the property of being Polishable is an intrinsic property of the topological group we are interested in. The reader can find more information on Polishable groups for example in Solecki [11].

As usual, $\mathbb{Q}$ and $\mathbb{P}$ denote the sets of rationals and irrationals, respectively.

The aim of this note is to show that Erdős space

$$E = \{x \in \ell^2 : (\forall\, n \in \mathbb{N})(x_n \in \mathbb{Q})\}$$

from [6] is Polishable. We will show that E with its Polish topology is homeomorphic to *complete* Erdős space

$$E_c = \{x \in \ell^2 : (\forall\, n \in \mathbb{N})(x_n \in (\{0\} \cup \{{}^1\!/_n : n \in \mathbb{N}\})\},$$

which was also considered in [6]. It is known, see [7] (and [2, 3]), that this space is homeomorphic to $\{x \in \ell^2 : (\forall\, n \in \mathbb{N})(x_n \in \mathbb{P})\}$.

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2. Erdős spaces

All topological spaces unders discussion are separable and metrizable.

Topological characterization theorems for E_c and E were obtained in [7, 3, 4]. These papers also contain various "models" for both E_c and E. Of particular importance here is the main result in [3]:

Theorem 2.1. *A nonempty space E is homeomorphic to E_c if and only if there is a zero-dimensional topology $\mathscr{W}$ on E that is coarser than the given topology on E such that for every $x \in E$ and neighborhood U of x in E there is a neighbourhood V of x in E with V closed in $(E, \mathscr{W})$, $(V, \mathscr{W})$ Polish, and V a nowhere dense subset of $(U, \mathscr{W})$.*

3. The Polishability of E

Let $\mathbb{Q}_d$ stand for the space of the rational numbers endowed with the discrete topology. We do not change the group structure on $\mathbb{Q}_d$. Hence $\mathbb{Q}_d^\infty$ is a Polish group with its standard operations. Observe that it is homeomorphic to the space of all irrational numbers. Let d be a complete metric on $\mathbb{Q}_d^\infty$ generating its topology, and put $G = \mathbb{Q}_d^\infty \times \ell^2$. The formula

$$\varrho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \|y_1 - y_2\| \qquad ((x_1, y_1), (x_2, y_2) \in G)$$

defines an admissible complete metric on G.

Put $E_{\mathrm{s}} = \{x \in \mathbb{Q}_d^\infty : \|x\| < \infty\}$. Hence as a set and as a group, E_{s} is nothing but E. It will be convenient to introduce the following notation. If $x \in E$, then $\hat{x}$ denotes x but considered to be an element of E_{s}. Define $i\colon E \to G$ by

$$i(x) = (\hat{x}, x) \qquad (x \in E).$$

Observe that i is an algebraic imbedding, that $\pi_1(i(E)) = E_{\mathrm{s}}$ and $\pi_2(i(E)) = E$, where π_1 and π_2 are the standard projection maps.

Lemma 3.1. *$i(E)$ is a closed subspace of G and hence is a Polish group.*

Proof. Let $((\hat{x}(n), x(n)))_n$ be a sequence in $i(E)$ converging to an element $(p, q) \in G$.

Claim 1. For every j there exists N such that for all $n \geq N$ we have $x(n)_j = \hat{x}(n)_j = p_j$.

This is clear since $\hat{x}(n)_j \to p_j$ and $\mathbb{Q}_d$ has the discrete topology.

Claim 2. $q \in E$ and $p = \hat{q}$.

Indeed, fix j and let N be as in Claim 1 for j. Observe that $x(n)_j \to q_j$. But for all $n \geq N$ we have $x(n)_j = p_j$ and so $q_j = p_j \in \mathbb{Q}$. □

Hence $i(E)$ is a closed subgroup of the Polish group G. From this we conclude that the formula

$$\varrho(x,y) = d(x,y) + \|x-y\| \qquad (x,y \in E_{\mathrm{s}})$$

defines a complete metric compatible with the topological group structure on E_{s}. Moreover, this topology on E_{s} is clearly stronger than the topology on E, hence E is Polishable.

Let $F \subseteq \mathbb{N}$ be finite, $\varepsilon > 0$, and put

$$V(F,\varepsilon) = \{x \in E_{\mathrm{s}} : (\forall\, n \in F)(x_n = 0)\,\&\,(\|x\| \le \varepsilon)\}.$$

The collection $\mathscr{V}$ of all $V(F,\varepsilon)$'s is a closed neighborhood base at the neutral element $e = (0,0,\dots)$ in E_{s}. Since E_{s} is a topological group, translates of members from $\mathscr{V}$ are closed neighborhood bases at arbitrary points of E_{s}.

Lemma 3.2. *For every $V \in \mathscr{V}$ and $p \in E_{\mathrm{s}}$, $p+V$ is closed in $\mathbb{Q}_d^\infty$. Hence every point x of E_{s} has arbitrarily small closed neighborhoods W such that W is a closed subspace of the zero-dimensional Polish space $\mathbb{Q}_d^\infty$.*

Proof. It clearly suffices to prove the first statement. Fix an arbitrary finite subset F of $\mathbb{N}$ and an arbitrary $\varepsilon > 0$. Observe that

$$p + V(F,\varepsilon) = \{z \in E_{\mathrm{s}} : (\forall\, n \in F)(z_n = p_n)\,\&\,\|z-p\| \le \varepsilon\}.$$

Pick an arbitrary element $y \in U = \mathbb{Q}_d^\infty \setminus (p + V(F,\varepsilon))$. If there exists $n \in F$ such that $y_n \ne p_n$, then $\{z \in \mathbb{Q}_d^\infty : z_n = y_n\}$ is a neighborhood of y which is contained in U. Assume therefore that $y_n = p_n$ for all $n \in F$. Then $\|y-p\| > \varepsilon$. Hence there is a finite subset G of $\mathbb{N}$ such that $\sum_{n\in G}(y_n - p_n)^2 > \varepsilon^2$. So

$$\{z \in \mathbb{Q}_d^\infty : (\forall\, n \in F \cup G)(z_n = y_n)\}$$

is a neighborhood of y which is contained in U, as required. □

Lemma 3.3. *Let $F \subseteq \mathbb{N}$ be finite, $\varepsilon > 0$ and $p \in E_s$. Then in $\mathbb{Q}_d^\infty$, $p + V(F, {}^1\!/\!_2\varepsilon)$ is a nowhere dense subset of $p + V(F,\varepsilon)$.*

Proof. First observe that by Lemma 3.2, $p + V(F, {}^1\!/\!_2\varepsilon)$ is a closed subset of $p+V(F,\varepsilon)$. Pick an arbitrary $y \in p+V(F, {}^1\!/\!_2\varepsilon)$. A basic neighborhood of y in $\mathbb{Q}_d^\infty$ has the form

$$U = \{z \in \mathbb{Q}_d^\infty : (\forall\, n \in G)(z_n = y_n)\},$$

for certain finite $G \subseteq \mathbb{N}$. Put $\lambda = \sqrt{\sum_{n\in G}(y_n - p_n)^2}$. Then $0 \le \lambda \le {}^1\!/\!_2\varepsilon$.

Moreover, let $t \in \mathbb{Q}$ be such that ${}^1\!/\!_4\varepsilon^2 < \lambda^2 + t^2 < \varepsilon^2$. Take an arbitrary $N > \max(F \cup G)$, and define $z \in \mathbb{Q}_d^\infty$ as follows:

$$z_n = \begin{cases} y_n & (n \in F \cup G), \\ t + p_N & (n = N), \\ p_n & (n \notin F \cup G \cup \{N\}). \end{cases}$$

Since $y_n = p_n$ for every $n \in F$, we get $z \in (U \cap (p + V(F, \varepsilon))) \setminus (p + V(F, {}^1\!/\!_2\varepsilon))$, as required. □

Corollary 3.4. *E is Polishable and E with its Polish topology is homeomorphic to E_c.*

Proof. Apply Lemmas 3.2 and 3.3 and Theorem 2.1. □

4. Examples

The question naturally arises whether all group structures on E that are compatible with its topology, are Polishable. Such a group structure is called *compatible* for short. Similarly, whether E with a compatible Polish topology is homeomorphic to E_c. It is not surprising that the answers to these questions are in the negative.

Example 4.1. There is a compatible Abelian group structure on E which is Polishable and for which E with its Polish topology is homeomorphic to E_c^∞ (and hence is not homeomorphic to E_c).

By [4, Corollary 9.4], E is homeomorphic to E^∞. Since E is Polishable, so is E^∞ and E^∞ with its Polish topology is clearly homeomorphic to E_c^∞ (Corollary 3.4). Finally, E_c is not homeomorphic to E_c^∞ by [5].

Example 4.2. There is a compatible Abelian group structure on E which is not Polishable.

Let K denote the Cantor group $\{0, 1\}^\infty$. Put

$$G = \{x \in K : (\exists\, m)(\forall\, n \geq m)(x_n = 0)\}.$$

Then G is clearly a σ-compact zero-dimensional topological group which is not Polishable since G is a countable increasing union of compact subgroups each with uncountable index, see [1, p. 12]. By [4, Theorem 9.2], $G \times E$ and E are homeomorphic. Since G is not Polishable, $G \times E$ is not Polishable either.

For a space X, let $\mathscr{H}(X)$ be the group of all homeomorphisms of X endowed with the compact-open topology. Moreover, a space X is called *countable dense homogeneous* provided that for all countable and dense subsets D and E of X there is an element $f \in \mathscr{H}(X)$ such that $f(E) = F$.

In [9, p. 150], the following simple necessary criterion for the Polishability of a topological group was proved. Let G be a Polish group. In addition, let H be a subgroup of G containing a countable collection $\mathscr{B}$ of subgroups such that

(1) every $B \in \mathscr{B}$ is closed in H,
(2) for every $B \in \mathscr{B}$ there are countable subsets A_B, A'_B of H such that
$$H = \bigcap_{B \in \mathscr{B}} A_B \overline{B} \cap \bigcap_{B \in \mathscr{B}} \overline{B} A'_B$$
(here closure means closure in G).

Then H is Polishable.

This gives us:

Theorem 4.3. *Let X be a connected space which is compact or locally compact and locally connected. Assume moreover that X is countable dense homogeneous space, and let D be an arbitrary countable dense subset of X. Then the subgroup*
$$\mathscr{H}(X, D) = \{f \in \mathscr{H}(X) : f(D) = D\}$$
of $\mathscr{H}(X)$ is Polishable.

Proof. For every $d \in D$, put $B_d = \{f \in \mathscr{H}(X) : f(d) = d\}$. Then every B_d is clearly a closed subgroup of the Polish group $\mathscr{H}(X)$. For all $d, e \in D$, pick $f_{d,e}$ in $\mathscr{H}(X, D)$ such that $f_{d,e}(d) = e$, [10]. Put
$$F = \{f_{d,e} : d, e \in D\};$$
we denote the subgroup generated by it by A. It is easy to see that
$$\mathscr{H}(X, D) = \bigcap_{d \in D} AB_d \cap \bigcap_{d \in D} B_d A.$$
Hence $\mathscr{H}(X, D)$ is Polishable by the criterion. □

Let X be a continuum. In many cases, the groups $\mathscr{H}(X, D)$ are homeomorphic to E. This is true for example if X is a topological n-manifold for some $n \geq 2$, a Hilbert cube manifold, or a Menger manifold. See [4, Chapter 10] for details. It is not clear what the Polish topology on the groups $\mathscr{H}(X, D)$ for such spaces X is. But it can be shown that these groups with their Polish topologies are not homeomorphic to E_c.

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