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Compact condensations and compactifications

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ABSTRACT

A Tychonoff space X will be called strongly bicompactly condensable (SBC) if there is a set $\mathcal{S}$ of compact Hausdorff topologies on the set X whose supremum in the lattice of topologies is the original topology. Such an $\mathcal{S}$ determines a compactification $K(\mathcal{S})$ of X. We examine which compactifications of an SBC X arise in this way: For some X, all do, and for others, some and not all; For some X, βX does, and for others, does not.

The second and third authors dedicate our share of this work to the memory of the first author, our friend Wis Comfort (1933 – 2016). See the obituary [9], reprinted in this issue.

1. Introduction

All spaces will be Tychonoff. For X a space, t_X will be its topology. We may hypothesize X, or t_X, without mentioning the other. An embedding is a map (continuous function) which is a homeomorphism onto its range.

Terminology, notation 1.1. Consider a space X and s a compact (Hausdorff) topology on the set X with $s \subseteq t_X$. Then s, or the space (X, s), or the continuous identity function $i_s : (X, t_X) \to (X, s)$ will be called

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a Bicompact Condensation (BC) of X (or, of (X, t_X), or of t_X). Or, a continuous bijection $X \xrightarrow{f} Y$ with Y compact, will be called a BC of X.

$$\mathrm{BC}(t_X) \equiv \{s \mid s \text{ is a BC of } t_X\}.$$

"BC" will be used as various parts of speech. "X is BC" means $\mathrm{BC}(t_X) \neq \varnothing$.

It is known (and see 1.6 below) that not all X are BC.

X is strongly BC (SBC) if $t_X = \bigvee \mathrm{BC}(t_X)$, this sup in the lattice of topologies. It is known (and see 1.6 below) that BC $\not\Longrightarrow$ SBC. There may be various $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ with $\bigvee \mathcal{S} = t_X$, as we shall see.

Suppose $\varnothing \neq \mathcal{S} \subseteq \mathrm{BC}(t_X)$. A sub-base for $\bigvee \mathcal{S}$ is $\bigcup \mathcal{S}$. Let $P = \prod\{(X, s) \mid s \in \mathcal{S}\}$, and let $X \xrightarrow{\delta} P$ be the diagonal map ($\pi_s \mathcal{S} = i_s \ \forall s \in \mathcal{S}$), which is obviously one-to-one and continuous as a map of (X, t_X). A sub-base for P is $\bigcup\{\pi_s(G) \mid s \in \mathcal{S}, G \in s\}$. It follows that $(X, \bigvee \mathcal{S}) \xrightarrow{\delta} P$ is an embedding and $\overline{\delta(X)}^P$ is a compactification of $(X, \bigvee \mathcal{S})$. We have

Theorem 1.2. *In the above setting,* $(X, t_X) \xrightarrow{\delta} P$ *is an embedding iff* $\bigvee \mathcal{S} = t_X$. *When this is so,* $\overline{\delta(X)}^P$ *is a compactification of* (X, t_X), *denoted* $K(\mathcal{S})$, *and called a compactification "of* SBC-*type".*

Thus, X is SBC iff X has a compactification of SBC-type. As we shall see many times, different $\mathcal{S}$ can produce different $K(\mathcal{S})$. Evidently, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ (each with $\bigvee \mathcal{S}_i = t_X$) entails $K(\mathcal{S}_1) \leq K(\mathcal{S}_2)$ in the semi-lattice of compactifications. If X is SBC, then $K(\mathrm{BC}(t_X))$ is the maximum SBC-type compactification.

The main thrust of this paper is "What are the compactifications of SBC-type?". We indicate some of our results. Various examples given later show that these results are, to some degree, sharp.

βX denotes the Čech-Stone compactification. "locally compact" is abbreviated LC. For X LC, we have the one-point compactification $\alpha X = X \cup \{\alpha\}$. clop$X$ denotes the family of clopen sets in X. X is called zero-dimensional (ZD) if clop$X$ is a basis. If X is ZD, there is a maximum ZD compactification ζX. X is called strongly ZD if $\zeta X = \beta X$.

Theorems 1.3 *(from §4). Suppose X is LC. The following compactifications, for spaces indicated, are* SBC-*type.*

(a) *αX (X is always* SBC*).*
(b) *Any ZD compactification of X. So, if X is strongly ZD, βX.*
(c) *If each non-void open set in X contains a copy of $[0, 1]$, every compactification.*

Theorems 1.4 *(from §5).*

(a) *Suppose X is ZD. If every $U \in \mathrm{clop}X$ is* BC, *then there is $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ with $\bigvee \mathcal{S} = t_X$ and $\xi X \leq K(\mathcal{S})$.*
(b) *(Corollaries of (a)) βX is* SBC-*type for the following.*
   (i) *X with at most one non-isolated point.*
   (ii) *$X =$ the irrationals.*
   (iii) *X countable scattered.*
   (iv) *$X = \sum\{Q_\alpha \mid \alpha < c\}$, each $Q_\alpha \approx \mathbb{Q}$ the rationals (while $\mathbb{Q}$ is not even* BC*).*

Finally, two classes of LC, ZD, pseudocompact spaces will be considered, called "maximal ψ", and "Dowker-type" (detailed description in the text).

Theorems 1.5.

(a) *(from §6). If X is maximal ψ, then βX is* SBC-*type.*

(b) *(from §7). If X is Dowker-type, then $\alpha X < \beta X$, and αX is the only compactification of* SBC*-type.*

Remarks 1.6. We comment briefly on the origins and history of BC issues. More appears in the text.

The question "What X are BC?" is attributed to P.S. Alexandroff in both [21] and [24]. For metrizable spaces, the question is raised by S. Banach [5].

The first paper of which we are aware is Parhomenko's [19], which contains (*inter alia*): LC implies BC (**Proof**. Take $p \in X$ and identify p with the point at infinity in αX (cf. 3.5 below); and, if X is BC and absolute F_σ, then the set of locally compact points in X is dense.). (So the rationals $\mathbb{Q}$ is not BC.)

Katětov [16] shows countable X is BC iff scattered (cf. 5.5 below).

Hewitt [14] translates BC for realcompact X into the existence of a certain kind of subring of $C^*(X)$, then into the existence of certain kind of family of zero-sets, and also notes (i) The class of BC spaces is closed under product formations. (**Proof**. If $\forall i\ X_i \to Y_i$ is a BC, then the resulting $\prod X_i \to \prod Y_i$ is a BC.), (ii) if X is BC with $|X| < 2^\omega$, then X has an isolated point (cf. 5.4 below).

Smirnov [25] translates BC into the existence of a certain kind of function $\beta X \twoheadrightarrow X$.

Reiter [23] observes that the class of BC spaces is closed under sum formation (**Proof**. If $\forall i\ X_i \to Y_i$ is a BC, then the resulting $\sum X_i \to \sum Y_i$ is a bijection, and $\sum Y_i$ is LC, thus BC.)

Bashkirov [6] has defined SBC spaces, in the following terms. Let $u(t_X) = \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq \mathrm{BC}(t_X), V\mathcal{S} = t_X\}$ understanding $u(t_X) = \infty$ if there is no such $\mathcal{S}$; so X is SBC iff $u(t_X) < \infty$. He gives an example of BC X with $u(t_X) = \infty$, various examples of $u(t_X) = 2$ (including X the irrationals), and various other $u(t_X) = m$.

Pytkeev [20] shows (*inter alia*) that $u(t_X) = 2$ for X completely metrizable and ZD.

Some other papers discussing various aspects of BC are [2], [18], [21], [15], [8], [22], [1], [4]. We apologize to neglected authors. We are not aware of any mention in the literature of the situation in our 1.2.

2. Comparison of compactifications

A compactification K of X is usually construed as a dense inclusion $X \hookrightarrow K$. Recall the quasi-order is ([12]) $K_1 \leq K_2$ means $X \hookrightarrow K$, extends continuously over K_2 (yielding a surjection $K_2 \twoheadrightarrow K_1$ which is the identity on X).

Definitions, etc. 2.1. Suppose X, $s \in \mathrm{BC}(t_X)$, $\varnothing \neq \mathcal{S} \subseteq \mathrm{BC}(t_X)$ with $\bigvee \mathcal{S} = t_X$; so $K(\mathcal{S})$ is the compactification of 1.1. Let K be another compactification of X.

(a) s is subordinate to K (sub$K$) if $X \to (X, s)$ has the continuous extension $a(s) : K \to (K, s)$. $\mathcal{S}$ is sub$K$ if each $s \in \mathcal{S}$ is. Then there is $a(\mathcal{S}) : K \to \prod_{\mathcal{S}}(X, s)$, given by $\pi_s a_{\mathcal{S}} = a(s)$ for all $s \in \mathcal{S}$, and the corestriction $a(\mathcal{S}) : K \twoheadrightarrow K(\mathcal{S})$ shows $K \geq K(\mathcal{S})$. And, $K = K(\mathcal{S})$ iff $a(\mathcal{S})$ is one-to-one.

(b) For any Y, $\mathscr{Z}(Y)$ denotes the family of zero-sets of Y. For any product $\prod_A Y_\alpha$ and $B \subseteq A$, we have the projection $\pi_B : \prod_A Y_\alpha \twoheadrightarrow \prod_B Y_\alpha$. For $B \subseteq \mathcal{S}$ we have $\pi_B : P \twoheadrightarrow P_B \equiv \prod_B(X, s)$.

(c) $\mathcal{S}$ finitely separates $\mathscr{Z}(K) \cap X$ if: $\forall$ disjoint $Z_0, Z_1 \in \mathscr{Z}(K)$ $\exists$ finite $B \subseteq \mathcal{S}$ for which the $\overline{\pi_B(Z_i \cap X)}$ are disjoint (which means: for $s = \bigvee B$, the $\overline{Z_i \cap X}^s$ are disjoint).

Theorem 2.2. *With data and definitions as in 2.1:*

(a) *$\mathcal{S}$ is* sub$K$ *iff $K \geq K(\mathcal{S})$.*

(b) *$\mathcal{S}$ finitely separates $\mathscr{Z}(K) \cap X$ iff $K \leq K(\mathcal{S})$.*

(c) *The following are equivalent.*

  (c_1) *$K = K(\mathcal{S})$.*

  (c_2) *$\mathcal{S}$ is* sub$K$ *and finitely separates $\mathscr{Z}(K) \cap X$.*

(c_3) *$\mathcal{S}$ is sub$K$ and $\{a_s \mid s \in \mathcal{S}\}$ separates points of K.*
Thus, K is of SBC*-type iff there is $\mathcal{S}$ satisfying (c_2) and/or (c_3).*

We defer the proof briefly.

Corollary 2.3. *Suppose $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ with $\bigvee \mathcal{S} = t_X$. $\beta X = K(\mathcal{S})$ iff $\forall$ disjoint $Z_0, Z_1 \in \mathscr{Z}(X)$ (n.b., $\mathscr{Z}(X)$) $\exists$ finite $B \subseteq \mathcal{S}$ for which the $\overline{\pi_B(Z_i)}$ are disjoint ("$\mathcal{S}$ finitely separates $\mathscr{Z}(X)$").*
Thus, βX is of SBC*-type iff there is $\mathcal{S}$ as above (or, the condition holds with $\mathcal{S} = \mathrm{BC}(t_X)$).*

Proof. (of 2.3 from 2.2). Any $\mathcal{S}$ is subβX and $K(\mathcal{S}) \subseteq \beta X$ so we just look at 2.2(b).
$\Longleftarrow$. If $Z_i \in \mathscr{Z}(\beta X)$ are disjoint, then so too $Z_i \cap X \in \mathscr{Z}(X)$. The B in the present condition shows the condition in 2.2(b), so $\beta X \leq K(\mathcal{S})$.
$\Longrightarrow$. If $Z_i \in \mathscr{Z}(X)$ are disjoint, then the $\overline{Z}_i^{\beta}$ are disjoint, so there are disjoint $E_i \in \mathscr{Z}(\beta X)$ with $E_i \supseteq \overline{Z}_i^{\beta}$ and $E_i \cap X \supseteq Z_i$ (see [13], if necessary). Now take B from 2.2(b). $\square$

Proof. (of 2.2). First the easy parts:
(a) We noted $\Longrightarrow$ is 2.1(a).
For $\Longleftarrow$: If $K \overset{f}{\twoheadrightarrow} K(\mathcal{S})$ witnesses $K \geq K(\mathcal{S})$, then, if $s \in \mathcal{S}$, $K \overset{K}{\twoheadrightarrow} (\mathcal{S}) \subseteq P \xrightarrow{\pi_s} (X, s)$ has $\pi_s f =$ the desired a_s.
(c_1) $\Leftrightarrow$ (c_3). $K = K(\mathcal{S})$ means $a(\mathcal{S})$ is one-to-one (see 2.1), which is equivalent to $\{a_s \mid s \in \mathcal{S}\}$ separating points. Also $K = K(\mathcal{S})$ entails $K \geq K(\mathcal{S})$, which is equivalent to $\mathcal{S}$ being sub$K$.
(c_1) $\Leftrightarrow$ (c_2), is the conjunction of (a) and (b).
It remains to prove (b). We need two lemmas, whose proofs we defer briefly.

Lemma 2.4. *For compactifications K_1, K_2 of X, $K_1 \leq K_2$ iff $\forall$ disjoint $Z_0, Z_1 \in \mathscr{Z}(K_1)$, the $\overline{Z_i \cap X}^{K_2}$ are disjoint.*

Lemma 2.5. *Suppose $Y = \prod_A Y_\alpha$, the Y_α compact. If F_0, F_1 are disjoint closed in Y, then there is finite $B \subseteq A$ for which the $\prod_B(F_i)$ are disjoint.*

We prove 2.2(b).
Suppose $K \leq K_{\mathcal{S}}$, and $Z_0, Z_1 \in \mathscr{Z}(K)$ are disjoint. By 2.4, the $\overline{Z_i \cap X}^{K_{\mathcal{S}}}$ are disjoint, so are disjoint closed sets in P. By 2.5, there is finite $B \subseteq \mathcal{S}$ with $\prod_B(F_i)$ disjoint. But, $\prod_B(Z_i \cap X) \subseteq \prod_B(F_i)$, so $\overline{\prod_B(Z_i \cap X)} \subseteq \prod_B(F_i)$.
Suppose $\mathcal{S}$ finitely separates $\mathscr{Z}(K) \cap X$. Toward showing $K \leq K(\mathcal{S})$ <u>via</u> 2.4, let $Z_0, Z_1 \in \mathscr{Z}(K)$ be disjoint. There is finite $B \subseteq \mathcal{S}$ with the $\overline{\prod_B(Z_i \cap X)}$ disjoint. Now, $\prod_B(\overline{Z_i \cap X}^P) \subseteq \overline{\prod_B(Z_i \cap X)}$, so the former are disjoint, thus the $\overline{Z_i \cap X}^P$ are disjoint. But, $\overline{Z_i \cap X}^P = \overline{Z_i \cap X}^{K(\mathcal{S}}$.
It remains to prove 2.4 and 2.5.

Proof. of 2.4. $K_1 \leq K_2$ means $X \overset{e}{\hookrightarrow} K_1$ extends over K_2. Taimonov's theorem ([12], p. 136) says e extends iff $\forall$ disjoint closed E_0, E_1 in K_1, the $\overline{e^{-1}(E_i)}^{K_2}$ are disjoint. Here, $e^{-1}(E_i) = E_i \cap X$.
Thus: $\Longrightarrow$ in 2.4 is immediate; $\Longleftarrow$ follows using Urysohn's Lemma (the E_i are contained in disjoint Z_i). $\square$

Proof. of 2.5. Given the F_i, there is $f \in C(Y)^*$ with $f(F_i) = \{i\}$ (by the Urysohn Lemma). Then, there is finite $B \subseteq A$ and $g \in C(\prod_B Y_\alpha)^{-1}$ with $|f(y) - g(\prod_B(y))| \leq 1/3\ \forall y \in Y$ (by the general Stone-Weierstrass Theorem, or the specific Dieudonné Theorem ([10]). Then $g \mid \prod_B(F_0) \leq 1/3$, $g(\prod_B(F_i)) \geq 2/3$, so the $\prod_B(F_i)$ are disjoint. $\square$

The proofs of 2.2 and 2.3 are complete. □

3. Some constructions of BCs

This section may be somewhat tedious. The reader could skip it, and refer back as needed.

We first note the following simple sufficient (not necessary) condition that $\bigvee \mathcal{S} = t_X$ (referring to 1.2).

Lemma 3.1. *Suppose $\mathcal{S} \subseteq \mathrm{BC}(t_X)$: If $\{(X, t_X) \to (X, s) \mid s \in \mathcal{S}\}$ separates points and closed sets in X ($x \notin F = \overline{F}^X \implies \exists s \in \mathcal{S}$ s.t. $x \notin \overline{F}^s$), then $\bigvee \mathcal{S} = t_X$ (so we have the compactification $X \subseteq K(S)$).*

Proof. Apply the Diagonal Theorem [12], 2.3.20. □

Next, for application to various zero-dimensional situations, we indicate a construction of BCs of X from BCs of clopen sets.

In the following, with $U \subseteq X$; t_U denotes the relative topology on U; $U' \equiv X - U$.

Upon stating the following, it becomes obvious.

Lemma 3.2.

(a) *Suppose $U \in \mathrm{clop}X$, so that $X = U + U'$. Suppose $s(U) \in \mathrm{BC}(t_U)$ and $s(U') \in \mathrm{BC}(t_{U'})$. Then $(U, s(U)) + (U', s(U'))$ is a* BC *of X. Denote the topology $s(U) + s(U')$. In this topology U and U' are compact and open.*

(b) *Suppose K is a compactification of X and $V \in \mathrm{clop}K$, so that $K = V + V'$. Then $V \cap X \equiv U \in \mathrm{clop}X$, and $U' = X - U = V' \cap X$, and V, V' are compactifications of U, U', resp.*

Let $s(U), s(U')$ be as in (a). These are sub$V$, subV', *resp., and $s(U) + s(U')$ is* sub$K$.

We adapt to our purposes material from [12], Chapts. 2 and 3, toward construction of BCs from upper continuous decompositions.

Let $\mathcal{E}$ be an equivalence relation on the set X with $X \xrightarrow{q} X/\mathcal{E}$ denoting the quotient onto the set of equivalence classes. In case $X \xrightarrow{f} Y$ is a function, we have the equivalence relation $x_1 \mathcal{E}(f) x_2$ meaning $f(x_1) = f(x_2)$.

When X is a space, we give $X/\mathcal{E}$ the quotient topology. $\mathcal{E}$, or really its decomposition of X into equivalence classes, is called upper semicontinuous (USC) if q is a closed map. For $X \xrightarrow{f} Y$ continuous, $\mathcal{E}(f)$ is USC iff f is a closed map.

We are assuming below all spaces are Tychonoff, even if that is not always needed.

From [12], 2.4.13 and 2.4.15, we have

Lemma 3.3. *Suppose F is closed in K and $F \xrightarrow{\gamma} Z$ is a closed map. Then the USC $\mathcal{E}(\gamma)$ on F extends to the USC $\mathcal{E} = \mathcal{E}(F, \gamma)$ on K as*

$$\mathcal{E} = \mathcal{E}(\gamma) \cup \{\{x\} \mid x \in K - F\},$$

and the resulting $K \xrightarrow{q} K/\mathcal{E}$ has the following features.

$$q^{-1}q(K - F) = K - F, \text{ and } q^{-1}q(F) = F.$$

$$q \mid (K - F) \text{ is a homeomorphic embedding into } K/\mathcal{E}.$$

$$q(F) = F/\mathcal{E}(\gamma) = \gamma(F).$$

This process constructs both compactifications (3.4) and BCs (3.5), as follows.

Corollary 3.4 *(Magill [17], [12], 3.5.13). Suppose $X \subseteq K$ with K compact, ($F \equiv$) $K - X$ closed in K, $Z \cap X = \varnothing$, and $K - X \twoheadrightarrow Z$ is continuous and onto. Let $\mathcal{E} = \mathcal{E}(K - X, \gamma)$, and $K \xrightarrow{q} K/\mathcal{E}$ be as in 3.3.*

Then, $q \mid X$ is a homeomorphic embedding. If X is dense in K, then $K/\mathcal{E}$ is a compactification of X with $K \geq K/\mathcal{E}$ and $K/\mathcal{E} - X \approx Z$.

Corollary 3.5 *(Parhomenko [19] (perhaps); [12], 3.3D). Again suppose $X \subseteq K$ with K compact, ($F \equiv$) $K - X$ closed in K. Now take $Z = X$, and $K - X \xrightarrow{\gamma} X$ continuous. Let $\mathcal{E} = \mathcal{E}(K - X, \gamma)$ as in 3.3.*

Then, $K \xrightarrow{q} K/\mathcal{E}$ has: $q \mid X$ one-to-one and onto $K/\mathcal{E}$: "$q \mid X$ is a BC *of X"; $q \mid (X - \gamma(K - X))$ is a homeomorphic embedding; $q(K - X) \approx \gamma(K - X)$.*

In the situation of 3.5, we have replaced t_X by a new compact topology, s, without changing the "pieces" $\gamma(K-X)$ and $X-\gamma(K-X)$, but these pieces fit together differently: Some t_X-closed E with $E\cap\gamma(K-X) = \varnothing$ will have $\overline{E}^s \cap \gamma(K - X) \neq \varnothing$.

Some applications below will combine 3.4 and 3.5.

A rather elaborate further refinement of 3.5 appears in §6.

The simplest instances of 3.5 have $K = \alpha X = X \cup \{\alpha\}$ the one-point compactification, then γ defined by picking $p \in X$, and $\gamma(\alpha) \equiv p$. Denote by s_p the resulting topology on the set X.

The following is evident.

Corollary 3.6. *Suppose X is LC, and $s \in \mathrm{BC}(t_X)$; s is subβX, i.e., we have $\beta X \overset{b}{\twoheadrightarrow} (X, s)$ extending the "identity" $(X, t_X) \to (X, s)$. These are equivalent.*

(a) *$\exists p \in X$ for which $s = s_p$.*
(b) *s is subαX.*
(c) $|b(\beta X - X)| = 1$.

4. Locally compact spaces

Theorem 4.1. *For X LC, αX is SBC-type (and X is* SBC*).*

Proof. ([6] shows X is SBC, by the same argument.)

We exhibit appropriate $\mathcal{S} = \{s_1, s_2\}$. Take $p_1 \neq p_2$ in X. By 3.6, we have the $s_i \equiv s_{p_i} \in \mathrm{BC}(t_X)$ which are subαX. 3.1 applies here because: if $x \notin F$, F closed in X, then for one of the p_i, $x \neq p_i$, and then $x \notin \overline{F}^{s_i}$. By 2.2, we have $K(\mathcal{S}) \leq \alpha X$, thus $=$ since αX is the minimum compactification. □

The following shows (as explained below) that frequently LC X has compactifications not SBC-type.

Theorem 4.2. *Suppose X (not assumed LC) has each of its* BC*s ZD. If K is a compactification of X with $K - X$ containing a connected set with at least two points, then K is not* SBC*-type.*

Proof. Using 2.2, consider an $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ which is sub$K$. The resulting $a(\mathcal{S}) : K \to \prod_{\mathcal{S}}(X, s) \equiv P$ cannot be one-to-one because each (X, s) is ZD, thus so is P, so the connected set $a(\mathcal{S})(C)$ must be a singleton. □

Corollary 4.3. *The countable discrete space $\mathbb{N}$ (which is SBC by 4.1) has compactifications not* SBC*-type. (In §5 below, we show $\beta\mathbb{N}$ is* SBC*-type. Since $\alpha\mathbb{N}$ is also, we see that "*SBC*-type" is inherited neither up nor down in the semi-lattice of compactifications of $\mathbb{N}$.)*

Proof. $\mathbb{N}$ is an X as in 4.2: Any BC is ZD because $\mathbb{N}$ is countable ([12]). $\mathbb{N}$ has compactifications K with $K - \mathbb{N} \approx [0,1]$ (for example), by a use of 3.4. $\square$

On the other hand

Theorem 4.4. *If X is LC, and each open $G \neq \varnothing$ contains (a copy of) $[0,1]$, then each compactification of X is* SBC*-type.*

Proof. Let K be a compactification of X. Per 2.2, we want $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ which is sub$K$ and finitely separates $\mathscr{Z}(K) \cap X$.

Suppose given disjoint $Z_0, Z_1 \in \mathscr{Z}(K)$. Case (i). $Z_0 \cup Z_1 \subset K$. Here, the $Z_i \in \mathrm{clop}X$. Consider 3.4(b) with $Z_0 = V$. The construct there, $s = s(U) + s(U')$ is sub$K$, and the $\overline{Z_i \cap X}^C$ are disjoint, using 3.4(a).

Case (ii). $Z_0 \cup Z_1 \neq K$. Then $(Z_0 \cap X) \cup (Z_1 \cup X) \neq X$ and there is $\varnothing \neq G$ open in X missing each $Z_i \cap X$. Take $g \in C(K, [0,1])$ with $\{y \in K \mid g(G) = i\}$ a neighborhood of Z_i. Then, take $I \approx [0,1]$ with $I \subseteq G$. Then an embedding $g(K - X) \overset{e}{\hookrightarrow} I$. Define $K - X \xrightarrow{\gamma} X$ as $\gamma = e \circ g$, and use 3.5 to produce from γ, $s \in \mathrm{BC}(t_X)$. This s is sub$K$ and has the $\overline{Z_i \cap X}^s$ disjoint. $\square$

4.1 shows any LC X is SBC. The following shows that weakening LC "slightly" can result in "not SBC" in a striking way.

Theorem 4.5. *Suppose $X = \{p\} \cup \sum_{\mathbb{N}} L_n$, where: each L_n is a continuum (compact, connected), not a point; p has no compact neighborhood in X (so $\sum L_n$ is dense in X). Then, the one-point compactification $\alpha \sum L_n = \sum L_n \cup \{\alpha\}$ is the unique* BC *of X.*

(Within $[0,1] \times [0,1]$, $X = \{(0,0)\} \cup \sum(\{\frac{1}{n} \times [0,1])$ is such a space.)

Proof. Recall Sierpinski's Theorem ([12], 6.1.27): If a continuum $C = \bigcup_n F_n$, F_n closed and disjoint, then $F_n \neq \varnothing$ for at most one n.

For our $X = \{p\} \cup \sum_n L_n$, the component of p, $C_p = \{p\}$, because: If $C_p \neq \{p\}$, then some $L_i \cap C_p \neq \varnothing$, so $L_i \subseteq C_p$ (otherwise, $C_p \cup L_i$ is a connected set bigger than C_p). Then, applying Sierpinski to $C_p = \{p\} \cup \bigcup\{L_i \mid L_i \subseteq C_p\}$ shows $C_p = \{p\}$.

Now let $X \xrightarrow{f} Y$ be a BC. Note that such restriction $L_n \xrightarrow{f} f(L_n)$ is a homeomorphism because L_n is compact and f is one-to-one. We claim that each $f(L_i)$ is clopen. That will mean $\bigcup f(L_n) = \sum L_n$, and $Y = \{f(p)\} \cup \bigcup_n f(L_n) = \alpha(\sum L_n)$ ($\alpha = f(p)$). Since $\{p\} = C_p$, $\{f(p)\} = f(C_p)$, and the latter is the component of $f(p)$. Since Y is compact, components and quasi-components coincide ([12], p. 357), so $\{f(p)\} = \bigcap\{U \mid f(p) \in U, U \text{ clopen}\}$. Take any L_n and $y \in f(L_n)$. There is clopen $U_n \ni f(p)$ with $y \in U_n$. Since $f(L_n)$ is connected, $f(L_n) \cap U_n = \varnothing$. Thus $\{f(p)\} = \bigcap_n U_n$. We arrange $U_{n+1} \subseteq U_n$ $\forall n$ and $U_n \cap f(L_i) = \varnothing$ $\forall i \leq n$.

So $Y - U_n = \bigcup_{i \leq n} f(L_i)$, for $i \leq n$ $f(L_i)$ is clopen, and $f(L_n)$ is clopen. $\square$

In the above, some hypothesis resembling "the L_n are continua" is needed: $\beta(\{p\} \cup \sum L_n)$ would be SBC-type were each L_n discrete (by 5.4 below), or countable scattered (5.5).

5. Zero-dimensional spaces

Recall that: X is ZD by definition if clop$X$ is a basis, and strongly ZD iff $\forall$ disjoint $Z_0, Z_1 \in \mathscr{Z}(X)$ $\exists U \in \mathrm{clop}X$ with $Z_0 \subseteq U$, $Z_1 \subseteq X - U \equiv U'$. And, Lindelöf ZD implies strongly ZD. See [12], 6.2.

Theorem 5.1. *If K is a ZD compactification of LC X, then K is of* SBC*-type.*

Proof. Given such $X \subseteq K$, from 2.2 we want $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ which is sub$K$, $\bigvee \mathcal{S} = t_X$, and $\mathcal{S}$ finitely separates $\mathscr{Z}(K) \cap X$.

Referring to 3.7: For $V \in \mathrm{clop}K$, $U = V \cap X$ is LC, with $U \subseteq V$ a compactification, and there is $s(U) \in \mathrm{BC}(t_U)$ which is sub$V$. Likewise, $s_V \equiv s(U) + s(U') \in \mathrm{BC}(t_X)$, is sub$K$ and U and U' are s_V-compact. Evidently, $\mathcal{S} \equiv \{s_V \mid V \in \mathrm{clop}X\}$ is sub$K$, and separates points and closed sets of X, so by 3.1, $\bigvee \mathcal{S} = t_X$.

For disjoint $Z_0, Z_1 \in \mathscr{Z}(K)$, there is $V \in \mathrm{clop}K$ with $Z_0 \subseteq V$, $Z_1 \subseteq V'$ (because K is strongly ZD). Then, $Z_0 \cap X \subseteq U$, $Z_1 \cap X \subseteq U'$, U and U' are s_V-compact and disjoint, so in s_V, the closures of the $Z_i \cap X$ are disjoint. □

4.2, say with $X = \mathbb{N}$, shows the need to have K ZD in 5.1.

We turn to "βX is SBC-type?"

In the following for $B \subseteq \mathrm{BC}(t_X)$, we have/denote $(X, t_X) \xrightarrow{e} \prod_B \{(X, s) \mid s \in B\} \equiv P_B$ (e given by $\pi_s e = (X, t) \to (X, s)\ \forall s \in B$).

Theorem 5.2. *Suppose X is ZD.*

(a) *Suppose $\mathcal{S} \subseteq \mathrm{BC}(t_X)$. Then, $\zeta X \leq K(\mathcal{S})$ iff*
  ($*$) *$\forall U \in \mathrm{clop}X\ \exists$ finite $B(U) \subseteq \mathcal{S}$ for which $e(U)$ and $e(X - U)$ have disjoint closures in $P_{B(U)}$.*

(b) *If X is strongly ZD, then βX is SBC-type iff for $\mathcal{S} = \mathrm{BC}(t_X)$, ($*$) holds.*

Proof. (a) ($*$) is easily seen to be equivalent to "$\mathcal{S}$ finitely separates $\mathcal{Z}(\zeta X) \cap X$". Apply 2.2(b).

(b) follows from (a). □

We are going to use 5.2 only for situations where the cardinals $|B(U)| = 1$ (5.3 – 5.6) or 2 (5.7). (We do not pursue other cases, which doubtless exist.)

We simplify the case $|B(U)| = 1$.

Corollary 5.3. *Suppose X is ZD, and that each $U \in \mathrm{clop}X$ is* BC. *Then,*

(a) *There is $\mathcal{S} \subseteq \mathrm{BC}(t_X)$ with $\zeta X \leq K(\mathcal{S})$.*

(b) *If X is strongly ZD, then βX is* SBC*-type.*

Proof. (a) Let $U \in \mathrm{clop}X$. Choose (in any way) $s(U) \in \mathrm{BC}(t_U)$, $s'(U') \in \mathrm{BC}(t_{U'})$, and let $s_U \equiv s(U) + s'(U') \in \mathrm{BC}(t_X)$ (per 3.6, as in 5.1). Here, U and U' are $s(U)$-compact. Then, $\mathcal{S} \equiv \{s_U \mid U \in \mathrm{clop}X\}$ satisfies 5.2(a) (with $B(U) = \{s_U\}$).

(b) follows. □

Corollary 5.4. *If X has ≤ 1 non-isolated point, then βX is* SBC*-type.*

Proof. If X has no non-isolated points, then X is discrete, thus strongly ZD, so 5.1 (or 5.2) gives βX SBC-type.

Suppose $X = D \cup \{p\}$, D discrete and p the non-isolated point. Again, X is strongly ZD: For disjoint $Z_i \in \mathscr{Z}(X)$, say $p \notin Z_0$. So Z_0 is open, and $\{Z_0, X - Z_0\}$ is a clopen partition separating Z_0, Z_1.

Now, the obvious map $D \cup \{p\} \to D \cup \{\alpha\} = \alpha D$ defines a BC for X.

If $U \in \mathrm{clop}X$, then: either $p \notin U$, whence U is discrete, thus BC; or $p \in U$, so U is "of the form $D \cup \{p\}$", thus BC.

Now apply 5.3. □

Let D be discrete and $p \in \beta D - D$. 5.4 applies to $X = D \cup \{p\}$. Obviously 5.4 extends to X with only finitely many non-isolated points.

X is called scattered if each $\varnothing \neq S \subseteq X$ contains an isolated point of S.

In the following, (a) $\Leftrightarrow$ (b) is due to Katětov [16], and (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) to Bashkirov [6] (who seems unaware of [16]).

Corollary 5.5. *For countable X, these are equivalent.*

(a) *X is scattered.*
(b) *X is* BC*.*
(c) *X is* SBC*.*
(d) *βX is* SBC*-type.*

Proof. We noted (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) above. Of course, (d) $\Rightarrow$ (c).

We show (a) & (b) $\implies$ (d). Now, any countable space is strongly ZD ([12], p. 362), and any subspace of a scattered space is scattered. So, if X satisfies (a) and (b), then 5.3 applies ((∗) holds), giving (d). □

[15] has an example in ZFC, of a scattered space which is not BC.

As noted in 1.6 (or by 5.5), the rationals $\mathbb{Q}$ is not BC. (However, see 5.7 below.) On the other hand, consider $P = \mathbb{R} - \mathbb{Q}$, the irrationals.

Corollary 5.6. *βP is* SBC*-type.*

Proof. 5.3 applies: (i) P is Lindelöf ZD, thus strongly ZD. (ii) P is BC, since $P \approx \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$ is BC and the class of BC-spaces is closed under product formation (1.6). (iii) $\forall \varnothing \neq U \in \mathrm{clop}P$, $U \approx P$ by [12], 6.2A(b). □

We consider an interesting case of 5.2 with $|B(K)| = 2$.

Let $m \neq 0$ be a cardinal number, and let $m \cdot \mathbb{Q}$ denote the sum of m copies of the rationals $\mathbb{Q}$, or, $m \cdot \mathbb{Q} = \sum_{\alpha<m} \mathbb{Q}_\alpha$ with each $\mathbb{Q}_\alpha \approx \mathbb{Q}$.

Theorem 5.7. *$m \cdot \mathbb{Q}$ is* BC *iff $c \leq m$.*

Proof. Suppose $m < c$. If $m \cdot \mathbb{Q} \xrightarrow{f} Y$ were a BC, then $|Y| < c$, Y is compact, thus Y has an isolated point y ([3], p. 30), and $f^{-1}(y)$ is an isolated point of $m \cdot \mathbb{Q}$ – which has none. (This argument is Hewitt's (see 1.6 here).)

Suppose $c \leq m$. Since $m \cdot c = m$, $m \cdot \mathbb{Q}$ is the sum of m copies of $c \cdot \mathbb{Q}$. Since the class of BC spaces is closed under sums, it suffices that $c \cdot \mathbb{Q}$ be BC. Write $[0,1] = \bigcup_{\alpha<c} \mathbb{Q}_\alpha$ ($\mathbb{Q}_\alpha \approx \mathbb{Q}$). Evidently, we have the BC $c \cdot \mathbb{Q} = \sum_{\alpha<m} \mathbb{Q}_\alpha \xrightarrow{f} \bigcup_{\alpha<c} \mathbb{Q}_\alpha = [0,1]$. □

Theorem 5.8. *$\beta(m \cdot \mathbb{Q})$ is* SBC*-type iff $c \leq m$.*

Proof. $\implies$. If $m < c$, $m \cdot \mathbb{Q}$ is not even BC (5.7).

$\impliedby$. This has several steps. Suppose $c \leq m$.

(1) Take any BC $m \cdot \mathbb{Q} \xrightarrow{f} Y$ (from 5.7).

In the following: Let X denote $m \cdot \mathbb{Q}$, sometimes expressed as $X = \sum_{\alpha<m} \mathbb{Q}_\alpha$. Always, $U \in \mathrm{clop}X$.

(2) Suppose $U \approx X$ and $X - U \approx X$. Then, there is a BC $X \xrightarrow{f} Y$ with $f(U)$ clopen, i.e., there is $U \in \mathrm{clop}Y$ with $u = f^{-1}V$. (I.e., here "$|B(U)| = 1$").

(By (1), there are BCs $U \to Y_1$ and $X - U \to Y_2$ which yield the BC $X = U + (X - U) \to Y_1 + Y_2 = Y$.)

(3) Put $E(U) \equiv \{\alpha \mid U \cap \mathbb{Q}_\alpha \neq \varnothing\}$. $|E(W)| = m$ iff $U \approx X$. ($\mathbb{Q}$ is the unique countable metrizable space with no isolated points ([12], p. 370). Thus, $\varnothing \neq U \cap \mathbb{Q}$ implies $U \cap \mathbb{Q}_\alpha \approx \mathbb{Q}$.)

(4) Suppose one of U and $X - U$ is not homeomorphic to X. Say $U \not\approx X$. Then $|E(U)| < m$ (by (3)), and there are BCs $X \xrightarrow{f_i} Z_i$ $(i = 0, 1)$ for which the $X \xrightarrow{e} Z_0 \times Z_1 \equiv P$ given by $f_i = \pi_i e$ has $\overline{e(U)}^P \cap \overline{e(X-U)}^P = \varnothing$. (I.e., here "$|B(U)| = 2$".)

Proof. of (4). Put $F = m - E(U)$ and write $F = F_0 \dot\cup F_1$ with each $|F_i| = m$. Then, each $U \cup \sum_{\alpha \in F_i} Q_\alpha \equiv U_i \in \mathrm{clop}X$ and $|E(U_i)| = m$. By (3) and (2), there are BCs $X \xrightarrow{f_i} Z_i$ with $V \in \mathrm{clop}Z_i$ for which $U_i = f_i^{-1}V_i$.

With $X \xrightarrow{e} Z_0 \times Z_1 \equiv P$ $(\pi_i e = f_i)$, we have $U = f_0^{-1}V_0 \cap f_1^{-1}V_1$; say $x \notin f_0^{-1}V_0 = (\pi_0 e)^{-r}V_0 = e^{-1}\pi_0^{-1}V_0$, i.e., $e(x) \notin \pi_0^{-1}V_0$. This shows $e(X - U) \subseteq P - V$. The latter being clopen. So $\overline{e(X-U)}^P \subseteq P - V$, and (4) is proved.

(2) and (4) show 5.2 applies, and since $\mathbb{Q}$ is strongly ZD, so is X. Thus βX is SBC-type. □

5.7 and 5.8 include the information: $m \cdot \mathbb{Q}$ is SBC iff $c \leq m$. We note that [6] Theorem 3 says that the class of SBC spaces is closed under $\sum$. (That does not give 5.8.)

6. More βX SBC-type

These X will be the "maximal Ψ-spaces" described in [13], [26], and [7] (among other places). Let $\mathcal{R}$ be a family of infinite subsets of $\mathbb{N}$ which is "almost disjoint" ($\forall a \neq b$ in $\mathcal{R}$, $a \cap b$ is finite). Denote $\mathcal{R} = \{d \mid d \in \mathcal{R}\} = D$. ("$d$" becomes a point).

Put $X = \mathbb{N} \cup D$ with the topology: points of $\mathbb{N}$ are isolated; a neighborhood of $d \in D$ is $\{d\} \cup (d - F)$ for F finite. Such X is LC, ZD, first countable, and D is closed and discrete. Further, X is pseudocompact iff $\mathcal{R}$ is maximal for almost-disjointness (which entails $|\mathcal{R}| > \omega$). We call these X "maximal Ψ"; there are 2^c such X which are strongly ZD, and 2^c which are not [7].

Theorem 6.1. *If X is maximal Ψ, then βX is* SBC-*type.*

Cf. §7, where we have another X which is LC, ZD, pseudocompact, but βX <u>not</u> SBC-type.

The proof of 6.1 proceeds in stages. For any X denote: $X^* = \beta X - X$; for $E \subseteq X$, $\overline{E}^\beta$ is closure in βX; $E^\# = \overline{E}^\beta \cap X^*$.

Consider, for any X, disjoint non-void $Z_0, Z_1 \in \zeta(X)$. Toward applying 2.3, we seek $s \in \mathrm{BC}(t_X)$ with the $\overline{Z_i}^s$ disjoint. We have the $\overline{Z_i}^\beta$ disjoint, and the $Z_i^\#$ disjoint. There are two cases: (i) (resp., (ii)) there is (resp., there is not) $W \in \mathrm{clop}X^*$ with $W \supseteq Z_0^\#$, $W \cap Z_1^\# = \varnothing$.

Proposition 6.2. *Suppose X is (merely) LC, ZD, and disjoint $Z_0, Z_1 \in \zeta(X)$ are "case (i)". Then, there is a ZD compactification K of X and $V \in \mathrm{clop}K$ with $V \supseteq Z_0^\#$, $V \cap Z_1^\# = \varnothing$ such that, for $U = V \cap X$, the $s = s(U) + s(U')$ from 3.2 has the $\overline{Z_i}^s$ disjoint.*

Proof. Define continuous $X^* \xrightarrow{\gamma} \{0, 1\}$ as $\gamma(V) = 0$, $\gamma(V') = 1$. Since X is LC, by 3.4, there is a compactification $\beta X \xrightarrow{f} K$ with $f \mid X^* = \gamma$, we have $\overline{Z_i}^* = f(\overline{Z_i}^\beta) = f(Z_i) \cup f(K - Z_i) = Z_i \cup \{i\}$, and these are disjoint. Since X is ZD, and $K = X \cup \{0, 1\}$, K is ZD, and there is $V \in \mathrm{clop}K$ as desired. □

One may note now: were X strongly ZD, then for <u>every</u> disjoint $Z_0, Z_1 \in \mathscr{Z}(X)$ case (i) obtains and 6.2 is applicable, so (5.2 again) βX is SBC-type. (5.2 is not proving 6.2 though.)

We turn to the more complicated case (ii). More general features of the argument will be isolated for possible later use.

Proposition 6.3. *Suppose (for any X), disjoint $Z_0, Z_1 \in \mathscr{Z}(X)$ are "case (ii)" (i.e., not case (i)). Then*

(a) *There is in βX a continuum C, $|C| \geq 2$, with each $Z_i \cap C \neq \varnothing$.*
(b) *Suppose X is LC. There is a compactification K of X with $K - X \approx [0,1]$, and $\overline{Z}_i^K = Z_i \cup \{i\}$ (thus disjoint), and there is a retraction $K \xrightarrow{r} K - X = [0,1]$.*
(c) *Suppose X is pseudocompact (as well as LC). Then, $\forall t \in K - X = [0,1]$, $r^{-1}(\{t\}) \cap X \neq \varnothing$ and $r^{-1}(\{t\})$ is uncountable.*

Proof. (a) In X^*, let C be the connected component of $Z_0^\#$. Then, $C = \bigcap\{W \in \text{clop}X^* \mid W \supseteq Z_0^\#\}$ (i.e., the component is the quasi-component because X^* is compact ([12], 6.1.23)). Since we are in "case (ii)", $C \cap Z^\# \neq \varnothing$, so $|C| \geq 2$ (thus $\geq c$).

(b) Now, since the $Z_i^\#$ are disjoint closed in compact X^*, there is $X^* \xrightarrow{\gamma} [0,1]$ with $\gamma(Z_i^\#) = \{i\}$, and since $\gamma(C)$ is connected with $0, 1 \in \gamma(C)$, $\gamma(C) = [0,1]$. By 3.4, there is a compactification $\beta X \xrightarrow{f} K$, with $f \mid X^* = \gamma$, so $K - X = [0,1]$. The identity function on $K - X = [0,1]$ extends over K (Tietz-Urysohn Theorem) to the desired retraction r.

(c) $\forall t \in [0,1]$, $\{t\}$ is G_δ in K. Since X is pseudocompact, this G_δ meets X (from [12], 3.10F). Moreover, $|r^{-1}(\{t\})| > \omega$, for if not, $r^{-1}(\{t\}) = \{t\} \cup A$ with $|A| = \omega$ and $t \notin A$, and then again, $\{t\} = r^{-1}(\{t\} - A$ is G_δ in K missing X. $\square$

Proof. of 6.1. Suppose X is "maximal Ψ", as $X = \mathbb{N} \cup D$, which is LC, ZD, pseudocompact, with D closed and discrete, and $|D| > \omega$.

Take disjoint $Z_0, Z_1 \in \mathscr{Z}(X)$. As noted, we want $s \in \text{BC}(t_X)$ with the $\overline{Z}_i^s$ disjoint.

If "case (i)", 6.2 gives such s.

If "case (ii)", we have the apparatus in 6.3 available ($K = X \dot\cup [0,1]$ and $K \xrightarrow{r} [0,1]$) which we refine further.

$\forall t \in [0,1]$, $r^{-1}(\{t\})$ is uncountable G_δ in K, so $r^{-1}(\{t\}) - \mathbb{N}$ is also, thus hits X, so we choose $d(t) \in D = X - \mathbb{N}$ with $r(d(t)) = t$. Then, $E \equiv \{d(t) \mid t \in [0,1]\}$ is closed and discrete (since D is), and $F = E \cup [0,1]$ is compact (gives $F = \overline{E}^K \cup [0,1]$).

Our situation now is an instance of 3.3, $F \subseteq K$, $F \xrightarrow{\gamma} Z$, for which $X \subseteq K$, $K - X = [0,1]$, $F = (F \cap X) \cup (F \cap (K - X))$, and $\gamma = r \mid F$. There is the resulting $k \xrightarrow{\gamma} K/\mathcal{E}$. Here we have $r(F \cap X) = K - X$, so $q(X) = K/\mathcal{E}$, $q(X) = q(K \cap X) = q((K - F) \cap X) \cup q(F \cap X)$. From 3.3, $q \mid (K - F)$ is a homeomorphism. Here, $q \mid F \cap X = r \mid F \cap X$, which is one-to-one. Thus, $X \to q(X) = K/\mathcal{E}$ is one-to-one and onto, and is a BC say s, in which $F \cap X$ (with the X-topology) is replaced by $r(F \cap X) = [0,1]$.

Looking back at 6.3(b), where $\overline{Z}_i^K = Z_i \cup \{i\}$, we see that $\overline{Z}_i^s = Z_i \cup \{i\}$, so these are disjoint. $\square$

7. X SBC with βX not SBC-type

We prove 1.6, proceeding in stages.

Theorem 7.1. *Suppose X is LC (so X is* SBC, *and αX is* SBC*-type (4.1)).*

A. *αX is the only* SBC*-type compactification of X iff the only $s \in \text{BC}(t_X)$ are the s_p ($p \in X$) of 3.6.*
B. *If that is true, and X is also ZD, then αX is the only ZD compactification of X.*

Proof. A. Evident from 2.2 and 3.6.

B. An easy argument shows X ZD $\Longrightarrow$ αX ZD. Now use A, and 5.1. $\square$

In a space Y, a subset A is called a P-set (resp., weak P-set) if every G_δ containing A is a neighborhood of A (resp., each countable E with $E \cap A = \varnothing$ has $\overline{E} \cap A = \varnothing$).

Theorem 7.2. *Suppose X is ZD, with X^* $(= \beta X - X)$ a metrizable continuum, and a weak P-set in βX. Then X satisfies 7.1.*

Proof. X is LC because $X = \beta X - X^*$.

Toward 7.1 A, suppose $X = (X, t) \xrightarrow{f} (X, s)$ is a BC ($f =$ the identity function on the set X). We have $\beta X \xrightarrow{\beta f} (X, s)$, and we want to show that $A \equiv \beta f(X^*)$ is a singleton (3.6).

$B \equiv f^{-1}(A)$ is closed in X, and the restriction $B \overset{f}{\twoheadrightarrow} A$ is one-to-one. Since X^* is a metrizable continuum, so is A ([12], 3.7.19), so A has a countable dense set D. Take countable $E \subseteq P$ with $f(E) = D$.

Since X^* is a weak P-set, $\overline{E}^\beta \subseteq X$, so $\overline{E}^\beta$ is ZD. We have $\beta f(\overline{E}^\beta) = \overline{\beta f(E)}^s$ (since βf is a closed surjection), and since $E \subseteq X$, $\overline{\beta f(E)}^s = \overline{f(E)}^s = \overline{D}^s = A$. Also, since $\overline{E}^\beta \subseteq X$, we have $\beta f(\overline{E}^\beta) = f(\overline{E}^\beta)$, and because f is one-to-one and $\overline{E}^\beta$ is compact, $f \mid \overline{E}^\beta$ is a homeomorphism, and onto A. Thus, A is ZD and connected, and is a singleton. $\square$

We now describe the space in 1.5, and show it satisfies 7.2. The example is based on a famous space of Dowker [11]. Our description and a few facts more-or-less follow [12], 6.2.20.

On $\mathbb{I} = [0, 1]$, $xEy \equiv |x - y| \in \mathbb{Q}$ is an equivalence relation with c equivalence classes. Take (any) pairwise disjoint classes $\{\mathbb{Q}_\alpha \mid \alpha < \omega_1\}$ with each $Q_\alpha \neq \mathbb{Q}$. Each Q_α is countable and dense in $\mathbb{I}$.

Let $W = [0, \omega_1]$, the ordinal space, and $P \equiv \{\omega_1\} \times \mathbb{I} \subseteq W \times \mathbb{I}$.

For each $\alpha < \omega_1$; $S_\alpha \equiv \mathbb{I} - \bigcup_{\gamma \geq \alpha} Q_\alpha$ is D; put $Y_\alpha \equiv \bigcup_{\gamma \leq \alpha} \{\gamma\} \times S_\gamma$. Then, (the Dowker space) $Y \equiv \bigcup_{\alpha < \omega_1} Y_\alpha = \bigcup_{\alpha < \omega_1} \{\alpha\} \times S_\alpha$, is dense in $W \times I$, and is ZD and normal.

Put $Y' \equiv Y \cup P$. This space is normal, and Y is C^*-embedded in Y', so that $\beta Y' = \beta Y$.

Let $K \equiv \beta Y' = \beta Y$. Since $P \subseteq K$, K is not ZD. Let $X \equiv K - P$; so $\beta X = K$.

The following shows 1.5.

Theorem 7.3. *X is ZD, $X^* = P$ is even a P-set in βX $(= K)$. Thus, X satisfies 7.2, thus 7.1, so αX $(< \beta X)$ is the only compactification of* SBC*-type (and the only ZD compactification).*

Proof. Some technicalities are needed.

(a) If U is open in K and $\overline{U}^K \cap P = \varnothing$, then there is $\gamma < \omega_1$ for which $U \cap Y \subseteq ([0, 1] \times \mathbb{I}) \cap Y$.

(b) For $\gamma < \omega_1$, $\overline{[0, \gamma] \times \mathbb{I}) \cap Y}^K \cap P = \varnothing$.

For (a): Suppose we have U open in K, but the conclusion of (a) fails, i.e., $\forall \gamma < \omega_1$, $\exists x_\gamma \in U \cap Y$ but $x_\gamma \notin P$. Then, there is $x \in W \times \mathbb{I}$ which is a complete accumulation point of $\{x_\gamma \mid \gamma < \omega_1\}$, i.e., every neighborhood of x contains uncountably many x_γ. But then $x = (x_1, x_2)$ cannot have $\pi_1(x) = x_1 < \omega_1$, which means $x_1 = \omega_1$, so $x \in P$, and $\overline{U}^K \cap P \neq \varnothing$.

For (b): For $\gamma < \omega_1$, P and $([0, \gamma] \times \mathbb{I}) \cap Y'$ are disjoint closed sets in the normal space Y', thus have disjoint closures in $\beta Y' = K$.

We show X is ZD. Since X is LC, it suffices that $\overline{V}^X$ be ZD whenever V is open in X with $\overline{V}^X$ compact (and thus $\overline{V}^X \cap P = \varnothing$). Now $V = U \cap X$ for U open in K, and $\overline{U}^K = \overline{V}^X$ since $\overline{V}^X$ is compact. By (a) above, $\exists \gamma < \omega_1$ with $U \cap Y \subseteq ([0, \gamma] \times \mathbb{I}) \cap Y \equiv E$, and by (b) above, $\overline{E}^K \cap P = \varnothing$.

Here E is clopen in Y and Y is normal, so E is C^*-embedded in Y and $\overline{E}^K$ is (a model of) βE. Since $K = \beta Y$, $E = F \cap Y$ for F clopen in K, and $F = \overline{E}^K = \beta E$. Since $[0, \gamma] \times \mathbb{I}$ is separable metrizable and ZD, it is strongly ZD ([12], 6.2), so $\beta E = F$ is ZD, and so is its subspace $\overline{V}^X = \overline{U}^K = \overline{(U \cap Y)}^K$.

We show P is a P-set in K. Suppose U is open F_σ in K with $U \cap P = \varnothing$. We want $\overline{U}^K \cap P = \varnothing$. Now, $U = \bigcup_{\mathbb{N}} F_n$ with each F_n compact and $F_n \cap P = \varnothing$. So, $\forall n$ $\exists V_n$ open in K with $V_n \supseteq F_n$ and

$\overline{V_n}^K \cap P = \varnothing$; then by (a) above, $\exists \gamma_n < \omega_1$ with $V_n \cap Y \subseteq ([0, \gamma_n] \times \mathbb{I}) \cap Y$. Then $\gamma \equiv \sup_{\mathbb{N}} \gamma_n < \omega_1$ and $\forall n$, $V_n \cap Y \subseteq ([0, \gamma] \times \mathbb{I}) \cap Y \equiv E$. We have $\overline{U}^K \subseteq \overline{E}^K$, and by (b) above, $\overline{E}^K \cap P = \varnothing$. $\square$

Remark 7.4. We actually have many X as above, since the construction proceeds from any pairwise disjoint $\{Q_\alpha \mid \alpha < \omega_1\}$. These X are all pseudocompact, since $X^* \approx [0, 1]$ fails to contain $\beta\mathbb{N} - \mathbb{N}$. Thus (for what it's worth), these X, and the many "maximal Ψ" from §6, share the features: LC, ZD, pseudocompact.

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