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Some aspects of dimension theory for topological groups

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Abstract

We discuss dimension theory in the class of all topological groups. For locally compact topological groups there are many classical results in the literature. Dimension theory for non-locally compact topological groups is mysterious. It is for example unknown whether every connected (hence at least 1-dimensional) Polish group contains a homeomorphic copy of $[0, 1]$. And it is unknown whether there is a homogeneous metrizable compact space the homeomorphism group of which is 2-dimensional. Other classical open problems are the following ones. Let G be a topological group with a countable network. Does it follow that $\dim G = \text{ind } G = \text{Ind } G$? The same question if X is a compact coset space. We also do not know whether the inequality $\dim(G \times H) \leq \dim G + \dim H$ holds for arbitrary topological groups G and H which are subgroups of σ -compact topological groups. The aim of this paper is to discuss such and related problems. But we do not attempt to survey the literature.

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1. Introduction

All topological spaces under discussion are Tychonoff.

Dimension theory deals with dimensional invariants of topological spaces. It is intuitively clear that a point is 0-dimensional, a line 1-dimensional, a plane 2-dimensional and the space that

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we live in 3-dimensional. But how should we define the dimension of an arbitrary topological space and is it a topological invariant, i.e. do homeomorphic topological spaces have the same dimension? A point is obviously not homeomorphic to a line, a line is not homeomorphic to a plane, but how about a plane and the 3-dimensional Euclidean space? This problem resisted all attacks at the beginning of the twentieth century and needed a breakthrough. That came from Luitzen Egbertus Jan Brouwer (1881–1966).

A good definition of dimension was not available at that time, although already in 1843–1844, Bolzano [19] wrote about it. His paper was published about 100 years after it was written, so it did not have a strong impact. Actually, Bolzano's definition was never used, but it is in essence strongly related to the one that we are used to today. After a number of attempts by various authors, the first serious attempt came from Poincaré (1854–1912) in [84]. Unfortunately, Poincaré died soon after the publication of his article, and so he could not continue his line of thinking. In 1913, in his famous paper '*Über den natürlichen Dimensionsbegriff*' [21], a translated version of which appears in this volume, building on the ideas of Poincaré, Brouwer presented the first formal definition of dimension for a very wide class of topological spaces. The definition was given in purely topological terms, so that it was immediately clear that the dimension is preserved by homeomorphisms. He called this function the *dimensionsgrad* of a topological space and proved that on \mathbb{R}^n , it takes the value n . Hence a plane is indeed not homeomorphic to 3-dimensional Euclidean space, but much more is true. These results and the techniques Brouwer used shocked the mathematical world of his days.

Definitions of dimension also surfaced in the work of Pavel Samuilovich Urysohn (1898–1924) and Karl Menger (1902–1985), see [106] and [68], who defined the so-called *small inductive dimension function* ind . A heated discussion arose between Brouwer and Menger concerning priority in defining the notion of dimension. But not so between Brouwer and Urysohn who right before his death in 1924 together with his friend and co-author Pavel Sergejevich Alexandroff (1896–1982) visited Brouwer in his hometown Blaricum.

The story about their visit was told by many authors. Less well-known is that it was also told by Alexandroff himself, 42 years after the meeting in Blaricum, at the time he was one of the world's leading mathematicians himself. The occasion was his visit to the Netherlands in 1968, shortly after Brouwer's death, during a lecture at Utrecht University about the role of Dutch Topology in the period 1920–1930. In that lecture he discusses at length his visit to Brouwer. The text of the lecture was published in *Nieuw Archief voor Wiskunde* [2] and we take the liberty of repeating here some of what Alexandroff said about Brouwer.

Alexandroff writes that the volume of the journal with Brouwer's article '*Über den natürlichen Dimensionsbegriff*' [21] got into the hands of Urysohn in the summer of 1923, when Urysohn and he arrived in Göttingen. After Urysohn read Brouwer's article, a lively correspondence started between them. Among other things, Urysohn informed Brouwer about his results on dimension theory obtained in the winter of 1921–1922. Brouwer was delighted by Urysohn's contributions, he fully appreciated the significance of his results. Urysohn also brought to Brouwer's attention a small discrepancy in how Brouwer formulated his definition of dimension. The discrepancy relates to the definition of the concept of separating two sets, which lies in the foundation of the concept of dimension. The correspondence between Brouwer and Urysohn was devoted not only to dimension theory, but also to many questions in general topology which constituted the essence of Alexandroff and Urysohn's *Mémoire sur les espaces topologiques compacts* [4], the manuscript of which already existed in the summer of 1923. Brouwer invited Alexandroff and Urysohn to visit him in Blaricum. The visit took place in the middle of July 1924 and lasted approximately one week (Fig. 1). It was decided that



Fig. 1. Paul Alexandroff, Bertus Brouwer en Paul Urysohn.

Alexandroff and Urysohn would visit Brouwer again in the spring of 1925 for an extended period of time—approximately for one year. With these plans in mind, Alexandroff and Urysohn left the Netherlands somewhere between July 20 and July 25 and went to France. Only a month later, on August 17, Urysohn died near Bourg de Batz in Bretagne while swimming in the ocean.

At the beginning of May 1925, Alexandroff visited Brouwer again, this time alone. Some time later Menger and Vietoris joined them, along with the young Englishman Wilfred Wilson. The mathematical life in Blaricum in the winter 1925–1926 was very intensive and even more so when in the middle of December Emmy Noether arrived there to take part in the discussions.

In Amsterdam, Alexandroff spent a lot of time preparing several of Urysohn's unfinished papers for publication. He was very grateful for the excellent support he got from Brouwer. For example, Brouwer read the papers and checked all the proofs. According to Alexandroff, his work on the mathematical heritage of Urysohn was in fact joint work with Brouwer.

Alexandroff discusses many of Brouwer's contributions to mathematics in his paper [2]. We concentrate solely on dimension theory here. In the period that lasted from 1909 until 1913, the theme was Brouwer's method of simplicial approximation and his notion of the degree of a mapping. Using his method, Brouwer made his most fundamental discoveries in topology. Of these, Alexandroff mentions the following ones related to dimension theory: the invariance of domain theorem (1910–1911), the fixed-point theorem (1911), and the first formal definition of dimension (1913). He speaks about Brouwer's breathtaking geometric intuition which he blended in his mind with powerful set-theoretic thinking and set-theoretic imagination. Many of Brouwer's results on dimension theory of this period can be found in the book by Alexandroff and Pasynkov [3].

Dimension theory is still a beautiful and vital area of mathematics today. For a survey on the dimension theory of compact metrizable spaces in this volume, see Dranishnikov [34]. In this

paper we will concentrate on the dimension theory of topological groups. For locally compact topological groups there are many classical results in the literature. But dimension theory for non-locally compact topological groups is mysterious, as we will demonstrate here. Many fundamental problems remain unsolved for decade after decade. It is clear that a breakthrough is needed, and we hope that the spirit of Brouwer will make this possible.

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2. Selected equality theorems for classical dimension functions on topological groups

It is well-known, and easy to prove, that in every compact space quasi-components and components coincide. This yields that if X is any non-empty locally compact hereditarily disconnected space X , then $\text{ind } X = 0$ (Engelking [38, Theorem 1.4.5]). Here a space is called *hereditarily disconnected* if it does not contain any connected subspace of size greater than 1.

For locally compact topological groups, the three basic dimension functions dim , ind and Ind take the same values, as was shown by Pasyukov [81] in 1960. Earlier, Arhangel'skii [8] had already shown equivalence of ind and dim . The proof is based on the fact that every locally compact topological group is strongly paracompact, which means that every open cover of the group can be refined by a star-finite open cover. Even in the class of metrizable spaces there are examples of spaces M having the property that $\text{ind } M \neq \text{dim } M$. The first, very complicated, example of such a space was constructed by Roy [87] in 1962. But in the class of all locally compact topological groups such examples do not exist. Later we will come back to the question whether in the class of all topological groups the three basic dimension functions also take the same values.

Theorem 2.1. *Let G be a locally compact group. Then*

- (a) $\text{dim } G = 0$ iff G is hereditarily disconnected.
- (b) If G is not hereditarily disconnected, i.e. if the component of the identity G_0 is not trivial, then $\text{dim } G \geq n \geq 1$ iff G contains a copy of the cube \mathbb{I}^n .

This result is folklore. For compact groups, there are explicit references for this result, but for locally compact groups we were unable to find any. We are indebted to Karl Hofmann for providing us with the necessary details and references.

A topological group is called a *pro-Lie group* if it is topologically isomorphic to a closed subgroup of a product of finite-dimensional real Lie groups.

A topological group G is *almost connected* if the factor group G/G_0 (here G_0 denotes the identity component of G) is compact. So compact groups and connected groups are almost connected.

Proof of Theorem 2.1(a). Clearly, G is hereditarily disconnected iff $\text{ind } G = 0$. And $\text{ind } G = 0$ iff $\text{dim } G = 0$ by Arhangel'skii's result just quoted. \square

Proof of Theorem 2.1(b). If G contains a copy of \mathbb{I}^n , then clearly $\text{dim } G \geq n$.

For the converse, first observe that by Montgomery and Zippin [70, Lemma 2.3.1], G has an open (hence closed) subgroup U which is almost connected. Since every locally compact almost connected group is a pro-Lie group (Yamabe [113,114]), we have that U is an almost connected

pro-Lie group. Hence by Hofmann and Morris [48, Theorem 8.4], U contains a maximal compact normal subgroup C , and a subspace E homeomorphic to \mathbb{R}^m for some $m \in \mathbb{N}$ such that the map

$$(e, c) \mapsto ec : E \times C \rightarrow U$$

is a homeomorphism. Now C is homeomorphic to $C_0 \times C/C_0$, where C_0 is the component of the identity of C (Hofmann and Morris [46, Corollary 10.38]). Observe that C/C_0 is zero-dimensional. Hence U is homeomorphic to a product of the form $\mathbb{R}^m \times K \times Z$, where K is a compact connected group, and Z is a compact zero-dimensional space. As a consequence,

$$\dim U = m + \dim K + 0 = m + \dim K = m + p,$$

where $p = \dim K$ (Hurewicz [50]). Hence if $\dim K = 0$, then $\dim U = m \geq n$, and so we are done. If $p = \dim K > 0$, then K contains a copy of \mathbb{I}^p by Hofmann and Morris [46, Proposition 9.56], and so U contains a copy of $\mathbb{I}^m \times \mathbb{I}^p \approx \mathbb{I}^{m+p}$. There is a discrete space D such that G and $U \times D$ are homeomorphic, simply because U is a clopen subgroup of G . Hence $\dim G = \dim U$, and so we are done. \square

So for finite-dimensional locally compact topological groups the picture is clear from the perspective of dimension theory.

Theorem 2.1(b) can be generalized for infinite-dimensional locally compact groups in the same way by using the fact that every compact infinite-dimensional group contains a copy of the Hilbert cube Q (Hofmann and Morris [47]).

If we try to generalize the above classical results for locally compact groups for a wider class of groups, we immediately run into problems. Let us say that a space X is *totally disconnected* if for all distinct $x, y \in X$ there is a clopen subset C of X such that $x \in C$ and $y \in X \setminus C$. It is clear that every totally disconnected space is hereditarily disconnected. The converse is not true, however, even for subsets of the plane (Engelking [38, Example 1.4.7]).

Let E denote the subspace of Hilbert space ℓ^2 consisting of all points having the property that all their coordinates are rational. This is the famed Erdős space from [39], and it is known to be both totally disconnected and 1-dimensional. Since it is clearly a subgroup of ℓ^2 , we see that **Theorem 2.1(a)** cannot be generalized to arbitrary topological groups. Let E_c denote the subspace of Hilbert space ℓ^2 consisting of all points having the property that all their coordinates are irrational. This is a Polish space, and is totally disconnected and 1-dimensional as well. It was shown to be a (Boolean) topological group in Dijkstra, van Mill and Steprāns [28, Proposition 4.3]. So even for Polish groups, **Theorem 2.1(a)** cannot be generalized.

Since there exist totally disconnected 1-dimensional groups, the question naturally arises whether there are totally disconnected n -dimensional groups for every n . This was answered in the affirmative by van Mill [109]; these examples are separable and metrizable but not complete (they are Borel, though). This prompts the following open problem.

Question 2.2. Are there for every $n \geq 1$ examples of totally disconnected n -dimensional Polish groups?

Now let us turn our attention to **Theorem 2.1(b)**. Even for 1-dimensional Polish groups, there are fundamental open problems.

Question 2.3 (Dobrowolski). Let G be a nontrivial connected Polish group. Does G contain a copy of \mathbb{I} ?

Let G be a topological group with a closed subgroup H . If $x, y \in G$ and $xH \cap yH \neq \emptyset$ then $xH = yH$. Hence the collection of all *left cosets* $G/H = \{xH : x \in G\}$ is a partition of G into closed sets. Let $\pi : G \rightarrow G/H$ be defined by $\pi(x) = xH$. We endow G/H by the quotient topology. In other words, if $A \subseteq G$ then $\{xH : x \in A\}$ is open in G/H if and only if $\bigcup\{xH : x \in A\} = AH$ is open in G .

A space X is a *coset space* provided that there is a closed subgroup H of a topological group G such that X and G/H are homeomorphic.

It was shown by Ungar [104] that every locally compact separable metrizable and homogeneous space is a coset space. His proof was based on the Effros Theorem from [36]. Pasynkov [82] showed that if Y is a compact coset space of some locally compact group, then the dimension functions \dim , ind and Ind take the same values on Y . Hence the dimension theory of coset spaces is promising. See Hofmann and Morris [47] for the foundation of a cardinal-valued dimension theory for coset spaces of the form G/H , where G is compact. This of course includes the dimension theory of compact groups.

It is not true that all homogeneous compact spaces are coset spaces of topological groups. Fedorchuk [40] constructed a homogeneous compactum X such that $\dim X = 1 < \text{ind } X = 2$. Hence by the result of Pasynkov just quoted, X is not a coset space of any locally compact group. In fact, it is known that X is not a coset space of any topological group whatsoever.

The following fundamental problem is still open [89]:

Question 2.4. Suppose that X is a compact coset space. Is $\text{ind } X = \dim X = \text{Ind } X$?

One of the fundamental theorems of classical dimension theory belongs to Tumarin and Hurewicz. It says that $\dim X = \text{ind } X = \text{Ind } X$, for every separable metrizable space X . Arhangel'skii asked in this connection whether these equalities hold for every space with a countable network [9]. In the special case that X is the union of a finite collection of separable metrizable subspaces, the three classical dimension functions indeed agree on X , as was shown by Charalambous [23]. But he also showed that in general, the answer to Arhangel'skii's question is in the negative. This situation motivates the next question which was posed in [11, Problem 6.12] and is still open.

Question 2.5. Suppose that G is a topological group with a countable network. Is it true that $\dim X = \text{ind } X = \text{Ind } X$?

It is known that for a positive answer, it is enough to show that $\text{ind } G \leq \dim G$.

Question 2.6. Suppose that X is a topological space which is a quotient space of a separable metrizable space. Is it true that $\dim X = \text{ind } X = \text{Ind } X$?

Question 2.7. Suppose that X is a coset space of a separable metrizable topological group. Is it true that $\dim X = \text{ind } X = \text{Ind } X$?

In 1961, Sklyarenko and Smirnov asked in [96] whether the three classical dimension functions coincide for every normal topological group. This was answered in the negative by Shakhmatov [89] in 1989: he constructed the first example of a normal topological group with non-coinciding dimensions. Recall that the class of *Lindelöf Σ -spaces* is the smallest class of (Tychonoff) spaces which contains all separable metrizable spaces, all compact Hausdorff spaces, and is closed under continuous images, closed subspaces and finite products. Shakhmatov constructed in [89] for each natural number n an example of a topological group G_n which is

precompact, Abelian, Lindelöf Σ -space and satisfies the following conditions: $\dim G_n = n$ and the other two dimensions $\text{ind } G_n$ and $\text{ind } G_n$ are infinite. It was also established by Shakhmatov in the same paper that $\text{ind } G = \text{Ind } G$ for every topological group G which is a Lindelöf Σ -space. In particular, it follows that $\text{ind } G = \text{Ind } G$ for every σ -compact topological group G . He asked in [88] whether in fact for every σ -compact topological group all three dimension functions take the same value. Another question he posed in [88, Question 1.6] is: does the equality $\text{ind } G = \text{Ind } G$ hold for every Lindelöf topological group G ?

A topological group G is called ω -narrow if for every open neighborhood V of the neutral element in G , there exists a countable subset A of G such that $AV = G$. A topological group G is called \mathbb{R} -factorizable if, for every continuous real-valued function f on G , there exist a continuous homomorphism $\pi : G \rightarrow K$ onto a second countable topological group K and a continuous real-valued function h on K such that $f = h \circ \pi$.

Question 2.8 (Shakhmatov [88]). Is it true that $\dim G \leq \text{ind } G$, for every precompact (\mathbb{R} -factorizable, ω -narrow) topological group G ?

Theorem 2.9 (Shakhmatov [88]). *If G is an \mathbb{R} -factorizable topological group, then the conditions $\text{ind } G = 0$ and $\dim G = 0$ are equivalent.*

In particular, it follows from this theorem that the conditions $\text{ind } G = 0$ and $\dim G = 0$ are equivalent for topological groups with a countable network and for subgroups of σ -compact topological groups. It also follows that the equivalence of $\text{ind } G = 0$ and $\dim G = 0$ holds for precompact topological groups and for arbitrary subgroups of Lindelöf Σ -groups.

3. Homeomorphism groups of metrizable compacta

A particularly nice class of topological groups with many questions unanswered is the class $\mathcal{H}(X)$ of homeomorphism groups of metrizable compacta X . Of course, $\mathcal{H}(X)$ is endowed with the compact-open topology, which is in this case equivalent to the topology of uniform convergence. If (X, ρ) is a compact metric space, then the formula

$$\hat{\rho}(f, g) = \max_{x \in X} \rho(f(x), g(x))$$

is a metric compatible with the topology on $\mathcal{H}(X)$, and

$$\sigma(f, g) = \hat{\rho}(f, g) + \hat{\rho}(f^{-1}, g^{-1})$$

is a compatible *complete* metric. Since it is easy to show that $\mathcal{H}(X)$ is separable, we conclude that $\mathcal{H}(X)$ is a Polish group.

If X is locally compact but not compact, then $\mathcal{H}(X)$ with the compact-open topology need not be a topological group. It is a classical result of Arens [7] that if X is moreover locally connected, then $\mathcal{H}(X)$ is a topological group. Arens' condition was relaxed by Dijkstra [25] to the condition that every point in X has a neighborhood that is a continuum. This prompts the following problem:

Question 3.1. For which locally compact separable metrizable spaces X is it true that $\mathcal{H}(X)$ endowed with the compact-open topology is a topological group?

Let us consider the group $\mathcal{H}(\mathbb{I})$. If $f \in \mathcal{H}(\mathbb{I})$ then clearly $f(\{0, 1\}) = \{0, 1\}$. A moments reflection shows that f is either (strictly) increasing or (strictly) decreasing. If f is increasing, and so $f(0) = 0$ and $f(1) = 1$, then

$$\{g \in \mathcal{H}(\mathbb{I}) : \hat{\varrho}(f, g) < 1/2\}$$

consists entirely of increasing homeomorphisms. Hence the collection of all increasing homeomorphisms is open, and so is the collection of all decreasing homeomorphisms. We conclude that

$$\mathcal{H}_0(\mathbb{I}) = \{f \in \mathcal{H}(\mathbb{I}) : f \text{ is increasing}\}$$

is a clopen subset of $\mathcal{H}(\mathbb{I})$. Clearly, $\mathcal{H}_0(\mathbb{I}) \approx \mathcal{H}(\mathbb{I}) \setminus \mathcal{H}_0(\mathbb{I})$.

Example 3.2. $\mathcal{H}_0(\mathbb{I}) \approx \mathbb{R}^\infty$ and $\mathcal{H}(\mathbb{I}) \approx \{0, 1\} \times \mathbb{R}^\infty$.

This is due to Anderson [5]. The proof presented here was taken from Keesling [54], who writes on Page 5, line -11: ‘*The following proof was communicated to me by James West who heard it from R. Connelly who heard it from Morton Brown*’.

Proof. We will show that $\mathcal{H}_0(\mathbb{I}) \approx \prod_{n=0}^\infty \prod_{i=1}^{2^n} (0, 1)_{n,i}$. Let

$$(x_{n,i}) \in \prod_{n=0}^\infty \prod_{i=1}^{2^n} (0, 1)_{n,i}$$

be arbitrary. We will define an increasing homeomorphism h of \mathbb{I} associated with $(x_{n,i})$. Suppose that n is given and that we have defined sets

$$A_n = \{0 = \alpha_0^n < \alpha_1^n < \dots < \alpha_{2^n}^n = 1\}$$

and

$$B_n = \{0 = \beta_0^n < \beta_1^n < \dots < \beta_{2^n}^n = 1\}$$

and a ‘partial’ homeomorphism h such that $h(\alpha_i^n) = \beta_i^n$ for $i = 0, 1, \dots, 2^n$. We extend h to a set $A_{n+1} \supseteq A_n$ onto $B_{n+1} \supseteq B_n$ with each of A_{n+1} and B_{n+1} having $2^{n+1} + 1$ points. If n is odd, then let z_i be the midpoint of the interval $[\alpha_{i-1}^n, \alpha_i^n]$ for $i = 1, 2, \dots, 2^n$, and let

$$y_i = h(z_i) = x_{n,i}(\beta_i^n - \beta_{i-1}^n) + \beta_{i-1}^n.$$

If n is even then let y_i be the midpoint of the interval $[\beta_{i-1}^n, \beta_i^n]$ for $i = 1, 2, \dots, 2^n$, and let

$$z_i = h^{-1}(y_i) = x_{n,i}(\alpha_i^n - \alpha_{i-1}^n) + \alpha_{i-1}^n.$$

Put

$$A_{n+1} = A_n \cup \{z_i : i = 1, \dots, 2^n\}$$

and

$$B_{n+1} = B_n \cup \{y_i : i = 1, \dots, 2^n\},$$

respectively.

Put $A = \bigcup_{n=1}^\infty A_n$, and $B = \bigcup_{n=1}^\infty B_n$. Then both A and B are dense in \mathbb{I} , and $h : A \rightarrow B$ is an order preserving bijection. Thus h has an order preserving extension $\bar{h} : \mathbb{I} \rightarrow \mathbb{I}$. It is not hard to prove that the assignment $(x_{n,i}) \rightarrow \bar{h}$ is a homeomorphism between $\prod_{n=0}^\infty \prod_{i=1}^{2^n} (0, 1)_{n,i}$ and $\mathcal{H}_0(\mathbb{I})$. \square

The Anderson Theorem states that \mathbb{R}^∞ and ℓ^2 are homeomorphic. See van Mill [110, Chapter 6] for details. So Example 3.2 implies that $\mathcal{H}_0(\mathbb{I}) \approx \ell^2$ and $\mathcal{H}(\mathbb{I}) \approx \{0, 1\} \times \ell^2$.

The question naturally arises what can be said about the homeomorphism groups $\mathcal{H}(\mathbb{I}^n)$ for $n \geq 2$. In fact, it is natural to think of the following closed subgroup of $\mathcal{H}(\mathbb{I}^n)$:

$$\mathcal{H}_\partial(\mathbb{I}^n) = \{h \in \mathcal{H}(\mathbb{I}^n) : h \upharpoonright \partial\mathbb{I}^n = 1_{\partial\mathbb{I}^n}\};$$

here $\partial\mathbb{I}^n$ is the union of the endfaces of \mathbb{I}^n . It was shown by Luke and Mason [66] that $\mathcal{H}_\partial(\mathbb{I}^2)$ is an AR, which implies that $\mathcal{H}_\partial(\mathbb{I}^2) \approx \ell^2$ (apply e.g., Dobrowolski and Toruńczyk [33]).

Question 3.3. Let $n \geq 3$. Is $\mathcal{H}_\partial(\mathbb{I}^n)$ an AR?

This is widely open and is one of the most interesting open problems in infinite-dimensional topology.

For $n = \infty$, the analogous problem was solved (observe that Q has no boundary).

Example 3.4. $\mathcal{H}(Q) \approx \ell^2$.

The proof of this is difficult. See Ferry [41] and Toruńczyk [103] for details.

Hence we see that already for a relatively simple space such as the closed unit interval \mathbb{I} its homeomorphism group $\mathcal{H}(\mathbb{I})$ is quite large (from almost every perspective, including the dimension theoretic one). If C denotes the Cantor set $\{0, 1\}^\omega$, then $\mathcal{H}(C)$ is easily seen to be zero-dimensional. In fact, $\mathcal{H}(C)$ is homeomorphic to the space of irrational numbers \mathbb{P} . Are there interesting spaces X for which $\mathcal{H}(X)$ is 1-dimensional, or, more generally, n -dimensional? The answer to this naive problem is not so simple.

A space X is called *almost zero-dimensional* if it has an open base \mathcal{B} such that every $B \in \mathcal{B}$ has the property that $X \setminus \overline{B}$ is the union of clopen subsets of X .

Almost zero-dimensional spaces were introduced by Oversteegen and Tymchatyn [80]. They proved that almost zero-dimensional spaces are at most 1-dimensional, and used this result to conclude that the homeomorphism groups of various interesting spaces such as Sierpiński's Carpet and Menger's Universal Curve, are at most 1-dimensional. For a simpler proof that almost zero-dimensional spaces are at most 1-dimensional, see Levin and Pol [63].

Dijkstra [26] proved that the homeomorphism group of the n -dimensional Sierpiński carpet M_n^{n+1} for $n \neq 3$ is at least 1-dimensional. And also that the homeomorphism group of the n -dimensional universal Menger continuum μ^n is at least 1-dimensional. He in fact proved that the complete Erdős space E_c can be embedded in these groups, which is of independent interest.

The universal Menger continua μ^n were characterized topologically by Bestvina [15] who also proved that they are homogeneous. Hence there exist homogeneous metrizable continua of arbitrarily large dimension whose homeomorphism groups are 1-dimensional. This prompts the following open problem, basically due to Brechner [20].

Question 3.5. Is there a homogeneous metrizable continuum X such that $\dim \mathcal{H}(X) = 0$? And is there a homogeneous metrizable continuum X such that $1 < \dim \mathcal{H}(X) < \infty$? Specifically, is there a homogeneous metrizable continuum X such that $\mathcal{H}(X)$ is of dimension 2?

If $\mathcal{H}(X)$ is locally compact and X is metrizable, then $\mathcal{H}(X)$ is zero-dimensional, by a result of Keesling [53]. See also Hofmann and Morris [49] for a related result: if the homeomorphism group $\mathcal{H}(X)$ of a (Tychonoff) space X is compact, then it is a profinite topological group (hence zero-dimensional).

A classical object in continuum theory is the so-called *pseudo-arc* P . It is planar and Bing showed in [16] that it is homogeneous. He proved moreover in [17] that each arc-like hereditarily indecomposable continuum is homeomorphic to the pseudo-arc. As was shown recently by Hoehn and Oversteegen [45], there are up to homeomorphism exactly 3 nondegenerate homogeneous plane continua: the circle, the pseudo-arc and the circle of pseudo-arcs (Bing and Jones [18]). That solved a problem that traces its history back to the 1920 paper of Knaster and Kuratowski [56].

It is unknown what the dimension is of $\mathcal{H}(P)$. The only thing that is known is that it does not contain nontrivial continua (Lewis [64]).

Question 3.6. What is the dimension of $\mathcal{H}(P)$, where P denotes the pseudo-arc?

4. Some new dimension theorems for metrizable groups obtained by O. V. Sipacheva

The Katětov-Morita Theorem [51,71] says that $\dim X = \text{Ind } X$, for any metrizable space X . On the other hand, there are metrizable spaces X with $\text{ind } X \neq \dim X$. We already mentioned that the first, very complicated, example of such a space was constructed by Roy [87] in 1962. Much simpler examples of metrizable spaces witnessing the same phenomenon have been found later (see, e.g., [60]). All these examples have small inductive dimension zero. See the very interesting work of Mrowka [73,74] and Kulesza [61] for the troubles one gets into when trying to get similar examples of larger small inductive dimension.

Miščenko [69] asked in 1964 whether the three classical dimensions coincide for metrizable topological groups. This natural question remained open for a long time and some versions of it were repeated by various authors.

Only in 2008, Miščenko's question was answered in a remarkable paper of Sipacheva [94]. On the way to the solution, she obtained a few additional interesting results of independent interest. First, she studied when a space X can be embedded as a closed subspace into a metrizable zero-dimensional topological group, and proved the following statement:

Theorem 4.1. *A space X can be embedded in a metrizable topological group G with $\text{ind } G = 0$ if and only if the topology of X is generated by a uniformity which has a countable base consisting of open-and-closed sets. Moreover, if X can be embedded in a zero-dimensional metrizable group, then it can be embedded in such a group as a closed subspace.*

The proof of this theorem is quite involved. It is given in Section 1 of [94] and occupies, approximately, half of this paper. Sipacheva mentions in [94] that Theorem 4.1 (without the closedness assertion) was formulated by Miščenko [69], but its proof has never been published.

The next step in Sipacheva's strategy is to use the above criterion to embed an appropriate metrizable space with non-coinciding dimensions ind and Ind in a zero-dimensional metrizable group. She describes here a space (a special case of Mrowka's space in [73]) which can be embedded as a closed subspace in a zero-dimensional metrizable group but is not strongly zero-dimensional. This gives an example of a metrizable group with noncoinciding dimensions ind and Ind . Thus, the next fact is established:

Corollary 4.2. *There exists a metrizable topological group G with $\dim G = \text{Ind } G > \text{ind } G = 0$.*

The last statement answers some questions of Shakhmatov in [88] (see, in particular, Question 1.26). It shows that Pasynkov's theorem on the coincidence of the three classical

dimensions for locally compact topological groups cannot be extended to all topological groups which are paracompact p -spaces. This is so, since metrizable spaces are paracompact p . However, we have the following generalization of the Katětov-Morita Theorem which is due to Pasynkov [83]:

Theorem 4.3. *If a topological group G is a paracompact p -space, then $\dim G = \text{Ind } G$.*

The next open problem, which is a part of Question 1.27 in [88], becomes especially interesting in the context of Sipacheva's results discussed above.

Question 4.4. Is it true that $\text{ind } G = \dim G = \text{Ind } G$ for any Čech-complete topological group?

Even in the metric case this problem is not solved.

Question 4.5. Is it true that $\text{ind } G = \dim G = \text{Ind } G$ for any topological group which is completely metrizable?

Question 4.6. Suppose that $X = G/H$, where G is a completely metrizable topological group, and H is a closed subgroup of G . Is it true that $\text{ind } X = \dim X = \text{Ind } X$?

A remarkable generalization of the dimension coincidence theorem for locally compact topological groups was given by Tkachenko (see [101]):

Theorem 4.7. *Suppose that H is a closed subgroup of a locally pseudocompact topological group G . Then:*

- (a) $\dim G = \text{ind } G = \text{Ind } G$;
- (b) $\dim G = \dim H + \dim(G/H)$ (and hence, $\dim(G/H) \leq \dim G$).

5. Embeddings in topological groups and dimension

It is well-known that every topological space X can be represented as a closed subspace of a topological group (Markov's Theorem, see [12, Chapter 9]). A very natural question is: can we achieve in Markov's Theorem that the dimension of G is the same as the dimension of X , where dimension is understood as one of the classical dimensions? In the preceding section, we have already touched upon the topic in the title of this section. Note also that the third section of [94] contains an example of a zero-dimensional metrizable space which cannot be embedded in a zero-dimensional metrizable group. In this section, we consider situations of this kind in a more systematic way.

In 1978, Bel'nov [14] proved that every space X can be represented as a closed subspace of a homogeneous space Y with $\text{ind } X = \text{ind } Y$, $\dim Y = \dim X$, and $\text{Ind } X = \text{Ind } Y$. This led him to ask whether every space X can be embedded in a topological group G such that $\dim G = \dim X$. Answering this question, Shakhmatov showed, in particular, that the sphere \mathbb{S}^2 cannot be embedded in a 2-dimensional topological group, no matter which classical dimension function is used. For his general result and the proofs, see [91]. See also the article of Kato [52] in which it is shown that Shakhmatov's results about the spheres \mathbb{S}^n are a special case of a remarkable general theorem on manifolds:

Theorem 5.1 ([52]). *A compact n -dimensional manifold M^n without boundary can be embedded in an n -dimensional topological group if and only if M^n itself is homeomorphic to a topological group.*

Every zero-dimensional space (in the sense of \dim) can be embedded as a closed subspace of a topological group G with $\dim G = 0$ [91]. Kimura [55] showed that the 1-dimensional bouquet of two circumferences cannot be embedded in a 1-dimensional topological group. This is a counterexample to Bel'nov's question in the 1-dimensional case.

Vopěnka [111] constructed a compact space X such that $\dim X$ is finite and $\text{ind } X$ is infinite. It is unknown whether this space X can be embedded in a topological group G such that $\dim X = \dim G$, [55].

Kulesza constructed in [59], for every $n \geq 1$, an example of a compact n -dimensional separable metrizable space which does not embed in an n -dimensional topological group. The same paper contains the following result: the Kowalskij hedgehog with ω_1 many "spines" J_{ω_1} is a 1-dimensional metric space such that every topological group, which contains it as a subspace, is infinite-dimensional.

6. Some natural questions on subgroups and products of groups

It is easily proved by induction that the small inductive dimension ind is monotone with respect to subspaces: if $Y \subseteq X$, then $\text{ind } Y \leq \text{ind } X$. However, for the dimension functions \dim and Ind , the same is not true. Indeed, there exist a compact topological group G having a subspace Y such that $\dim G < \dim Y$. Even in hereditarily normal spaces the dimensions \dim and Ind need not be monotone. The first to show this was Filippov. He proved in [42] that if there exists a Souslin continuum, then there exists a zero-dimensional hereditarily normal space X , which contains subspaces with any prescribed finite dimension \dim or Ind . See also Pol and Pol [85] for a similar result in ZFC: there exists a hereditarily normal space X with $\dim X = \text{Ind } X = 0$ such that, for every $n \in \omega$, there exists a subspace A_n of X with $\dim A_n = \text{Ind } A_n = n$. It is also known [91] that every Tychonoff space X such that $\text{ind } X = 0$ can be represented as a closed subspace of an Abelian topological group G such that $\dim G = 0$. It follows easily from this that the dimension \dim of topological groups is not monotonous with respect to closed subspaces.

Shakhmatov [88, Problem 2.1] stated in 1989 that the next general question is open:

Question 6.1. Suppose that H is a subgroup of a topological group G . Is it true that $\dim H \leq \dim G$?

So far as we know, this question remains open until now. Tkachenko in [101, Problem 6.9] also mentions this problem and especially emphasizes the case when G is normal and H is closed in G . Shakhmatov collected in [88] many concrete cases in which the above inequality holds. We mention some of them.

Theorem 6.2. *Suppose that H is a precompact subgroup of a topological group G . Then $\dim H \leq \dim G$.*

In particular, the inequality holds whenever the group G is precompact. Shakhmatov also showed that the subgroup dimension inequality holds if the group G is Lindelöf or locally pseudocompact (Theorems 2.4 and 2.5 in [88]). Hence, the inequality holds when H is a subgroup of a σ -compact topological group G .

Question 6.3 (Shakhmatov [88]). Suppose that H is an ω -narrow subgroup of a topological group G . Is it true that then $\dim H \leq \dim G$?

A positive answer to this question would, in our opinion, be an amazing general result.

An important subclass of the class of all ω -narrow topological groups is the class of so-called \mathbb{R} -factorizable topological groups (see the paragraph preceding Question 2.8). In [88, Theorem 2.6], Shakhmatov attributes the following result to Tkachenko:

Theorem 6.4. *If H is an \mathbb{R} -factorizable subgroup of a topological group G , then the inequality $\dim H \leq \dim G$ holds.*

Indeed, we find this theorem in [102] and also in [101, Theorem 6.11]. In the second of these two articles, while discussing Theorem 6.11, Tkachenko mentions that it can be also proved using the fact that every \mathbb{R} -factorizable subgroup is z -embedded in the ambient group and applying a result of Chigogidze saying that \dim is monotonous with respect to z -embedded subspaces.

In connection with Problem 6.3, it is natural to pose the following question:

Question 6.5. Given an arbitrary $n \in \omega$, does there exist a separable metrizable topological group G_n such that $\dim G_n = n$ and each separable metrizable n -dimensional topological group H is topologically isomorphic to a topological subgroup of G_n ?

Recall that a universal separable metrizable topological group with respect to topological monomorphisms was constructed by Uspenskij [107] (see Shkarin [93] for the Abelian case).

A quite nontrivial situation also occurs with respect to the natural product inequality in dimension theory. In 1930, Pontryagin [86] constructed 2-dimensional compact metrizable spaces X and Y such that

$$\dim(X \times Y) = 3 < \dim X + \dim Y = 4.$$

Question 6.6. Is it possible to embed $X \times Y$ in the product of three 1-dimensional metrizable compacta?

In this connection, we note that, for every $n \in \mathbb{N}$, the Euclidean space \mathbb{R}^n has an $(n - 1)$ -dimensional subspace M_n such that $\dim(M_n)^k = n - 1$, for every positive integer k , Anderson and Keisler [6] (see Kulesza [58] for similar examples that are even Polish). This statement can be strengthened as follows:

Theorem 6.7. *For every $n \in \mathbb{N}$, there exists a separable metrizable topological group G_n such that $\dim(G_n)^k = n - 1$, for every positive integer k .*

Question 6.8. Given any $n \in \omega$, does there exist a Polish topological group G_n such that $\dim(G_n)^k = n - 1$, for every positive integer k ?

The inequality $\dim(X \times Y) \leq \dim X + \dim Y$ (which we call below *the sum-product inequality*) holds for compact spaces and for metrizable spaces. It also holds for paracompact p -spaces (see [79], where it is shown that the finite products of such spaces have a certain special structure which guarantees that the inequality holds). It even holds for paracompact Σ -spaces (see [57]). The question of whether the sum-product inequality holds for all paracompact spaces is still open.

However, spaces satisfying the opposite inequality $\dim(X \times Y) > \dim X + \dim Y$ do exist, [112]. The following problem can be found in print in [88]:

Question 6.9. Does the inequality $\dim(G \times H) \leq \dim G + \dim H$ hold for arbitrary topological groups G and H ? What if the factors are equal?

The answer to this question is not known even if the groups are Lindelöf.

Theorem 6.10 ([88, Theorem 3.3]). *If G and H are precompact topological groups, then $\dim(G \times H) \leq \dim G + \dim H$.*

In connection with this result, the following two questions seem to be in order.

Question 6.11. Does the inequality $\dim(G \times H) \leq \dim G + \dim H$ hold for arbitrary topological groups G and H which are subgroups of σ -compact topological groups?

Observe that if both G and H are σ -compact, then the answer to the last question is “yes”, since it is in the affirmative for all Lindelöf Σ -groups.

Question 6.12. Does the inequality $\dim(G \times H) \leq \dim G + \dim H$ hold for arbitrary topological groups G and H with countable Souslin number?

Question 6.13. Does the inequality $\dim(G \times H) \leq \dim G + \dim H$ hold for arbitrary Lindelöf topological groups G and H with countable Souslin number?

A much more general situation is covered by the next question. Thus, a positive answer to it would be, in our opinion, an important theorem.

Question 6.14 ([88, Theorem 3.4]). Does the inequality $\dim(G \times H) \leq \dim G + \dim H$ hold for arbitrary ω -narrow topological groups G and H ?

We would like also to mention one more remarkable result of Nagata [75,76]: any n -dimensional metric space can be topologically embedded in a topological product of $n + 1$ metric spaces that are all 1-dimensional.

Question 6.15. Is it possible to embed any n -dimensional (possibly, non-metrizable) compactum in a product of $n + 1$ 1-dimensional compacta?

Some deep results on the dimension of product spaces, which are relevant to the results we are discussing in this section, can be found in Morita’s paper [72]. In particular, it is proved there that if X is a normal space and Y is a locally compact paracompact space, then $\dim(X \times Y) \leq \dim(X) + \dim(Y)$. Furthermore, if, in addition, Y is a polyhedron, then, in the preceding inequality, equality holds. Hence, if X is a topological space such that the product space $X \times \mathbb{I}$, where \mathbb{I} is the closed unit interval, is homeomorphic to a subspace of the Euclidean space \mathbb{R}^3 , then $\dim(X) = 2$.

Several basic questions of this kind are open for the other classical dimension functions. Does the inequality $\text{Ind}(G \times H) \leq \text{Ind } G + \text{Ind } H$ hold for arbitrary topological groups G and H ? This natural general question was also posed by Shakhmatov in [88, Question 3.10]. He observed that even in the precompact case the answer to it is unknown. See however [Theorem 6.10](#). Note also the next result of Shakhmatov [88]: if G and H are any Lindelöf Σ -groups, then $\text{Ind}(G \times H) \leq \text{Ind } G + \text{Ind } H$. See also the discussion of this theorem in [101, Theorem 6.12] where some other interesting questions are posed.

7. L.G. Zambakhidze's problem

At the early stage of development of topology and dimension theory, it was not immediately clear how to define zero-dimensionality in topological spaces. We have already discussed this problem in the preceding sections. Let us add here that Brouwer was interested in scattered spaces as well. Recall that a space X is *scattered* if every nonempty subspace Y of X contains an isolated point (of Y). Clearly, scatteredness can be considered as an exotic version of zero-dimensionality. Note, however, that there are scattered spaces which are not zero-dimensional. See Solomon [99] and Terasawa [100].

A base \mathcal{B} of a space X is called a *bc-base* [13] if the boundary $B(U) = \overline{U} \setminus U$ of every member U of \mathcal{B} is compact. Spaces with a *bc-base* are also called *rimcompact*. A separable metrizable space has a *bc-base* if and only if it can be compactified by a zero-dimensional remainder (de Groot [24], Freudenthal [43,44]; see also [1]).

In this section, we call a nonempty space X *zero-dimensional* if X has a base consisting of clopen subsets, that is, if $\text{ind } X = 0$. Notice that a base consisting of clopen subsets is, obviously, a *bc-base*. Thus, every zero-dimensional space and every locally compact space, as well as the free topological sum of such spaces, have a *bc-base*.

Outside the class of separable metrizable spaces, the existence of a *bc-base* for X is no longer equivalent to the existence of a compactification bX with a zero-dimensional remainder. This was demonstrated by Sklyarenko [95] who proved that a Tychonoff space of countable type has a compactification bX with a zero-dimensional remainder if and only if X has a *bc-base*, and Smirnov [98] constructed a space X which does not have a *bc-base* but has a compactification bX such that $\text{ind}(bX \setminus X) = 0$. For every positive $n \in \omega$, Charalambous [22] constructed a space X_n with a *bc-base* such that, for every compactification $b(X_n)$ of X_n , the remainder $b(X_n) \setminus X_n$ is normal and n -dimensional.

For topological groups, the situation is different:

Theorem 7.1 (L.G. Zambakhidze [115]). *If a topological group G has a zero-dimensional remainder in some compactification, then G has a *bc-base*.*

About 12 years ago, in a conversation with Arhangel'skii, Zambakhidze asked whether every non-zero-dimensional topological group with a *bc-base* is locally compact. Several partial results in this direction were obtained in [13]. Some of them we list below.

We note that the product of two spaces with a *bc-base* need not have a *bc-base*.

Proposition 7.2. *Let $G \times H$ be the product of a zero-dimensional non-locally compact topological group G with a locally compact non-zero-dimensional topological group H . Then $G \times H$ does not have a *bc-base*.*

Proof. By Theorem 2.1, H contains a copy of \mathbb{I} . Hence $G \times H$ contains a closed homeomorph of $G \times \mathbb{I}$. Striving for a contradiction, assume that $G \times H$ has a *bc-base*. Then so does $G \times \mathbb{I}$ since the property of having a *bc-base* is hereditary with respect to closed subsets. Let e denote the neutral element of G . There is an open neighborhood U of $(e, 0)$ whose boundary R is compact and has the property that $\pi_{\mathbb{I}}(U)$ is a proper subset of \mathbb{I} . Here $\pi_{\mathbb{I}} : G \times \mathbb{I} \rightarrow \mathbb{I}$ denotes the projection. Clearly, $\pi_G(R)$ is a compact subset of the open subset $\pi_G(U)$ of G . Here $\pi_G : G \times \mathbb{I} \rightarrow G$ denotes the projection. Since G is nowhere locally compact, there is $g \in \pi(U) \setminus \pi_G(R)$. Observe that $\{g\} \times \mathbb{I}$ is connected and intersects U as well as the complement of the closure of U , but misses R . This contradicts the connectivity of \mathbb{I} . \square

Hence, the topological group $\mathbb{R} \times \mathbb{Q}$, where \mathbb{R} is the usual topological group of real numbers and \mathbb{Q} is the usual topological group of rational numbers, does not have a bc -base.

Question 7.3. Does there exist a non-locally compact non-zero-dimensional semitopological (paratopological) group G with a bc -base?

Subsets A and B of a topological group G will be called *translation-disjoint* if for any open neighborhood O of the neutral element e of G there exists $c \in O$ such that cA and B are disjoint.

We will call a base \mathcal{B} of a space X a *bcs-base* if the boundary $B(U) = \overline{U} \setminus U$ of every member U of \mathcal{B} is σ -compact. The spaces with a *bcs-base* are called *rim- σ -compact*.

Theorem 7.4 ([13]). *Suppose that G is a non- σ -compact topological group with a bcs -base, and that $G = \bigcup_{i < \omega} Y_i$, where each Y_i is a separable metrizable F_σ -subspace of G . Then*

- (1) G can be written as $A \cup B$, where A and B are zero-dimensional and A is σ -compact,
- (2) $\text{ind}(G) = \text{Ind}(G) = \text{dim}(G) \leq 1$,
- (3) any σ -compact subspace of G is zero-dimensional.

Corollary 7.5 ([13]). *Every σ -compact non-locally compact topological group with a bc -base is zero-dimensional.*

Theorem 7.6 ([13]). *Suppose that G is a non-locally compact topological group with a bc -base. Then every compact subspace of G is zero-dimensional.*

Corollary 7.7. *Let X be a space such that the free topological group $F(X)$ of X has a bc -base. Then the subspace A_n of $F(X)$ consisting of reduced words of length $\leq n$ is zero-dimensional.*

However, we do not know the answer to the next question:

Question 7.8. Suppose that the free topological group $F(X)$ of a Tychonoff space X has a bc -base. Is $F(X)$ zero-dimensional?

Theorem 7.9 ([13]). *Suppose that a non-locally compact topological group G has a zero-dimensional remainder in a compactification $b(G)$. Then*

- (a) $\text{ind}(G) \leq 1$;
- (b) $\text{ind}(b(G)) \leq 2$.

Recall that, under the assumptions in the above statement, the topological group G has a bc -base by Zambakhidze's [Theorem 7.1](#).

As an application, let us consider compactifications of the space of rational numbers \mathbb{Q} . The 1-dimensional sphere \mathbb{S}^1 can be interpreted as a compactification of \mathbb{Q} . The remainder $\mathbb{S}^1 \setminus \mathbb{Q}$ of \mathbb{Q} in this compactification is homeomorphic to the space \mathbb{P} of irrational numbers. Notice that $\text{ind}(\mathbb{P}) = 0$, and \mathbb{P} is homeomorphic to a topological group. In this connection we mention the next easy to establish but rather unexpected (in our opinion) fact:

Proposition 7.10 ([13]). *If a zero-dimensional remainder of \mathbb{Q} is homeomorphic to a topological group, then it is homeomorphic to the space \mathbb{P} of irrational numbers.*

Corollary 7.11 ([13]). *If $b(\mathbb{Q})$ is any compactification of \mathbb{Q} such that the remainder satisfies the condition $\text{ind}(b(\mathbb{Q}) \setminus \mathbb{Q}) \geq 2$, then it is not homeomorphic to any topological group.*

Question 7.12. Does there exist a compactification $b\mathbb{Q}$ of \mathbb{Q} such that the remainder $Y = b(\mathbb{Q}) \setminus \mathbb{Q}$ is homeomorphic to a 1-dimensional topological group?

The next question of Zambakhidze remains the main open problem here:

Question 7.13 ([13]). Is every non-locally compact topological group with a bc -base zero-dimensional? What if, in addition, the group is assumed to be precompact?

A related question is:

Question 7.14. Suppose that G is a non-locally compact topological group with a zero-dimensional remainder. Is it true that G itself is zero-dimensional? What if, in addition, the group is assumed to be precompact?

If the answer to Problem 7.12 is in the affirmative, then the answer to Zambakhidze's question is in the negative.

Question 7.15 ([13]). Is every metrizable non-locally compact topological group with a bc -base zero-dimensional?

The answer to the next question may very well be in the affirmative.

Question 7.16. Is every non-locally compact topological group with a bc -base totally disconnected?

Question 7.17. Can a non-locally compact topological group with a bc -base contain a topological copy of the usual space of reals?

8. Miscellanea

1. A fundamental fact concerning the behavior of the dimension of topological groups under open continuous homomorphisms was established in [10]:

Theorem 8.1. *Every topological group can be represented as a quotient group of some topological group G such that $\dim G = 0$.*

In his paper [91, Corollary 4.3], Shakhmatov proves a more precise 'precompact' version of this theorem: Every precompact (Abelian) group G is a quotient group of a precompact (Abelian) group H so that $\dim H = 0$ and $w(H) = w(G)$. His proof uses the technique of free precompact groups. Dikranjan [32, Theorem 1.2] gave a direct constructive proof (free of free topological groups) of the 'abelian' option of Shakhmatov's theorem.

A similar statement holds for Abelian topological groups. The next question was posed in [10, Question 12].

Question 8.2. Let τ be an infinite cardinal number and H be a topological group with $w(H) \leq \tau$. Is it true that there exists a topological group G such that $w(G) \leq \tau$, $\dim G \leq 0$, and H is a quotient group of G ?

The version of this question with $\text{ind } G \leq 0$ instead of $\text{dim } G \leq 0$ (also formulated in [10]) was answered positively by Tkachenko (see [88]).

The next general question, listed as Question 7.3 in [88], seems to be folklore. The first listed author of the present paper remembers that in the sixties, P.S. Alexandroff regularly discussed version (i) of it during his Topology Seminars at Moscow University.

Question 8.3. Is it possible to represent an arbitrary nonempty space Y as an image of a space X under a continuous mapping f such that $\text{dim } X = 0$, $w(X) \leq w(Y)$, and f satisfies one of the following conditions: (i) f is perfect; (ii) f is open; (iii) f is quotient?

This can be related to another general problem which was also posed by Shakhmatov in [88]:

Question 8.4. Suppose that f is a continuous mapping of a space X to a space Y . Then do there exist a Tychonoff space Z , and continuous mappings $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ such that $f = h \circ g$, $\text{dim } Z \leq \text{dim } X$, and $w(Z) \leq w(Y)$?

Some other versions of this fundamental open problem are also considered by Shakhmatov [88], where several examples and partial positive results in this direction are given.

2. Recall that a topological space X is said to be *totally disconnected* if each point in X is the intersection of a family of subsets of X which are both open and closed.

It is a natural question whether for every totally disconnected topological group (G, \mathcal{T}) there exists a topology \mathcal{T}^* on G such that $\mathcal{T}^* \subseteq \mathcal{T}$, $\text{ind}(G, \mathcal{T}^*) = 0$, and (G, \mathcal{T}^*) is a topological group. But such a topology does not always exist since Megrelishvili [67] constructed a totally disconnected minimal topological group which is not zero-dimensional, using the famed Erdős example (see below).

Dikranjan in [30] established that every countably compact hereditarily disconnected topological group is zero-dimensional; he also proved in [29] that this theorem does not extend to all pseudocompact groups, there are counterexamples that are even not totally disconnected. Dikranjan also mentions in [31] the following result of Shakhmatov: every totally disconnected pseudocompact topological group G admits a weaker zero-dimensional group topology.

Question 8.5 (Arhangel'skii, see [88, Question 8.4]). Suppose that (G, \mathcal{T}) is a topological group such that $\text{ind } G = 0$. Then does there exist a topology \mathcal{T}^* on G such that $\mathcal{T}^* \subseteq \mathcal{T}$, $\text{dim}(G, \mathcal{T}^*) = 0$, and (G, \mathcal{T}^*) is a topological group?

A similar question has been asked by Arhangel'skii about topological spaces (see [88]):

Question 8.6. Suppose that (X, \mathcal{T}) is a topological space such that $\text{ind } X = 0$. Does there exist a topology \mathcal{T}^* on X such that $\mathcal{T}^* \subseteq \mathcal{T}$ and $\text{dim}(X, \mathcal{T}^*) = 0$?

Shakhmatov published in [88] the following general versions of the above questions (see Questions 8.6 and 8.7):

Question 8.7. Suppose that (X, \mathcal{T}) is a topological space. Does there exist a topology \mathcal{T}^* on X such that $\mathcal{T}^* \subseteq \mathcal{T}$ and $\text{dim}(X, \mathcal{T}^*) \leq \text{ind}(X, \mathcal{T})$?

Question 8.8. Suppose (G, \mathcal{T}) is a topological group. Does there exist a topology \mathcal{T}^* on G such that $\mathcal{T}^* \subseteq \mathcal{T}$, $\text{dim}(G, \mathcal{T}^*) \leq \text{ind}(G, \mathcal{T})$, and (G, \mathcal{T}^*) is a topological group?

In the same paper, Shakhmatov asked whether $\dim G = \text{ind } G = \text{Ind } \beta G$ for every minimal topological group.

Again, let E denote Erdős space, the subspace of Hilbert space ℓ^2 consisting of all points having the property that all of its coordinates are rational. It is a totally disconnected 1-dimensional subgroup of ℓ^2 . As was shown by Erdős [39], every clopen subspace of E is unbounded in norm. Take the collection of all clopen subsets of E as the basis for a new topology \mathcal{T} on E . This topology is obviously weaker than the usual separable metrizable topology on E . It is not metrizable, though. It was shown by Dijkstra and van Mill [27, Remark 5.7] that the character of (E, \mathcal{T}) is uncountable.

Question 8.9. Is the topology \mathcal{T} compatible with the group structure on E ?

Observe that (E, \mathcal{T}) is a homogeneous space.

3. The class of topological spaces represented by topological fields is very special. See [92] for details. At present, it is not much known about dimensional aspects of the theory of topological fields.

For every $n \geq 1$, Ursul [105] constructed an example of a separable metric n -dimensional topological field P_n . The field P_n is not homeomorphic to \mathbb{R}^n or \mathbb{C}^n . This remarkable result shows that the dimension theory of topological fields can be quite nontrivial. Shakhmatov in [92] asked the following question:

Question 8.10. Does there exist a topological field F such that not all classical dimensions of F coincide?

It is well-known that non-metrizable compacta cannot be embedded in topological fields.

4. The spheres in the Euclidean space contributed to the birth of the idea of inductive definition of the classical dimension ind . Of course, this refers to a very special situation. However, Jun-iti Nagata proved the next remarkable general theorem [77,78]:

Theorem 8.11. *If X is a metrizable space, and $\text{Ind } X \leq n$ for some $n \in \omega$, then the space X is metrizable by a metric ρ such that for every closed subset F of X and every $\varepsilon > 0$, the dimension Ind of the boundary of the ε -neighborhood $O_\varepsilon(F)$ of F does not exceed $n - 1$.*

Question 8.12. Suppose that G is a metrizable topological group with $\text{Ind } G \leq n$ for some $n \in \omega$. Is the topological group G metrizable by a left-invariant metric ρ such that for every closed subset F of X and every $\varepsilon > 0$, the dimension Ind of the boundary of the ε -neighborhood $O_\varepsilon(F)$ does not exceed $n - 1$?

5. Duda noticed that in the definition of inductive dimension one may interchange the steps corresponding to ind and Ind . In this way, one obtains continuum many new inductive dimension functions. For details, see [35].

Egorov and Podstavkin studied in [37] the dimension function Dind whose definition is due to Arhangel'skii. In the definition of Dind the ideas of the definitions of the dimensions ind and dim are unified in a natural way. We put $\text{Dind } X = -1$ if $X = \emptyset$. Proceeding by induction, we let $\text{Dind } X \leq n$ if every open cover γ of X can be refined by a disjoint family η of open sets such that $\text{Dind}(X \setminus \bigcup \eta) \leq n - 1$. If in this definition we also assume that each γ is finite, and require that each η be finite, then the dimension function so obtained is denoted by Dind_f . Their investigation was extended by Kulpa [62] who proved, in particular, that $\text{Dind } X = \text{Dind } \beta X$, for every normal space X .

Question 8.13. How are $\text{Dind } G$ and $\text{Ind } G$ ($\text{Dind}_f G$ and $\text{Ind } G$) related, when G is a topological group?

6. Any connected linearly ordered topological space without endpoints can be considered as a natural generalization of the usual space \mathbb{R} of real numbers. Below we call such spaces *generalized lines*. Taking the topological product of any two generalized lines, we obtain a space which may be called a *generalized plane*. Similarly we can define generalizations of the usual three-dimensional space \mathbb{R}^3 , and so on.

Löttgren and Wagner [65] proved a generalized Jordan curve theorem for any generalized plane. Moreover, Slye [97] introduced five axioms for an abstract set S which imply the generalized Jordan curve theorem in S . It is not difficult to see that these axioms hold for the generalized plane. Hence, the main result in [65] can be easily derived from the results in [97].

Van Dalen used the techniques and results of Löttgren and Wagner in his Ph.D. thesis [108] to prove the following theorem on the invariance of domain:

Theorem 8.14 ([108, Chapter 3]). *Let X be a generalized plane and U be an open subset of X . Then, for every one-to-one continuous mapping f of U into the space X , the set $f(U)$ is open in X .*

For other interesting results on generalized planes and their n -dimensional versions, see [108]. It was shown there, in particular, that to study and understand some basic phenomena occurring in the Euclidean finite-dimensional spaces, we do not have to rely upon the concept of distance; in fact, we can completely exclude the real numbers from our considerations.

7. The construction of a free topological group is a powerful method for defining a topological group starting with an arbitrary topological space. Results we mention below concern dimension of free topological groups. Some of them play an essential role in the proofs of the results mentioned in 1. A space is called a *paracompact σ -space* if it is paracompact and has a σ -discrete network. It was established in [10] that the free topological group $F(X)$ of any metrizable space X is a paracompact σ -space. Using this result, Arhangel'skii established in [10] the following fact:

Theorem 8.15. *If X is any metrizable space (paracompact σ -space) such that $\dim X = 0$, then $\dim F(X) = 0$.*

We now refer the reader to [101] where the next theorem due Tkachenko and Sipacheva is discussed:

Theorem 8.16. *If X is any space such that $\dim X = 0$, then $\text{ind } F(X) = \text{ind } A(X) = 0$.*

It has been asked by Arhangel'skii whether $\text{ind } X = 0$ alone suffices to show that $\text{ind } F(X) = 0$. The negative answer to this question was obtained by Shakhmatov [90]. He established that there exists a space X such that $\text{ind } X = 0$, $\text{ind } F(X) \neq 0$, and $\text{ind } A(X) \neq 0$. In addition, X can be chosen to be normal or pseudocompact. However, the following basic problem posed by Arhangel'skii [10] in 1981 still remains open:

Problem 8.17. Is it true for arbitrary space X that if $\dim X = 0$, then $\dim F(X) = 0$ ($\dim A(X) = 0$)?

Another challenging old open problem from [10] is this one:

Problem 8.18. Is it true for arbitrary metrizable space X that if $\text{ind } X = 0$, then $\text{ind } F(X) = 0$ ($\text{ind } A(X) = 0$)?

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