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NONHOMOGENEITY OF REMAINDERS

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Abstract. We present a cardinal inequality for the number of homeomorphisms of the remainders of compactifications of nowhere locally compact spaces. As a consequence, we obtain that if X is countable and dense in itself, then the remainder of any compactification of X has at most continuum many homeomorphisms.

1. Introduction

All topological spaces under discussion are Tychonoff.

A space X is *homogeneous* if for any two points $x, y \in X$ there is a homeomorphism h from X onto itself such that $h(x) = y$.

In 1956, Walter Rudin [22] proved that the Čech-Stone remainder $\beta\omega \setminus \omega$, where ω is the discrete space of nonnegative integers, is not homogeneous under CH. This result was later generalized considerably by Frolík [14] who showed in ZFC that $\beta X \setminus X$ is not homogeneous, for any nonpseudocompact space X. For other results that are in the same spirit, see e.g. [9], [10], [18].

Hence the study of (non)homogeneity of Čech-Stone remainders has a long history. In this note we are interested in homogeneity properties of arbitrary remainders of topological spaces. We address the following general problem: *when does a space have a homogeneous remainder?* If X is locally compact, then the Alexandroff one-point compactification αX of X has a homogeneous remainder. Hence for locally compact spaces, our question has an obvious answer. If X is not locally compact, however, then it need not have a homogeneous remainder, as the topological sum of the space of rational numbers $\mathbb{Q}$ and the space of irrational numbers $\mathbb{P}$ shows (for details, see §5).

Hence we consider questions of the following type: if X is homogeneous, and not locally compact, does X have a homogeneous remainder? In particular, special attention is given to remainders of nonlocally compact topological groups. For some recent facts on such remainders, see Arhangel'skii [4] and [5]. One of them, established in [4], is: every remainder of a topological group is either Lindelöf or pseudocompact.

However, the main result below concerns remainders of any nowhere locally compact space, not necessarily homogeneous. The aim of this note is to show that if

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X is nowhere locally compact and bX is an arbitrary compactification of X, then the number of homeomorphisms of the remainder $bX \setminus X$ is bounded by $|\mathsf{RO}(X)|^{\mathrm{Hsw}(X)}$ (for definitions, see §2), which in turn is bounded by $2^{|X|}$. So if X is countable and dense in itself, then any remainder of X has at most $\mathfrak{c}$ homeomorphisms, where $\mathfrak{c}$ denotes the cardinality of the continuum. From this we get several examples, among them a countable topological group and a countably compact topological group no remainders of which are homogeneous. We also get new and very simple proofs that familiar Čech-Stone remainders such as $\beta\mathbb{Q} \setminus \mathbb{Q}$ and $\beta\mathbb{P} \setminus \mathbb{P}$ are not homogeneous.

By Rudin [22], $\beta\omega \setminus \omega$ has $2^{\mathfrak{c}}$ homeomorphisms under CH. Hence the assumption on nowhere local compactness in our results is essential. (Interestingly, Shelah [23, §41] proved the consistency that every homeomorphism of $\beta\omega \setminus \omega$ is trivial, hence in his model there are only $\mathfrak{c}$ of them; see also [25].)

2. Preliminaries

For a space X, we let $\mathscr{H}(X)$ denote its group of homeomorphisms. If $A \subseteq X$, then $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ denote its closure and interior, respectively. Similarly, $\overline{A}$ denotes the closure of A if no confusion can arise.

We let $\mathsf{RO}(X)$ denote the complete Boolean algebra of all *regular open* subsets of X, where a set is *regular open* if it is the interior of its own closure. By Pierce [20], it follows that $|\mathsf{RO}(X)|^\omega = |\mathsf{RO}(X)|$. Moreover, it is easy to see (and well known) that for every space X we have

$$(\dagger) \qquad |\mathsf{RO}(X)| \leq 2^{d(X)},$$

where $d(X)$ denotes the density of X.

For a space X, let $\tau(X)$ denote its topology and put $o(X) = |\tau(X)|$. The *super cardinality* of a space X is the minimum cardinality of a compactification of X. Let βX denote the Čech-Stone compactification of X.

A space X can be *condensed* on a space Y if there is a continuous bijection from X onto Y.

The *Hausdorff separating weight* of a space X, abbreviated $\mathrm{Hsw}(X)$, is the least infinite cardinal κ for which there exists a family $\mathscr{U}$ of open subsets of X such that for all distinct $x, y \in X$, there exist disjoint $U, V \in \mathscr{U}$ such that $x \in U$ and $y \in V$. Observe that $\mathrm{Hsw}(X)$ is rather "small" since it is obviously bounded by $|X|$. There is a related cardinal function called the *i-weight* of a space X, abbreviated $\mathrm{iw}(X)$, which is the minimal weight of all spaces onto which X can be condensed (Arhangel'skii [3]). It is clear that $\mathrm{Hsw}(X) \leq \mathrm{iw}(X)$ for all spaces X. Example 1 in Popov [21] has the property that its Hausdorff separation weight is countable, while its i-weight is uncountable (since it does not have a G_δ-diagonal). So the inequality in $\mathrm{Hsw}(X) \leq \mathrm{iw}(X)$ cannot be replaced by equality.

A collection of subsets $\mathscr{N}$ of a space X is called a *network* for X if every open subset of X is the union of a subfamily of $\mathscr{N}$. The *netweight*, $\mathrm{nw}(X)$, of X is the least infinite cardinal κ for which there exists a network for X of cardinality κ. Since 1959 (Arhangel'skii [1] and [2]), it is known that $\mathrm{iw}(X) \leq \mathrm{nw}(X)$ (see also [7, 5.2.10]). Clearly, $\mathrm{nw}(X) \leq |X|$.

Hence for the cardinal functions $\mathrm{Hsw}(X)$, $\mathrm{iw}(X)$ and $\mathrm{nw}(X)$ for a given space X, the following inequalities hold:

$$(\ddagger) \qquad \mathrm{Hsw}(X) \leq \mathrm{iw}(X) \leq \mathrm{nw}(X) \leq |X|.$$

We have already seen that the first inequality cannot be replaced by equality. The same is true for the second inequality. To see this, let D be a discrete space of size ω_1. Then $\mathrm{iw}(D) = \omega$ but $\mathrm{nw}(D) = \omega_1$. That the third inequality cannot be replaced by equality is trivial.

If $f\colon X \to Y$ is a multivalued function, and $A \subseteq Y$, then $f^{-1}(A) = \{x \in X : f(x) \subseteq A\}$. We say that f is *upper semi-continuous* provided that $f^{-1}(U)$ is open in X for every open subset U of Y.

We refer to Juhász [16] for undefined terminology on cardinal functions.

3. A bound on the number of homeomorphisms of remainders

We will use the following triviality: if D is dense in X, and $U \subseteq X$ is nonempty and open, then $\mathrm{cl}_X(U) \cap D = \mathrm{cl}_D(U \cap D)$.

Here is the main result in this note. In its proof, we make use of an idea in Krivoručko [17] (see also Hodel [15, §10]). We are indebted to Tony Hager for bringing Krivoručko's paper to our attention.

Theorem 3.1. *Let X be a nowhere locally compact space with a compactification bX. Then*

$$|\mathscr{H}(bX \setminus X)| \le |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} \le 2^{d(X)\mathrm{Hsw}(X)} \le 2^{\mathrm{nw}(X)} \le 2^{|X|}.$$

Proof. First observe that both X and the remainder $Y = bX \setminus X$ are dense in bX. If $x \in X$, then $\mathscr{U}_x$ denotes the family of all neighborhoods of x in bX. Now let $f \in \mathscr{H}(Y)$ be arbitrary, and define for every $x \in X$, the set $f^{\#}(x)$, as follows:

$$f^{\#}(x) = \bigcap_{U \in \mathscr{U}_x} \overline{f(U \cap Y)}$$

(here "closure" denotes closure in bX). Observe that by the denseness of Y in bX we have that the collection

$$\{f(U \cap Y) : U \in \mathscr{U}_x\}$$

has the finite intersection property, hence $f^{\#}(x)$ is a nonempty compact subset of bX.

Claim 1. $f^{\#}(x) \subseteq X$.

Indeed, pick an arbitrary $p \in Y$, and consider the point $q = f^{-1}(p)$. There is an open neighborhood U of x such that $q \notin \overline{U}$. Hence $q \notin \overline{U} \cap Y$, and so

$$p \notin f(\overline{U} \cap Y) = f(\mathrm{cl}_Y(U \cap Y)) = \mathrm{cl}_Y(f(U \cap Y)) = \overline{f(U \cap Y)} \cap Y,$$

as required.

Hence $f^{\#}\colon X \to X$ is a well-defined compact-valued multivalued function.

Claim 2. $f^{\#}$ is upper semi-continuous.

Pick $x \in X$, and let V be an open subset of X such that $f^{\#}(x) \subseteq V$. Let V' be an open subset of bX such that $V' \cap X = V$. Since $f^{\#}(x)$ is compact (Claim 1), there is an element $U \in \mathscr{U}_x$ such that $\overline{f(U \cap X)} \subseteq V'$, hence $f^{\#}(U \cap X) \subseteq V$.

Claim 3. If $f, g \in \mathscr{H}(Y)$ and $f \neq g$, then there exists $x \in X$ such that $f^{\#}(x) \cap g^{\#}(x) = \emptyset$.

Pick $p \in Y$ such that $f(p) \neq g(p)$. Let U and V be disjoint closed neighborhoods of $f(p)$ and $g(p)$ in bX. Let W be an open neighborhood of p in Y such that $f(W) \subseteq U$ and $g(W) \subseteq V$. Let W' be an open subset of bX such that $W' \cap Y = W$. Pick $x \in W' \cap X$. Then, clearly, $f^{\#}(x) \cap g^{\#}(x) = \emptyset$.

Now take a family $\mathscr{U}$ of open subsets of X which separates the points of X in the Hausdorff sense, and assume that $|\mathscr{U}| = \mathrm{Hsw}(X)$. We may assume without loss of generality that $\mathscr{U}$ is closed under finite intersections and finite unions. For every $U \in \mathscr{U}$ and $f \in \mathscr{H}(Y)$, let $H(U, f) = \mathrm{int}_X(\mathrm{cl}_X((f^{\#})^{-1}(U)))$. Clearly, $H(U, f) \in \mathsf{RO}(X)$. Denote the function $\mathscr{U} \to \mathsf{RO}(X)$ that assigns to $U \in \mathscr{U}$ the set $H(U, f) \in \mathsf{RO}(X)$ by F_f.

Claim 4. The function $\mathscr{H}(Y) \to \mathsf{RO}(X)^{\mathscr{U}}$ defined by $f \mapsto F_f$ is one-to-one.

Take distinct $f, g \in \mathscr{H}(Y)$. By Claim 3, there exists $x \in X$ such that $f^{\#}(x) \cap g^{\#}(x) = \emptyset$. Since $\mathscr{U}$ is closed under finite intersections and finite unions, the compactness of $f^{\#}(x)$ and $g^{\#}(x)$ implies that there exist disjoint $U, V \in \mathscr{U}$ such that $f^{\#}(x) \subseteq U$ and $g^{\#}(x) \subseteq V$. Then $x \in (f^{\#})^{-1}(U)$ and so $x \in F_f(U)$. We will show that $x \notin F_g(U)$ from which it will follow that $F_f \neq F_g$. Striving for a contradiction, assume that $x \in F_g(U) \subseteq \mathrm{cl}_X((g^{\#})^{-1}(U))$. Since $x \in (g^{\#})^{-1}(V)$ and $(g^{\#})^{-1}(V)$ is open, there exists $p \in (g^{\#})^{-1}(V) \cap (g^{\#})^{-1}(U)$. Hence $\emptyset \neq g^{\#}(p) \subseteq U \cap V$, which is a contradiction.

So this completes the proof of the inequality $|\mathscr{H}(Y)| \leq |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)}$.

For every space X we have

$$|\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} \leq 2^{d(X)\mathrm{Hsw}(X)} \leq 2^{\mathrm{nw}(X)} \leq 2^{|X|}.$$

This is a direct consequence of the inequalities (†) and (‡) in Section 2 and the trivial observation that $d(X) \leq \mathrm{nw}(X)$. So we are done. $\square$

Our cardinal inequality $|\mathscr{H}(bX \setminus X)| \leq |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)}$ raises several natural questions. The first one is whether it is possible to replace $|\mathsf{RO}(X)|$ by the weight $\mathrm{w}(X)$ of X. The following example shows that this is impossible.

Example 3.2. There is a nowhere locally compact space X with a compactification bX such that $|\mathscr{H}(bX \setminus X)| = 2^{\mathfrak{c}}$, while $\mathrm{Hsw}(X) = \omega$ and $\mathrm{w}(X) = \mathfrak{c}$.

Indeed, let D be a discrete space of cardinality $\mathfrak{c}$, and let $\alpha D = D \cup \{\infty\}$ denote its Alexandroff one-point compactification. Put $Z = \alpha D \times \mathbb{I}$, where $\mathbb{I}$ denotes the closed unit interval $[0, 1]$. Let

$$X = \{(d, q) : d \in D, q \in \mathbb{Q} \cap \mathbb{I}\}, \quad Y = Z \setminus X.$$

Then X is homeomorphic to the topological sum of $\mathfrak{c}$ many copies of $\mathbb{Q}$, hence is nowhere locally compact and $\mathrm{w}(X) = \mathfrak{c}$. It is clear that the real line $\mathbb{R}$ can be split into $\mathfrak{c}$ many disjoint copies of $\mathbb{Q}$. Hence X can be condensed onto $\mathbb{R}$, and so $\mathrm{Hsw}(X) = \mathrm{iw}(X) = \omega$. Any permutation π of D induces a homeomorphism $\bar{\pi}$ of Z in the obvious way and satisfies $\bar{\pi}(X) = X$ and $\bar{\pi}(Y) = Y$. Hence $\mathscr{H}(Y) = 2^{\mathfrak{c}} > \mathrm{w}(X)^{\mathrm{Hsw}(X)} = \mathrm{w}(X)^{\mathrm{iw}(X)} = \mathfrak{c}$.

Another interesting inequality for the number of homeomorphisms of remainders can be obtained along the following lines. Let X be a nowhere locally compact space with a compactification bX. Then

$$|\mathscr{H}(bX \setminus X)| \leq o(X \times X).$$

The proof is parallel to the proof of Theorem 3.1 until the end of the proof of Claim 3. It can then be completed as follows.

Consider the graph $\Gamma(f)$ of the function $f^{\#}$, i.e.,

$$\Gamma(f) = \{(x, y) \in X \times X : y \in f^{\#}(x)\}.$$

Since $f^{\#}$ is upper-semicontinuous, $\Gamma(f)$ is a closed subset of $X \times X$. For completeness sake, we present the easy proof. Assume that $(x, y) \notin \Gamma(f)$. Then $y \notin f^{\#}(x)$, and hence there is a closed neighborhood V of $f^{\#}(x)$ that does not contain y. Let U be a neighborhood of x such that for every $z \in U$ we have $f^{\#}(z) \subseteq V$. Then $U \times (X \setminus V)$ is a neighborhood of (x, y) that misses $\Gamma(f)$. The assignment $f^{\#} \mapsto \Gamma(f)$ is clearly one-to-one by Claim 3, which completes the proof.

This result prompts the following open problem.

Problem 3.3. Let X be a nowhere locally compact space with a compactification bX. Is it true that $|\mathscr{H}(bX \setminus X)| \leq o(X)$?

If $\mathrm{Hsw}(X) = \omega$, in particular if X has a countable network, then the answer to Problem 3.3 is in the affirmative. Simply observe that by Theorem 3.1, inequality (‡) in Section 2 and the result of Pierce quoted in Section 2, we have

$$|\mathscr{H}(bX \setminus X)| \leq |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} \leq |\mathsf{RO}(X)|^{\mathrm{nw}(X)} = |\mathsf{RO}(X)|^{\omega} = |\mathsf{RO}(X)| \leq o(X).$$

If X is uncountable, then the inequality $|\mathscr{H}(bX \setminus X)| \leq |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)}$ seems sharper than the inequality $|\mathscr{H}(bX \setminus X)| \leq o(X \times X)$. As an illuminating example, consider the Sorgenfrey line $\mathbb{S}$ and its compactification $b\mathbb{S}$. Clearly, $|\mathsf{RO}(\mathbb{S})| = \mathfrak{c}$ and $\mathrm{Hsw}(\mathbb{S}) = \omega$, hence $|\mathscr{H}(b\mathbb{S} \setminus \mathbb{S})| \leq \mathfrak{c}$. But $o(\mathbb{S} \times \mathbb{S}) = 2^{\mathfrak{c}}$.

The netweight is naturally related to the bound $o(X \times X)$ by the inequalities

$$o(X) \leq o(X \times X) \leq 2^{\mathrm{nw}(X \times X)} = 2^{\mathrm{nw}(X)},$$

so that

$$\mathscr{H}(bX \setminus X) \leq o(X \times X) \leq 2^{\mathrm{nw}(X)}.$$

4. Applications to spaces with a countable network

In this section, we prove some results for spaces with countable netweight.

If X has a countable network and is nowhere locally compact, and bX is any compactification of X, then $|\mathscr{H}(bX \setminus X)| \leq \mathfrak{c}$ by Theorem 3.1. This will show that it is very rare that such a space has a homogeneous remainder.

Theorem 4.1 ($2^{\omega} < 2^{\omega_1}$). *If X has a countable network and is nowhere locally compact, and bX is a compactification of X such that $bX \setminus X$ is homogeneous, then $bX \setminus X$ is first-countable and $|bX| \leq \mathfrak{c}$.*

Proof. We have already observed that $|bX \setminus X| \leq \mathfrak{c}$. But also $|X| \leq \mathfrak{c}$ by Juhász [16, 2.3(a)].

Pick an arbitrary $p \in Y = bX \setminus X$. There clearly is a compact G_{δ}-subset S of bX such that $p \in S \subseteq bX \setminus X$. Since S is compact, and has cardinality at most $\mathfrak{c}$, it must have a point at which it is first-countable. This is a (well-known) consequence of our assumptions and the classical Čech-Pospišil Theorem (see [16, 3.16]) that if Z is compact and if for some κ, $\chi(q, Z) \geq \kappa$ for every $q \in Z$, then $|S| \geq 2^{\kappa}$. Hence S is first-countable at some point, and since S is a compact G_{δ} in bX, this means that bX is first-countable at some point of S. So we conclude that $bX \setminus X$, being homogeneous, is first-countable. □

Unfortunately, this is not true in ZFC, as we will now show.

Theorem 4.2 (MA+$\neg$CH)**.** *For every countable dense subset X of the Cantor cube 2^{ω_1}, the remainder $2^{\omega_1} \setminus X$ is homogeneous.*

Proof. It was shown by Steprāns and Zhou [24] that 2^{ω_1} is countable dense homogeneous under MA+$\neg$CH. That is, if A and B are arbitrary countable dense subsets of 2^{ω_1}, then there is a homeomorphism $f\colon 2^{\omega_1} \to 2^{\omega_1}$ such that $f(A) = B$. Hence we may assume without loss of generality that the countable dense subset X in our theorem is a subgroup of 2^{ω_1}. Now take arbitrary elements $x, y \in Y = 2^{\omega_1} \setminus X$. Let ξ and η be homeomorphisms of 2^{ω_1} such that $\xi(X \cup \{x\}) = X$ and $\eta(X \cup \{y\}) = X$. Since X is a subgroup of 2^{ω_1}, there is a translation γ of 2^{ω_1} such that $\gamma(\xi(x)) = \eta(y)$. Observe that $\gamma(X) = X$. Hence the function

$$f = \eta^{-1} \circ \gamma \circ \xi$$

is a homeomorphism of 2^{ω_1} such that $f(x) = y$ and $f(Y) = Y$. □

So we arrive at the conclusion that the "concrete" space $2^{\omega_1} \setminus X$, where X is any countable dense subset of 2^{ω_1}, behaves very differently in various models of set theory. Under MA+$\neg$CH it is homogeneous, but not so under CH. See also van Mill [19], where a "concrete" compact space was constructed that behaves similarly.

Problem 4.3. Assume that $2^\omega < 2^{\omega_1}$. If X is hereditarily Lindelöf and nowhere locally compact, and bX is a compactification of X such that $bX \setminus X$ is homogeneous, is $bX \setminus X$ first-countable?

If $bX \setminus X$ in this problem is a topological group, then from Arhangel'skii [5] we get in ZFC that $bX \setminus X$ is separable metrizable and X is separable. Hence Problem 4.3 is "really" about homogeneous spaces that do not have the structure of a topological group.

The authors recently found another bound for the number of homeomorphisms of remainders that implies the following: if X is nowhere locally compact, Lindelöf, separable and of pseudo-character ω, then for any compactification bX of X we have that $|\mathscr{H}(bX \setminus X)| \le \mathfrak{c}$. This implies that the answer to Problem 4.3 is in the affirmative provided that X is separable. Details will appear elsewhere.

5. Examples

We now present some examples.

Example 5.1. There is a countable topological group, no remainder of which is homogeneous.

Let X be a countable space of super cardinality $2^\mathfrak{c}$. That such a space exists is due to Efimov [13]. See also van Douwen and Przymusiński [11]. Now let G denote the free topological group $F(X)$ over X, [7, Chapter 7]. Then G is countable, and contains a closed copy of X. Hence G has super cardinality $2^\mathfrak{c}$ and so no remainder of it can be homogeneous by Theorem 3.1.

There has been quite an interest in pseudocompact topological groups in the last decades. We contribute to the topic by constructing an example of a pseudocompact topological group no remainder of which is homogeneous.

Theorem 5.2. *No pseudocompact separable topological group of cardinality* $\mathfrak{c}$ *whose character is also equal to* $\mathfrak{c}$ *has a homogeneous remainder.*

Proof. Let G be a topological group which has the properties stated in the theorem. By Comfort and Ross [8], βG has the structure of a topological group and G is a subgroup of it. A compact group of character $\mathfrak{c}$ has cardinality $2^{\mathfrak{c}}$ by [7, 5.2.7], hence G is not compact since G has cardinality $\mathfrak{c}$. So G is not locally compact either since otherwise G would be an open subgroup of βG and hence G would be closed in βG and compact, a contradiction.

We conclude that G is nowhere locally compact and hence the remainder G^* is dense in βG. Since G^* contains a translate of G, it also follows that G^* is C^*-embedded in βG, i.e., $\beta G^* = \beta G$.

Now let bG denote an arbitrary compactification of G such that $R = bG \setminus G$ is homogeneous. Let $f\colon \beta G \to bG$ denote the canonical continuous mapping that restricts to the identity on G. We first claim that the remainder R is C^*-embedded in bG. (This is a consequence of Arhangel'skii [6, Theorem 2.1], for completeness sake we include the argument.) Indeed, let Z_0 and Z_1 be any two disjoint zero-sets in R. Then $f^{-1}(Z_0)$ and $f^{-1}(Z_1)$ are disjoint zero-sets of G^*, hence their closures $\overline{f^{-1}(Z_0)}$ and $\overline{f^{-1}(Z_1)}$ in $\beta G^* = \beta G$ are disjoint as well. Since f is the identity on G, this obviously implies that

$$f(\overline{f^{-1}(Z_0)}) \cap f(\overline{f^{-1}(Z_1)}) = \emptyset,$$

hence Z_0 and Z_1 have disjoint closures in bG. From this we conclude that $\beta R = bG$.

This implies that every element $h \in \mathscr{H}(R)$ can be extended to a homeomorphism $\bar{h} \in \mathscr{H}(bG)$. Hence $\bar{h}$ restricts to a homeomorphism of G. The assignment $h \mapsto \bar{h} \restriction G$ is clearly one-to-one, hence $|\mathscr{H}(R)| \leq \mathfrak{c}$ since G is separable and has cardinality $\mathfrak{c}$. The homogeneity of R consequently implies that $|R| \leq \mathfrak{c}$.

We conclude that $|bG| = \mathfrak{c}$. Suppose that for every $p \in R$ we have that $\chi(p, bG) = \mathfrak{c}$. Then $\chi(x, bG) = \mathfrak{c}$ for every $x \in bG$. Hence by the classical Čech-Pospišil Theorem (see [16, 3.16]) it would follow that $|bG| \geq 2^{\mathfrak{c}}$, which is a contradiction. Hence there exists $p \in R$ such that $\chi(p, bG) < \mathfrak{c}$. Put $\kappa = \chi(p, bG)$. Since R is homogeneous, and every homeomorphism of R can be extended to a homeomorphism of bG, this implies that $\chi(q, bG) = \kappa$ for every $q \in R$.

By the famous Ivanovskij-Kuzminov Theorem ([7, §4.1]), bG is dyadic, being a continuous image of the compact topological group βG. Since R is dense in bG, by Efimov [12], we obtain that the weight of bG is at most $\kappa < \mathfrak{c}$. Hence the weight of G is at most $\kappa < \mathfrak{c}$. But the character of G is $\mathfrak{c}$, and hence we have reached a contradiction. □

This leads us to:

Example 5.3. There is a countably compact topological group, no remainder of which is homogeneous.

We endow the Tychonoff cube $2^{\mathfrak{c}}$ with its standard Boolean group structure. Since $2^{\mathfrak{c}}$ is separable, a standard closure argument gives a countably compact separable subgroup of $2^{\mathfrak{c}}$ of cardinality $\mathfrak{c}$. Hence by Theorem 5.2, this is the example we are looking for.

Example 5.4. $\beta\mathbb{Q} \setminus \mathbb{Q}$ and $\beta\mathbb{P} \setminus \mathbb{P}$ are not homogeneous.

This is clear from Theorem 3.1 since both remainders have cardinality $2^{\mathfrak{c}}$. As we wrote in the introduction, this is not new since it is a consequence of Frolík's Theorem. It also follows from van Douwen's Theorem from [9] that $|X| \leq 2^{\pi(X)}$ for every homogeneous space X since obviously both $\beta\mathbb{Q}\setminus\mathbb{Q}$ and $\beta\mathbb{P}\setminus\mathbb{P}$ have countable π-weight.

Example 5.5. If X is the topological sum of $\mathbb{Q}$ and $\mathbb{P}$, then no remainder of X is homogeneous.

Indeed, let $X = A \cup B$, where A and B are disjoint clopen subsets of X such that $A \approx \mathbb{Q}$ and $B \approx \mathbb{P}$. Fix a compactification bX of X. There are disjoint open subsets U and V of bX such that $U \cap X = A$ and $V \cap X = B$. Since neither A nor B is locally compact, we may pick $p \in U \setminus A$ and $q \in V \setminus B$. Let E and F be closed neighborhoods of p and q in bX such that $E \subseteq U$ and $V \subseteq F$. Then E is a compactification of $E \cap A$ which is homeomorphic to $\mathbb{Q}$. Hence $E \setminus A$ is Čech-complete and hence a Baire space, A being countable. Similarly, F is a compactification of $F \cap B$ which is homeomorphic to $\mathbb{P}$. Hence $F \setminus B$ is first category in itself, $\mathbb{P}$ being completely metrizable. It is hence clear that no homeomorphism of $bX \setminus X$ can take p onto q.

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