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Every crowded pseudocompact ccc space is resolvable

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We prove that every pseudocompact crowed ccc space is $\mathfrak{c}$-resolvable. This gives a partial answer to problems posed by Comfort and García-Ferriera, and Juhász, Soukup and Szentmiklóssy.

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1. Introduction

All spaces under discussion are Tychonoff.

Let $\kappa \geq 2$ be a cardinal. A space X is called *κ-resolvable* if there is a family of κ-many pairwise disjoint dense subsets of X. By a *resolvable* space we mean a space that is 2-resolvable. Observe that a resolvable space is *crowded*, i.e., has no isolated points. A space is called *irresolvable* if it is not resolvable. The notion of (κ-)resolvability is due to Hewitt [4] and Ceder [1], respectively. It is known that every locally compact crowded space is $\mathfrak{c}$-resolvable, where $\mathfrak{c}$ denotes the cardinality of the continuum (for details and some historical comments, see Comfort and García-Ferriera [2]). It is also known that there are irresolvable crowded spaces (Hewitt [4]).

Kunen, Szymański and Tall [6] proved assuming $V = L$, that every crowded Baire space is resolvable. Moreover, they showed that if ZFC is consistent with the existence of a measurable cardinal, then ZFC is consistent with the existence of an irresolvable (zero-dimensional) crowded Baire space.

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It was shown in Comfort and García-Ferriera [2, Theorem 6.9] that every countably compact crowded space is ω-resolvable. This result was improved by Pytkeev [8] (see also [5, Theorem 3.6]). He showed that any countably compact crowded space is ω_1-resolvable, and Juhász, Soukup and Szentmiklóssy [5, Theorem 3.6] left open the question whether ω_1 can be improved to $\mathfrak{c}$. It was asked by Comfort and García-Ferriera [2, §7] whether every pseudocompact crowded space is resolvable. Since every pseudocompact space is Baire, the answer is yes if one assumes $V = L$.

We prove here that every pseudocompact crowded space which satisfies the countable chain condition (abbreviated: ccc) is $\mathfrak{c}$-resolvable. This is a partial answer to the aforementioned problems of Comfort and García-Ferriera, and Juhász, Soukup and Szentmiklóssy.

2. Preliminaries

A space satisfies the *countable chain condition* (abbreviated: ccc) provided that any family consisting of pairwise disjoint nonempty open subsets of it is countable. A space is *crowded* if it has no isolated points.

Lemma 2.1. *Let X be crowded ccc space, and let W be a nonempty open subset of X. Then there is a countably infinite family $\mathscr{U}$ of open F_σ-subsets of X such that*

(1) *for every $U \in \mathscr{U}$, $\overline{U} \subseteq W$,*
(2) *if $U, V \in \mathscr{U}$ are distinct, then $\overline{U} \cap \overline{V} = \emptyset$,*
(3) *$\bigcup \mathscr{U}$ is dense in W.*

Proof. Pick an arbitrary point $x \in W$. Since X is Tychonoff, its open F_σ-subsets form a base. Hence we simply let $\mathscr{U}$ be a maximal family of open F_σ-subsets of X satisfying (1) and (2) and with the additional condition that for every $U \in \mathscr{U}$, $x \notin \overline{U}$. Then (3) follows by maximality, and $\mathscr{U}$ is countable by ccc. It is clear that $\mathscr{U}$ is infinite since X is crowded. □

It is a well-known result of Souslin that every uncountable completely metrizable separable space contains a copy of the Cantor set 2^ω [7, p. 437].

A space is *pseudocompact* if every real valued continuous function on X is bounded. A subspace Y of X is called *G_δ-dense in X* provided that every nonempty G_δ-subset of X meets Y. A useful characterization of pseudocompactness was obtained by Gillman and Jerison [3, p. 95, 6I.1]. They showed that a space X is pseudocompact if and only X is G_δ-dense in βX. Here βX denotes the Čech–Stone-compactification of X. If X is a space, then a subset Z of X is called a *zero-set* of X if there is a continuous function $f\colon X \to [0,1]$ such that $f^{-1}(\{0\}) = Z$. See [3] for more information on these concepts.

Lemma 2.2. *Let X be pseudocompact space and let Z be a zero-set of X. Then $\mathrm{cl}_{\beta X}(Z)$ is a zero-set of βX.*

Proof. This is Gillman and Jerison [3, 8B.5]. □

Corollary 2.3. *Let X be pseudocompact and let Z be a zero-set of X. If $f\colon X \to K$ is continuous, and K is metrizable, then $f(Z)$ is compact.*

Proof. Observe that $f(X)$ is compact, so the function f extends (uniquely) to a continuous function $\beta f\colon \beta X \to K$. Pick an arbitrary $x \in \overline{f(Z)}$. Consider the set $(\beta f)^{-1}(\{x\})$. It is a compact G_δ-subset of βX. If $(\beta f)^{-1}(\{x\}) \cap \mathrm{cl}_{\beta X}(Z) = \emptyset$, then $x \notin \beta f(\mathrm{cl}_{\beta X}(Z)) \subseteq \overline{f(Z)}$, which is a contradiction. Since X is G_δ-dense in βX, it consequently follows by Lemma 2.2 that

$$\emptyset \neq ((\beta f)^{-1}(\{x\}) \cap \mathrm{cl}_{\beta X}(Z)) \cap X = f^{-1}(\{x\}) \cap Z,$$

hence $x \in f(Z)$. □

Lemma 2.4. *Let X be pseudocompact and let $\mathscr{A}$ be a countable collection of closed subsets of X. If $U \subseteq X$ is open and nonempty and $U \subseteq \bigcup \mathscr{A}$, then there exists $A \in \mathscr{A}$ such that $A \cap U$ has nonempty interior in X.*

Proof. Assume that for every $A \in \mathscr{A}$ we have that $A \cap U$ has empty interior in X. List $\mathscr{A}$ as $\{A_n : n < \omega\}$. Observe that $U \setminus A_0$ is nonempty and open, hence there is a nonempty open subset U_0 of X such that $U_0 \subseteq \overline{U}_0 \subseteq U \setminus A_0$. Similarly, $U_0 \setminus A_1 \neq \emptyset$. Hence there is a nonempty open subset U_1 of X such that $U_1 \subseteq \overline{U}_1 \subseteq U_0 \setminus A_1$. Continuing in this way inductively, we obtain a decreasing sequence of nonempty open set $(U_n)_n$ such that $U_n \subseteq \overline{U}_n \subseteq U_{n-1} \setminus A_n$. By Gillman and Jerison [3, Lemma 9.13] we have that $\bigcap_{n<\omega} \overline{U}_n \neq \emptyset$ since X is pseudocompact. But this contradicts the fact that $\mathscr{A}$ covers U. □

Observe that this lemma implies the well-known fact that every pseudocompact space is Baire.

If X is a space, then τX denotes its topology. A family $\mathscr{A}$ of subsets of X is called *cellular* if its members are pairwise disjoint.

Let $\mathbb{I}$ denote the closed interval $[0, 1]$. Let κ denote an infinite cardinal. If $A \subseteq B \subseteq \kappa$, then $\pi^B_A : \mathbb{I}^B \to \mathbb{I}^A$ denotes the projection. If $B = \kappa$, then π^B_A will be denoted by π_A.

Let X be a separable metrizable space. A subset A of X is called a *Bernstein set in X* if A and $X \setminus A$ intersect every uncountable compact subset of X. The following lemma and its proof are well-known, it is included for the sake of completeness.

Lemma 2.5. *Every separable metrizable space can be partitioned into a family $\{A_\eta : \eta < \mathfrak{c}\}$ consisting of Bernstein subsets of X (some members of this family may be empty).*

Proof. If X contains no uncountable compact subsets, then we can split X into $|X|$ singleton sets. This gives us the sets A_η for $\eta < |X|$. If $|X| < \mathfrak{c}$, then the other A_η's are defined to be empty.

Hence assume that there is at least one uncountable compact subset of X. Then there are exactly $\mathfrak{c}$ Cantor sets in X since each Cantor set contains $\mathfrak{c}$ Cantor subsets and by Souslin's Theorem, X contains a Cantor set. Let $\{K_\xi : \xi < \mathfrak{c}\}$ enumerate all uncountable compact sets in X such that each of them is listed $\mathfrak{c}$ times. By transfinite induction on $\xi < \mathfrak{c}$, we pick

$$x_\xi \in K_\xi \setminus \{x_\eta : \eta < \xi\}.$$

The set $S_\xi = \{x_\eta : x_\eta \in K_\xi\}$ has size $\mathfrak{c}$ for each $\xi < \mathfrak{c}$. Redefine S_0 as

$$S_0 \cup (X \setminus \{x_\xi : \xi < \mathfrak{c}\}).$$

Enumerate S_ξ for $\xi < \mathfrak{c}$ in a one-to-one way as $\{y^\xi_\eta : \eta < \mathfrak{c}\}$, and for $\eta < \mathfrak{c}$, put $A_\eta = \{y^\xi_\eta : \xi < \mathfrak{c}\}$. It is clear that $\{A_\eta : \eta < \mathfrak{c}\}$ is as required. □

Observe that if X in the above lemma contains an uncountable compact subset K, then every A_η is nonempty since it has to meet K.

3. Countable chain condition spaces

In this section we present the proof of our main result that every pseudocompact crowded ccc space is $\mathfrak{c}$-resolvable. We will first construct a certain tree of nonempty open sets in a given compact ccc space Z. If Z is 2^{ω_1}, then the height of the tree is ω, while it is ω_1 if Z is a (compact) Souslin line. So in general we do not know what its height is, but we do know that it is either ω or ω_1. Then for a crowded ccc pseudocompact space X, we use the tree for $Z = \beta X$ to split X into $\mathfrak{c}$-many pairwise disjoint dense subsets. The tree gives us an inverse system of compact metrizable spaces and the actual splitting is done using that system and partitions of its members into Bernstein sets. An important ingredient in our proof is Souslin's Theorem that we formulated in §2.

(A) compact ccc spaces. We let Z denote any compact crowded ccc space. We assume that Z is a subspace of $\mathbb{I}^\kappa$, for some infinite cardinal κ.

The following lemma is well-known and its proof is included for the sake of completeness.

Lemma 3.1. *For every nonempty open F_σ-subset U of Z there are a countable $A(U) \subseteq \kappa$ and an open subset V of $\mathbb{I}^{A(U)}$ such that $\pi_{A(U)}^{-1}(V) \cap Z = U$.*

Proof. Write U as $\bigcup_{n<\omega} S_n$, where each S_n is compact. By compactness, for each n there exists a finite subset $F(n)$ of κ such that $\pi_{F(n)}(S_n) \cap \pi_{F(n)}(Z \setminus U) = \emptyset$. Now put $A(U) = \bigcup_{n<\omega} F(n)$, and let $V = \mathbb{I}^{A(U)} \setminus \pi_{A(U)}(Z \setminus U)$. $\square$

Let $\mathscr{U}_0$ be a family of open F_σ-subsets of Z given by Lemma 2.1 with $W = Z$. For each $U \in \mathscr{U}_0$, let $A(U)$ be the countable subset of κ we get from Lemma 3.1 for U, and put $A_0 = \bigcup_{U\in\mathscr{U}_0} A(U)$. Let $Y_0 = \pi_{A_0}(Z)$, and let $f_0 \colon Z \to Y_0$ denote the restriction of the projection π_{A_0} to Z.

The crucial property of Y_0 is that it is compact and metrizable while moreover for every $U \in \mathscr{U}_0$ there exists an open subset V_U^0 in Y_0 such that $f_0^{-1}(V_U^0) = U$.

Observe that if we enlarge A_0 to a countable subset B of κ, then $\pi_B(Z)$ has the same 'crucial' property.

Put $F_0 = Y_0 \setminus \bigcup_{U\in\mathscr{U}_0} V_U^0$ and observe that F_0 and $f_0^{-1}(F_0)$ are nowhere dense in Y_0 respectively Z.

Now consider a fixed $U \in \mathscr{U}_0$. Let $\mathscr{U}_1(U)$ be the family of open F_σ-subsets of Z given by Lemma 2.1 with $W = U$. Put $\mathscr{U}_1 = \bigcup_{U\in\mathscr{U}_0} \mathscr{U}_1(U)$. Then $\mathscr{U}_1$ is a countable cellular family in Z consisting of open F_σ-subsets of Z and it 'refines' $\mathscr{U}_0$. Now do the same construction as above to obtain $Y_1 = \pi_{A_1}(Z)$ and f_1 and make sure that $A_0 \subseteq A_1$. Hence there is a continuous surjection $f_0^1 \colon Y_1 \to Y_0$ such that the diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\
Y_0 & & Y_1 \arrow[ll, "f_0^1"]
\end{tikzcd}$$

commutes. The crucial property of Y_1 is that for every $U \in \mathscr{U}_1$ there exists an open subset V_U^1 in Y_1 such that $f_1^{-1}(V_U^1) = U$. Since the diagram commutes, for every $U \in \mathscr{U}_0$ there also exists an open subset V_U^1 in Y_1 such that $f_1^{-1}(V_U^1) = U$. Put $F_1 = Y_1 \setminus \bigcup_{U\in\mathscr{U}_1} V_U^1$ and observe that F_1 and $f_0^{-1}(F_1)$ are nowhere dense in Y_1 respectively Z. Moreover, $(f_0^1)^{-1}(F_0) \subseteq F_1$.

We continue this process in exactly the same way for all $n < \omega$, thus obtaining an increasing sequence of countable subsets $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of κ, collections of open subsets $\mathscr{U}_n$ of Z which refine one another, spaces $Y_0, Y_1, \ldots$ corresponding to $A_0, A_1, \ldots$, and mappings such that all subdiagrams of the following diagram commute:

$$\begin{tikzcd}
 & & & Z \arrow[dlll, "f_0"'] \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr, "f_n"] & & & \\
Y_0 & Y_1 \arrow[l, "f_0^1"] & Y_2 \arrow[l, "f_1^2"] & \cdots \arrow[l] & Y_{n-1} \arrow[l] & Y_n \arrow[l, "f_{n-1}^n"] & \cdots \arrow[l]
\end{tikzcd}$$

For every $n > 1$ and $U \in \mathscr{U}_n$, let $V_U^n \subseteq Y_n$ be open such that $f_n^{-1}(V_U^n) = U$, and put $F_n = Y_n \setminus \bigcup_{U\in\mathscr{U}_n} V_U^n$. Observe that F_n and $f_n^{-1}(F_n)$ are nowhere dense in Y_n respectively Z and that $(f_{n-1}^n)^{-1}(F_{n-1}) \subseteq F_n$.

Put $A = \bigcup_{n<\omega} A_n$, and let $Y^\omega = \pi_A(Z)$. Observe that Y^ω is the inverse limit of the sequence $\{Y_n : n < \omega\}$ and hence the restriction of the projection π_A to Z 'completes' the above diagram.

The collections $\mathscr{U}_n$ for $n < \omega$ form an ω-branching tree under reverse inclusion. The intersection of each path trough the tree is a closed G_δ-subset of Z. Moreover, these closed G_δ-subsets are obviously pairwise disjoint. By ccc, there are at most countably many of them that have nonempty interior. If there are no such paths, the process stops and we put $Y_\omega = Y^\omega$. If there are such paths, we continue the construction exactly in the same way with each of the countably many nonempty interiors. The collection $\mathscr{U}_\omega$ of pairwise disjoint nonempty open F_σ-subsets of Z that we obtain 'refines' $\mathscr{U}_n$ for every $n < \omega$. Now enlarge A to a countable subset B of κ that 'deals' with every element of $\mathscr{U}_\omega$, and put $Y_\omega = \pi_B(Z)$ and let f_ω be the restriction of the projection π_B to Z. The crucial property of Y_ω is that for every $U \in \bigcup_{n\leq\omega}\mathscr{U}_n$ there exists an open subset V_U^ω in Y_ω such that $f_\omega^{-1}(V_U^\omega) = U$. Observe that Y_ω is the inverse limit of the previous Y's precisely when the process stop at stage ω. Put $F_\omega = Y_\omega \setminus \bigcup_{U\in\mathscr{U}_\omega} V_U^\omega$. There is no reason that that F_ω should be nowhere dense, but it is obvious that for every $n < \omega$ we have $(f_n^\omega)^{-1}(F_n) \subseteq F_\omega$. Since Z is crowded, the process does not stop at stage $\omega + 1, \omega + 2, \ldots$, etc. Hence this brings us to stage $\omega+\omega$ and we test again whether there are paths in our tree corresponding to an intersection with nonempty interior. If there are no such paths, the process stops and $Y_{\omega+\omega}$ is the inverse limit of the previous Y's. If there are such paths, we continue until the next limit ordinal. The process definitely stops at stage ω_1, since by ccc, each path in our tree 'dies' before ω_1. But there is no reason why there could not be paths of arbitrarily large countable length.

In any case, the process stops at an infinite limit ordinal $\lambda \leq \omega_1$. This brings us to the following inverse system

$$\begin{tikzcd}
 & & & & Z \arrow[dllll, "f_0"'] \arrow[dlll] \arrow[dll] \arrow[d] \arrow[dr] \arrow[drrr, "f_\lambda"] & & & \\
Y_0 & Y_1 \arrow[l, "f_0^1"] & Y_2 \arrow[l, "f_1^2"] & \cdots \arrow[l] & Y_\alpha \arrow[l] & Y_{\alpha+1} \arrow[l, "f_\alpha^{\alpha+1}"] & \cdots \arrow[l] & Y_\lambda \arrow[l]
\end{tikzcd}$$

and the corresponding tree of open subsets $\mathbb{T} = \{U : (\exists\,\alpha < \lambda)(U \in \mathscr{U}_\alpha)\}$, ordered in the obvious way. Observe that $\mathbb{T}$ does not have uncountable chains nor anti-chains. Also observe that Y_α for $\alpha < \lambda$ is compact metrizable, and that Y_λ may have weight ω_1.

As before, for every $\alpha < \lambda$ and $U \in \mathscr{U}_\alpha$, let V_U^α in Y_α be open such that $f_\alpha^{-1}(V_U^\alpha) = U$, and put $F_\alpha = Y_\alpha \setminus \bigcup_{U\in\mathscr{U}_\alpha} V_U^\alpha$. Observe that $f_\alpha^{-1}(F_\alpha) = Z \setminus \bigcup \mathscr{U}_\alpha$, and that $(f_\beta^\alpha)^{-1}(F_\beta) \subseteq F_\alpha$ if $\alpha > \beta$.

Lemma 3.2. *If $\lambda = \omega_1$, then $\{f_\alpha^{-1}(F_\alpha) : \alpha < \lambda\}$ covers Z.*

Proof. If there exists an $x \in Z \setminus \bigcup_{\alpha<\lambda} f_\alpha^{-1}(F_\alpha)$, then for each $\alpha < \lambda$ there exists $U_\alpha \in \mathscr{U}_\alpha$ such that $x \in U_\alpha$. But then $(U_\alpha)_{\alpha<\lambda}$ is an uncountable chain in $\mathbb{T}$. $\square$

Lemma 3.3. *If $p \in F_\alpha \setminus \bigcup_{\beta<\alpha}(f_\beta^\alpha)^{-1}(F_\beta)$ for some $\alpha < \lambda$, then $f_\alpha^{-1}(\{p\})$ has empty interior in Z.*

Proof. If $\alpha = 0$, then there is nothing to prove since $f_0^{-1}(F_0)$ is nowhere dense by construction. Assume therefore that $\alpha > 0$. Let $q \in Z$ be such that $f_\alpha(q) = p$. Then $f_\beta(q) \notin F_\beta$ for every $\beta < \alpha$. Hence for every $\beta < \alpha$ there exists $U_\beta \in \mathscr{U}_\beta$ such that $q \in U_\beta$. For every $\beta < \alpha$ there also exists an open subset V_β of Y_α such that $f_\alpha^{-1}(V_\beta) = U_\beta$. Since $f_\alpha(q) = p$, this means that $f_\alpha^{-1}(\{p\}) \subseteq \bigcap_{\beta<\alpha} U_\beta$. If $f_\alpha^{-1}(\{p\})$ has nonempty interior, then there exists $U \in \mathscr{U}_\alpha$ such that $U \subseteq \bigcap_{\beta<\alpha} U_\beta$ and $U \cap f_\alpha^{-1}(\{p\}) \neq \emptyset$. But then $f_\alpha(q) = p \in V_U^\alpha \subseteq Y_\alpha \setminus F_\alpha$, which is a contradiction. $\square$

(B) Pseudocompact ccc spaces. Now let X be an arbitrary crowded ccc pseudocompact space. We put $Z = \beta X$ and assume that we created for this Z the inverse system in (A). If $A \subseteq X$, then $\overline{A}$ and $\mathrm{cl}_Z(A)$ denote the closures of A in X and Z, respectively.

Observe that X is G_δ-dense in Z. (Here we use Gillman and Jerison [3, p. 95, 6I.1], see §2.)

Let $\mathscr{Z}'(X)$ be the collection of all zero-sets of X with nonempty interior (in X).

For every nonempty open subset V of X, we let $\kappa(V) < \omega_1$ be the least ordinal $\mu < \omega_1$ for which $f_\mu^{-1}(F_\mu) \cap V$ has nonempty interior if such an ordinal exists, and ∞ otherwise.

Lemma 3.4. *Let $V \subseteq X$ be nonempty and open and assume that $\kappa = \kappa(V) < \omega_1$. Then there exists $\mu \le \kappa$ such that $f_\mu(V) \cap \big(F_\mu \setminus \bigcup_{\delta<\mu} (f_\delta^\mu)^{-1}(F_\delta)\big)$ contains a Cantor set.*

Proof. Pick $B \in \mathscr{Z}'(X)$ such that $B \subseteq V$ and $f_\kappa(B) \subseteq F_\kappa$. By Corollary 2.3, $f_\mu(B)$ is compact for every $\mu \le \kappa$. Assume that for every $\mu \le \kappa$ we have that

$$A_\mu = f_\mu(B) \cap \big(F_\mu \setminus \bigcup_{\delta<\mu} (f_\delta^\mu)^{-1}(F_\delta)\big)$$

is countable. By Lemma 3.3 it consequently follows that B is a first category subset of X, which contradicts the fact that X is Baire. Hence there exists $\mu \le \kappa$ such that A_μ is uncountable. But then the G_δ-set A_μ contains a Cantor set by Souslin's Theorem [7, p. 437]. $\square$

Lemma 3.5. *Suppose that some nonempty open set $V \subseteq X$ has the following property: for every $B \in \mathscr{Z}'(X)$ such that $B \subseteq V$ and every $\alpha < \lambda$, the interior of B in X meets $\bigcup \mathscr{U}_\alpha$. Then if $\lambda < \omega_1$,*

$$f_\lambda(V) \cap (Y_\lambda \setminus \bigcup_{\beta<\lambda} (f_\beta^\lambda)^{-1}(F_\beta)) \text{ contains a Cantor set,}$$

and if $\lambda = \omega_1$,

$$f_\alpha(V) \cap (F_\alpha \setminus \bigcup_{\beta<\alpha} (f_\beta^\alpha)^{-1}(F_\beta)) \text{ contains a Cantor set}$$

for some $\alpha < \omega_1$.

Proof. Pick $B \in \mathscr{Z}'(X)$ such that $B \subseteq V$ and let E be a nonempty open subset of X that is contained in B.

Claim 1. *For every $\beta < \lambda$ there exist $\beta < \alpha < \lambda$ and disjoint elements $W_\alpha^0, W_\alpha^1 \in \mathscr{U}_\alpha$ that both meet E.*

For every $\xi < \lambda$, pick $U_\xi \in \mathscr{U}_\xi$ such that $U_\xi \cap E \neq \emptyset$. If $E \subseteq \mathrm{cl}_Z(U_\alpha)$ for every $\beta < \alpha < \lambda$, then $\mathrm{cl}_Z(E) \subseteq \bigcap_{\beta<\alpha<\lambda} U_\alpha$. This is a contradiction if $\lambda < \omega_1$ since $\mathrm{cl}_Z(E)$ has nonempty interior in Z. It is also a contradiction if $\lambda = \omega_1$ since in that case $(U_\alpha)_{\beta<\alpha<\lambda}$ is an uncountable chain in $\mathbb{T}$. Hence there exists $\beta < \alpha < \lambda$ such that $E' = E \setminus \mathrm{cl}_Z(U_\alpha) \neq \emptyset$. By assumption, there exists an element $W \in \mathscr{U}_\alpha$ that meets E'. Then clearly $W \neq U_\alpha$. Hence we conclude that there exist disjoint elements W_α^0 and W_α^1 in $\mathscr{U}_\alpha$ that both meet E.

Pick α_0 as in Claim 1 for $\beta = 0$. Observe that for any $x \in W_{\alpha_0}^0$ and $y \in W_{\alpha_0}^1$ we have that $f_{\alpha_0}(x) \neq f_{\alpha_0}(y)$.

Let V_0 and V_1 be nonempty open subsets of X such that $V_i \subseteq \overline{V}_i \subseteq W_{\alpha_0}^i \cap E$ for $i = 0, 1$. By Claim 1, there are a countable ordinal number $\alpha_0 < \xi_0 < \lambda$ and elements $W_{\xi_0}^{00}, W_{\xi_0}^{01} \in \mathscr{U}_{\xi_0}$ such that $W_{\xi_0}^{00}, W_{\xi_0}^{01}$ are disjoint and both meet V_0. Again by Claim 1, there are a countable ordinal number $\alpha_0 < \xi_1 < \lambda$ and elements $W_{\xi_1}^{10}, W_{\xi_1}^{11} \in \mathscr{U}_{\xi_1}$ such that $W_{\xi_1}^{10}, W_{\xi_1}^{11}$ are disjoint and both meet V_1. We may assume without loss of generality that $\xi_0 \le \xi_1$. There exist by our assumptions elements $W_{\xi_1}^{00}, W_{\xi_1}^{01} \in \mathscr{U}_{\xi_1}$ such that $W_{\xi_1}^{00} \cap (V_0 \cap W_{\xi_0}^{00}) \neq \emptyset$ and $W_{\xi_1}^{01} \cap (V_0 \cap W_{\xi_0}^{01}) \neq \emptyset$. Since $\mathscr{U}_{\xi_1}$ 'refines' both $\mathscr{U}_{\xi_0}$ and $\mathscr{U}_{\alpha_0}$ and $\mathscr{U}_{\xi_0}$ 'refines' $\mathscr{U}_{\alpha_0}$, we get $W_{\xi_1}^{10} \cup W_{\xi_1}^{11} \subseteq W_{\alpha_0}^1$ and $W_{\xi_1}^{00} \cup W_{\xi_1}^{01} \subseteq W_{\xi_0}^{00} \cup W_{\xi_0}^{01} \subseteq W_{\alpha_0}^0$. Hence the collection $W_{\xi_1}^{00}, W_{\xi_1}^{01}, W_{\xi_1}^{10}, W_{\xi_1}^{11}$

is pairwise disjoint. Put $\alpha_1 = \xi_1$. Now pick nonempty open sets $V_{00}, V_{01}, V_{10}, V_{11}$ of X such that $\overline{V}_{00} \subseteq V_0 \cap W^{00}_{\alpha_1}, \overline{V}_{01} \subseteq V_0 \cap W^{01}_{\alpha_1}, \overline{V}_{10} \subseteq V_1 \cap W^{10}_{\alpha_1}$, and $\overline{V}_{11} \subseteq V_1 \cap W^{11}_{\alpha_1}$. Observe that for $a \in V_{00}, b \in V_{01}, c \in V_{10}$, and $d \in V_{11}$, the elements $f_{\alpha_1}(a), f_{\alpha_1}(b), f_{\alpha_1}(c), f_{\alpha_1}(d)$ are pairwise distinct.

Continuing in this way inductively, we build a Cantor tree of open subsets of X in the standard way and a sequence of countable ordinals $\beta < \alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$ each corresponding to a level of the tree. Let $\alpha = \sup_{n<\omega} \alpha_n$; we can easily arrange that $\alpha = \lambda$ if $\lambda < \omega_1$. Each path in the Cantor tree has nonempty intersection, again by pseudocompactness of X and Gillman and Jerison [3, p. 95, 6I.1]. Also, each path in the Cantor tree corresponds to a chain in $\mathbb{T}$. Only countably many intersections of these chains have nonempty interior in Z. The intersections of all but these countably many chains are contained in $f_\alpha^{-1}(F_\alpha) \setminus \bigcup_{\beta<\alpha} f_\beta^{-1}(F_\beta)$ if $\lambda = \omega_1$, and in $Z \setminus \bigcup_{\beta<\lambda} (f_\beta)^{-1}(F_\beta)$ if $\lambda < \omega_1$. This shows that

$$f_\alpha(B) \cap \big(F_\alpha \setminus \bigcup_{\beta<\alpha} (f^\alpha_\beta)^{-1}(F_\beta)\big)$$

is uncountable if $\lambda = \omega_1$, and

$$f_\lambda(B) \cap \big(Y_\lambda \setminus \bigcup_{\beta<\lambda} (f^\lambda_\beta)^{-1}(F_\beta)\big)$$

is uncountable if $\lambda < \omega_1$. But an uncountable G_δ-set contains a Cantor set by Souslin's Theorem [7, p. 437]. $\square$

(C) The proof. We will now present the proof of our main theorem. To this end, let X be any pseudocompact crowded ccc space. We adopt the notation in the previous sections for X and $Z = \beta X$. We will use the following fact several times. Suppose that $B \in \mathscr{Z}'(X)$ and there exists $\beta < \lambda$ such that $f_\beta(B)$ contains a Cantor set. Then $f_\alpha(B)$ contains a Cantor set for every $\beta \le \alpha < \lambda$. The proof is easy. Simply observe that $f_\alpha(B)$ is compact by Corollary 2.3 and uncountable since it maps onto $f_\beta(B)$. As a consequence, $f_\alpha(B)$ contains a Cantor set, again by Souslin's Theorem [7, p. 437].

As to be expected, we have to distinguish between two subcases.

Case 1. *$\lambda < \omega_1$.*

This is the easy case.

Lemma 3.6. *For each nonempty open subset $V \subseteq X$ we have that $f_\lambda(V)$ contains a Cantor set.*

Proof. By Lemma 3.5 we may assume without loss of generality that there exist $B \in \mathscr{Z}'(X)$ and $\alpha < \lambda$ such that $B \subseteq V$ and the interior of B misses $\bigcup \mathscr{U}_\alpha$. By shrinking B if necessary, we may assume that B misses $\bigcup \mathscr{U}_\alpha$ from which it follows that $f_\alpha(B) \subseteq F_\alpha$. Hence we are done by Lemma 3.4. $\square$

Now split Y_λ into a family $\mathscr{A}$ consisting of $\mathfrak{c}$ nonempty Bernstein sets in Y_λ (Lemma 2.5). There are many Cantor sets in Y_λ by the previous result. Hence this is indeed possible.

Lemma 3.7. *For every $A \in \mathscr{A}$, $f_\lambda^{-1}(A) \cap X$ is dense in X.*

Proof. Indeed, let $V \subseteq X$ be dense and open, and pick an arbitrary $A \in \mathscr{A}$. By Lemma 3.6, $f_\lambda(V) \cap A \neq \emptyset$, and so $V \cap (f_\lambda^{-1}(A) \cap X) \neq \emptyset$. $\square$

We now turn to the more complicated case.

Case 2. $\lambda = \omega_1$.

For every $\alpha < \omega_1$, let $\{A^\alpha_\xi : \xi < \mathfrak{c}\}$ be a partition of $F_\alpha \setminus \bigcup_{\beta<\alpha}(f^\alpha_\beta)^{-1}(F_\beta)$ into $\mathfrak{c}$ Bernstein sets in $F_\alpha \setminus \bigcup_{\beta<\alpha}(f^\alpha_\beta)^{-1}(F_\beta)$ (recall that some members of this family may be empty). For $\xi < \mathfrak{c}$, put

$$B_\xi = \bigcup_{\alpha<\omega_1} f_\alpha^{-1}(A^\alpha_\xi) \cap X.$$

Then $\mathscr{B} = \{B_\xi : \xi < \mathfrak{c}\}$ partitions X by Lemma 3.2. We claim that every B_ξ is also dense. To this end, pick arbitrary $\xi < \mathfrak{c}$ and nonempty open $V \subseteq X$. By Lemma 3.5 we may assume without loss of generality that there exist $B \in \mathscr{Z}'(X)$ and $\alpha < \lambda$ such that $B \subseteq V$ and the interior of B misses $\bigcup \mathscr{U}_\alpha$. By shrinking B if necessary, we may assume that B misses $\bigcup \mathscr{U}_\alpha$ from which it follows that $f_\alpha(B) \subseteq F_\alpha$. Hence by Lemma 3.4 there exists $\gamma \le \alpha$ such that $f_\gamma(B) \setminus \bigcup_{\beta<\gamma}(f^\gamma_\beta)^{-1}(F_\beta)$ contains a Cantor set and consequently meets A^α_ξ.

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